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Constant Parameter Identification: An Accelerated Heavy–Ball–based Approach

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Abstract—**This paper contributes to designing a parameter identification algorithm for linear regression systems with constant unknown parameters. The proposed algorithm is based on an accelerated version of the heavy–ball method and uses a nonlinear version of Kreisselmeier's regressor extension. Moreover, it can identify constant parameters in a finite time under a persistent excitation condition. The local stability analysis is developed using a Lyapunov function approach. The applicability and effectiveness of the proposed parameter identification algorithm are illustrated through simulation results.**

Index Terms—**Parameter Identification, Heavy–Ball Method, Finite–Time.**

I. Introduction

In control theory, the problem of control design, in the presence of parametric uncertainties and unknown inputs, remains challenging. The tools developed by the parameter identification theory allow us to deal with such a lack of knowledge. Online parameter identification has attracted considerable attention in recent decades. Many techniques address the identification problem, *e.g.*, least–squares (LS) algorithms, gradient descent–based algorithms, and adaptive estimation, to mention a few popular techniques (see, for instance, [1], [2], and [3]).

Most proposed approaches in parameter identification have addressed the identification of constant parameters. To recall some recent works, in [4], a recursive LS method is designed to identify constant parameters for flexible joints. Under some persistence of excitation conditions, the parameter identification error converges to a region around the origin. In [5], a switching adaptive parameter identification algorithm is proposed for robot manipulators, with the error converging to zero in finite time. In [6], a parameter identification algorithm is proposed for homogeneous systems based on a class of artificial neural networks. The parameter identification error converges to a region around the origin. In [7], a linear regression model is used to ensure the boundedness of the parameter identification error in a finite time without the persistent excitation condition. In [8], an adaptive identification method is proposed to estimate constant parameters for sinusoidal signals, *e.g.*, the unknown offset, amplitude, frequency, and phase, with exponential error convergence to a region around the origin. Similarly, in [9], an adaptive algorithm based on a gradient–descent method solves the problem of constant parameter identification. However, most of the above–mentioned algorithms can only ensure exponential convergence rates.

One method has been proposed to improve the rate of convergence of gradient descent–based algorithms: the heavy–ball method, which is the first optimization approach using momentums [10]. Nowadays, it is one of the most popular methods for dealing with optimization problems and designing machine learning algorithms that provide a fast convergence rate when objective functions are smooth and strongly convex. For instance, in [11], the authors provide an analytical characterization of the convergence regions of accelerated gradient descent algorithms based on the frequency domain. The heavy–ball method is also used for distributed optimization problems. For example, in [12], a distributed heavy–ball method, which combines the $\mathcal{AB}$ algorithm, *i.e.*, a generalization of distributed first–order methods, with gradient tracking and momentum term, is proposed to minimize a sum of smooth and strongly–convex functions. A similar problem is studied in [13], where the authors develop an accelerated distributed gradient method for fixed networks, which relies on gradient tracking techniques and local memory to accelerate the convergence speed. In [14], a generalized heavy–ball method is introduced to deal with the global convergence for some non–convex problems. A family of distributed momentum methods is proposed in [15] for distributed optimization problems over directed graphs. The authors develop a global R–linear convergence analysis for smooth and strongly convex functions. In [16], a second–order differential equation is introduced from the control theory framework, called heavy–ball dynamics with displaced gradient, and possesses the same convergence properties as the original heavy–ball dynamics. The authors use a Lyapunov function approach to characterize the asymptotic convergence properties of the resulting discrete–time algorithms. In [17], a hybrid control approach is proposed to ensure fast convergence and global asymptotic stability

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for an optimization problem of the unique minimizer of a convex function. The proposed approach is based on a hysteresis mechanism that switches between Nesterov's accelerated gradient descent and the heavy–ball method to ensure the asymptotic convergence employing a Lyapunov function approach. However, none of those mentioned above works offer a finite–time convergence rate. Moreover, only two provide a Lyapunov function characterization of the accelerated convergence rate.

A novel parameter identification algorithm is proposed in this paper for linear regression systems with constant unknown parameters. The proposed algorithm is based on an accelerated heavy–ball method that uses a nonlinear version of Kreisselmeier's regressor extension. This accelerated heavy–ball–based algorithm can identify constant unknown parameters in finite time under a persistence of excitation condition. The parameter identification error's local finite–time stability is ensured through a Lyapunov function approach. To the best of our knowledge, this is the first time, despite a number of its modifications, that such an accelerated version of the heavy–ball method is proposed, ensuring finite–time convergence.

The rest of this manuscript is organized as follows. Some preliminary knowledge is presented in Section II. The problem statement is formulated in Section III. The accelerated heavy–ball–based parameter identification algorithm is presented in Section IV. The convergence analysis of the parameter identification error is given in Section V, while we provide some simulation results in Section VI. Finally, Section VII gives the concluding remarks.

Notation: We denote $\lceil s \rfloor^{\alpha} := |s|^{\alpha}\mathrm{sign}(s)$, for $s \in \mathbb{R}$ and $\alpha \geq 0$. For a vector $s \in \mathbb{R}^n$, $\lceil s \rfloor^{\alpha}$ is understood componentwise. For $p \geq 1$, the L^p–norm of $s \in \mathbb{R}^n$ is defined by $|s|_p = (\sum_{i=1}^{n} |s_i|^p)^{\frac{1}{p}}$. So $|s|_{\alpha+1}^{\alpha+1} = s^{\top}\lceil s \rfloor^{\alpha}$, for any $\alpha > 0$. The Euclidean norm of $s \in \mathbb{R}^n$ is denoted by $|s| := |s|_2$, and for a matrix $A \in \mathbb{R}^{m\times n}$, the induced norm is the spectral norm, *i.e.,* $|A| = \sqrt{\lambda_{\max}(A^{\top}A)}$, where $\lambda_{\max}$ (respectively, $\lambda_{\min}$) is the maximum (respectively, the minimum) eigenvalue. The set of all inputs $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^p$ such that its L_{∞} norm on $[0,\infty]$ is finite, *i.e.,* $|u|_{\infty} :=$ ess $\sup_{t\geq 0}|u(t)| < \infty$, is denoted as $\mathcal{L}_{\infty}$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$.

II. Preliminaries

Consider the system

$$\dot{x}(t) = f(t, x(t)),\ t \in \mathbb{R}_{\geq 0},\ x(0) = x_0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector. The function $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be locally bounded uniformly in t and locally Lipschitz or Hölder continuous function in x, and $f(t, 0) = 0$. For an initial condition $x_0 \in \mathbb{R}^n$, the solution of system (1) is denoted as $x(t, x_0)$, and for brevity we assume that it is defined for any $t \geq 0$.

Definition 1. [18], [19]. *At the steady state $x = 0$, the system* (1) *is* finite–time stable (FTS) *if there exists a function $\alpha \in \mathcal{K}$ such that $|x(t, x_0)| \leq \alpha(|x_0|)$, for all $t \geq 0$ and $x_0 \in \mathbb{R}^n$, and there exists $0 \leq T_s < +\infty$ such that $x(t, x_0) = 0$, for all $t \geq T_s$. The function $T_0 : x_0 \mapsto \inf\{T_s \geq 0 : x(t, x_0) = 0,\ \forall t \geq T_s\}$ is called the* settling–time *of the system* (1).

The following lemmas are used.

Lemma 1. [20]. *Let $x_1, x_2 \in \mathbb{R}^n$, and $W(x_1, x_2) = |x_1|^{\alpha} + |x_2|^{\beta} - c|x_1|^{\gamma}|x_2|^{\delta}$, with some positive constants $c, \alpha, \beta, \gamma, \delta > 0$. Then, the following statements hold:*

1) For any value of c, W is positive definite if

$$\frac{\gamma}{\alpha} + \frac{\delta}{\beta} > 1 \ \text{and}\ \max\left\{|x_1|^{\alpha}, |x_2|^{\beta}\right\} \leq c^{\frac{1}{1-\left(\frac{\gamma}{\alpha}+\frac{\delta}{\beta}\right)}}.$$

2) For any value of c, W is positive if

$$\frac{\gamma}{\alpha} + \frac{\delta}{\beta} < 1 \ \text{and}\ \max\left\{|x_1|^{\alpha}, |x_2|^{\beta}\right\} \geq c^{\frac{1}{1-\left(\frac{\gamma}{\alpha}+\frac{\delta}{\beta}\right)}}.$$

3) For sufficiently small values of c and any $x_1, x_2 \in \mathbb{R}^n$, W is positive definite if

$$\frac{\gamma}{\alpha} + \frac{\delta}{\beta} = 1.$$

Lemma 2. [21]. *Let $x_1, x_2 \in \mathbb{R}^n$, and $W(x_1, x_2) = |x_1|^{\alpha} + |x_2|^{\beta} + c_1|x_1|^{\eta}|x_2|^{\zeta} - c_2|x_1|^{\gamma}|x_2|^{\delta}$, with some positive constants $c_1, c_2, \alpha, \beta, \gamma, \delta, \eta, \zeta > 0$. Then, for any values of $c_1, c_2 > 0$, W is positive definite if and only if*

$$\frac{\eta}{\alpha} + \frac{\zeta}{\beta} < 1,\ \gamma + \delta\frac{(\alpha-\eta)}{\zeta} > \alpha,\ \delta + \gamma\frac{(\beta-\zeta)}{\eta} > \beta.$$

III. Problem Statement

Let us consider the following linear regression system:

$$y(t) = \phi^{\top}(t)\theta,\ t \geq 0, \tag{2}$$

where $\theta \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}$ are the unknown vector of constant parameters and the output, respectively, and $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is a continuously differentiable function of time, so–called *regressor*, which is known, bounded, and persistently excited (PE). This work aims to identify the constant parameter vector θ under the following assumption:

Assumption 1. *The regressor ϕ is PE, bounded (i.e., $|\phi\phi^{\top}|_{\infty} \leq \bar{\phi}$), and it satisfies $\left|\frac{d(\phi\phi^{\top})}{dt}\right|_{\infty} \leq \bar{\phi}_d$, for some known positive constants $\bar{\phi}$, $\bar{\phi}_d > 0$.*

IV. Heavy–ball–based Algorithm

The proposed accelerated heavy–ball–based parameter identification algorithm is given as

$$\dot{\hat{\theta}}_1 = \hat{\theta}_2, \tag{3a}$$

$$\dot{\hat{\theta}}_2 = \tau k_1 z_2^{\top}\lceil z_1 - z_2\hat{\theta}_1\rfloor^{\alpha} - k_2\lceil\hat{\theta}_2\rfloor^{\beta}, \tag{3b}$$

$$\dot{z}_1 = \tau^{-1}\rho(z,\hat{\theta})\left[\phi y - z_1\right],\ z_1(0) = 0, \tag{3c}$$

$$\dot{z}_2 = \tau^{-1}\rho(z,\hat{\theta})\left[\phi\phi^{\top} - z_2\right],\ z_2(0) = 0, \tag{3d}$$

$$\rho(z,\hat{\theta}) = \begin{cases} 1, & \forall t \in [0, T), \\ |z_1 - z_2\hat{\theta}_1|_{\gamma+1}^{\gamma+1} + |\hat{\theta}_2|_{\mu+1}^{\mu+1}, & \forall t \geq T, \end{cases} \tag{3e}$$

where $\hat{\theta}_1 \in \mathbb{R}^n$ is the identified value of the parameter vector θ, $\hat{\theta}_2 \in \mathbb{R}^n$ is an auxiliary velocity vector, $z_1 \in \mathbb{R}^n$ provides a filtered version of the term $\phi\, y$, $z_2 = z_2^\top \in \mathbb{R}^{n\times n}$ represents a filtered version of the matrix $\phi\,\phi^\top$, the filter time constant is given by $\tau > 0$, the positive function ρ represents a filter nonlinear gain, with some excitation time $T > 0$, the parameters of the algorithm are $\alpha, \beta \in (0, 1)$, and γ, μ, k_1, $k_2 > 0$.

Note that the dynamics in (3b) describes the accelerated heavy–ball method, where the nonlinear term $\lceil \hat{\theta}_2 \rfloor^\beta$ represents the added "velocity" or "momentum" term. Moreover, the dynamics (3c)–(3e) provide a nonlinear version of Kreisselmeier's regressor extension.

A. Parameter Identification Error Dynamics

Let us define the parameter identification error as

$$e_1 = \theta - \hat{\theta}_1,\ e_2 = -\hat{\theta}_2. \tag{4}$$

Based on (2) and (3), the error dynamics can be written as follows

$$\dot{e}_1 = e_2, \tag{5a}$$
$$\dot{e}_2 = -\tau k_1 z_2^\top \lceil z_1 - z_2\hat{\theta}_1 \rfloor^\alpha + k_2 \lceil \hat{\theta}_2 \rfloor^\beta. \tag{5b}$$

From (3c) and (3d), it follows that $z_1(t) = z_2(t)\theta$, for all $t \geq 0$ (the initial conditions of the filters are chosen to be zero). Therefore, the error dynamics (5) is rewritten as

$$\dot{e}_1 = e_2,\ e_1(0) := e_{10}, \tag{6a}$$
$$\dot{e}_2 = -\tau k_1 z_2^\top \lceil z_2 e_1 \rfloor^\alpha - k_2 \lceil e_2 \rfloor^\beta,\ e_2(0) := e_{20}. \tag{6b}$$

The parameter identification error dynamics (6) can be viewed as a dynamical system, which characterizes an accelerated time–dependent version of the heavy–ball method. Clearly, $e_1 = e_2 = 0$ is an equilibrium of the system (6), uniformly in z_2. A characterization of the finite–time stability of this error dynamics at the origin, in terms of Lyapunov functions, is not straightforward.

B. Some Properties of z_2 and $\rho(z, \hat{\theta})$

First, since ϕ is PE, it follows that [22]

$$\alpha_1 I \geq \int_t^{t+T} \phi(\sigma)\phi^\top(\sigma) d\sigma \geq \alpha_0 I,\ \forall t \geq 0,$$

with a level of excitation $\alpha_0 > 0$, an excitation time $T > 0$, and a positive constant $\alpha_1 > 0$, where I is the identity matrix of respective dimension. The following result will help us to deal with the stability analysis of the parameter identification error dynamics.

Lemma 3. *Let Assumption 1 be satisfied. Then, the trajectories of the system (3d) satisfy*

$$\underline{\phi} := e^{-\frac{1}{\tau}T}\frac{\alpha_0}{\tau} \leq \inf_{t\geq T}(\lambda_{\min}(z_2(t))),\ |z_2|_\infty \leq \bar{\phi}. \tag{7}$$

Moreover, the approximation error for $\phi(t)\phi^\top(t)$, i.e., $\varepsilon(t) = z_2(t) - \phi(t)\phi^\top(t)$, satisfies

$$|\varepsilon|_\infty \leq \bar{\phi} + \tau\bar{\phi}_d := \varepsilon^+. \tag{8}$$

Note that if we show that the error dynamics (6) is FTS, then the function $\rho(z, \hat{\theta})$ will satisfy

$$\lim_{t\to T_s} \rho(z(t), \hat{\theta}(t)) = 0,$$

where T_s is the convergence time of the error dynamics (6), which is dependent on the initial conditions of the system. The switched structure of the function $\rho(z, \hat{\theta})$ allows us to avoid it reaching zero before the excitation time T, and for the system (3), $\rho(z, \hat{\theta}) = 0$ is the invariant subspace containing equilibria. Moreover, if z_2 is non–singular, then this set of equilibria is a singleton, and it corresponds to the convergence of $\hat{\theta}$ to the actual value of θ.

Additionally, when $\rho(z(T_s), \hat{\theta}(T_s)) = 0$, we will have that $\dot{z}_2(T_s) = 0$, and hence $z_2(t)$ will get constant at $z_2(T_s)$, for all $t \geq T_s$. At this time T_s, we already have that $z_2(T_s) \succ 0$, and we will have that $e_1(t, e_{10}) = e_2(t, e_{20}) = 0$, for all $t \geq T_s \geq T$.

V. Stability Analysis

Let us define the following real–valued functions that are useful in the stability analysis:

$$G : t \mapsto \frac{\tau k_1}{\alpha + 1}|z_2(t)e_1(t)|_{\alpha+1}^{\alpha+1}, \tag{9a}$$
$$E : t \mapsto G(t) + \frac{1}{2}e_2(t)^\top e_2(t), \tag{9b}$$
$$F : t \mapsto |z_2(t)e_1(t)|_{q-1}^{q-1} e_1(t)^\top e_2(t),\ q > 1. \tag{9c}$$

Note that the functions G and E are continuously differentiable and positive definite. Based on (6), we have

$$\dot{G} = \tau k_1 \left(\lceil z_2 e_1 \rfloor^\alpha\right)^\top \left[z_2 e_2 - \tau^{-1}\rho(z, \hat{\theta})\varepsilon e_1\right], \tag{10a}$$
$$\dot{E} = -k_1 \rho(z, \hat{\theta})\varepsilon\, e_1^\top \lceil z_2 e_1 \rfloor^\alpha - k_2 e_2^\top \lceil e_2 \rfloor^\beta, \tag{10b}$$
$$\begin{aligned}\dot{F} = &-\tau k_1 |z_2 e_1|_{q-1}^{q-1} |z_2 e_1|_{\alpha+1}^{\alpha+1} - k_2 |z_2 e_1|_{q-1}^{q-1} e_1^\top \lceil e_2 \rfloor^\beta \\ &+ |z_2 e_1|_{q-1}^{q-1}|e_2|^2 + (q-1)\left(\lceil z_2 e_1 \rfloor^{q-2}\right)^\top \times \\ &\times \left[z_2 e_2 - \tau^{-1}\rho(z, \hat{\theta})\varepsilon e_1\right] e_1^\top e_2.\end{aligned} \tag{10c}$$

Based on [23], we propose the following Lyapunov function candidate

$$V = E^{r+1} + \ell F, \tag{11}$$

for some r, $\ell > 0$. The stability analysis, based on (11), is performed in two steps. First, to demonstrate finite–time stability and convergence for $t \geq T$, *i.e.*, to show, based on lemmas 1, 2, and 3, that $\dot{V} \leq -\bar{\eta}V^u$, with some $\bar{\eta} > 0$ and $u \in (0, 1)$, for $t \geq T$. Then, to show that a finite–time escape is impossible for $t \in [0, T)$. Due to space limitations, the complete proof is omitted.

The main result of this work is presented in the following theorem.

Theorem 1. *Consider the system (2) under Assumption 1. If the powers of the accelerated heavy–ball–based algorithm (3) are designed such that the set of conditions*

$$\gamma \geq q - 2 - (\alpha+1)r, \quad (12)$$
$$\frac{\alpha+\gamma+2}{q+\alpha} + \frac{2r}{2r+\beta+1} > 1, \quad (13)$$
$$\frac{(\alpha+1)(r+1)}{q+\alpha} + \frac{\mu+1}{2r+\beta+1} > 1, \quad (14)$$
$$\frac{\alpha+1}{q+\alpha} + \frac{2r+\mu+1}{2r+\beta+1} > 1, \quad (15)$$
$$\frac{\gamma+q+1}{q+\alpha} + \frac{1}{2r+\beta+1} > 1, \quad (16)$$
$$\frac{q}{q+\alpha} + \frac{\mu+2}{2r+\beta+1} > 1, \quad (17)$$
$$q > (\alpha+1)r + \max\left(\frac{\alpha+1}{2}, \frac{\alpha}{\beta}, \frac{\beta(\alpha+1)+(1-\alpha)}{2}\right), \quad (18)$$
$$1 > \alpha > \frac{q-r-1}{r} > 0, \ \alpha < \beta, \quad (19)$$

holds with some $\alpha, \beta \in (0,1)$, and the gains k_1 and k_2 are designed sufficiently big; then, the origin of the parameter identification error dynamics (6) is locally FTS*, with a settling–time function satisfying*

$$T_s \leq T + \frac{V^{1-u}(e_1(T), e_2(T))}{\bar{\eta}(1-u)}.$$

Remark 1. *The set of conditions (12)–(19) always hold for sufficiently big values of q, γ, and μ. Additionally, the finite–time stability is always ensured for sufficiently big values of k_1 and k_2.*

VI. Simulation Results

The simulations have been done in MATLAB with the Euler explicit discretization method and sampling time equal to 0.01[s].

Let us consider an academic example for system (2) with:

$$\phi(t) = \begin{pmatrix} 2\cos(6t) + 3\sin(10t) + 2\sin(5t)\cos(10t) \\ -6\sin(t) + \cos(20t) + 3\sin(10t)\cos(5t) \end{pmatrix},$$
$$\theta = \begin{pmatrix} \theta_{11} \\ \theta_{12} \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}.$$

Fixing $\alpha = 0.8$, $\beta = 0.96$, $\gamma = 0.5$, $\mu = 1$, and $r = 1.5$, we can prove that the set of conditions (12), (13), (18), and (19), hold for $q = 3.674$. Then, taking $\tau = 0.01$, $k_1 = 0.08$, $k_2 = 8$, and $T = 5$, which satisfy the statements of Theorem 1, we apply the parameter identification algorithm (3), considering $\hat{\theta}_1(0) = \hat{\theta}_2(0) = (3, -5)^\top$.

For comparison purposes, we also implement the algorithm (3), when the classic heavy–ball method is considered, *i.e.,* when $\alpha = \beta = 1$ and $\rho(z, \hat{\theta}) = 1$, with the same values for the rest of the parameters. The nomenclature to the legends is as follows: *aHB* corresponds to the proposed accelerated heavy–ball–based algorithm, while *HB* corresponds to the classic heavy–ball–based algorithm. The obtained results are depicted by Figs. 1, 2, and 3.

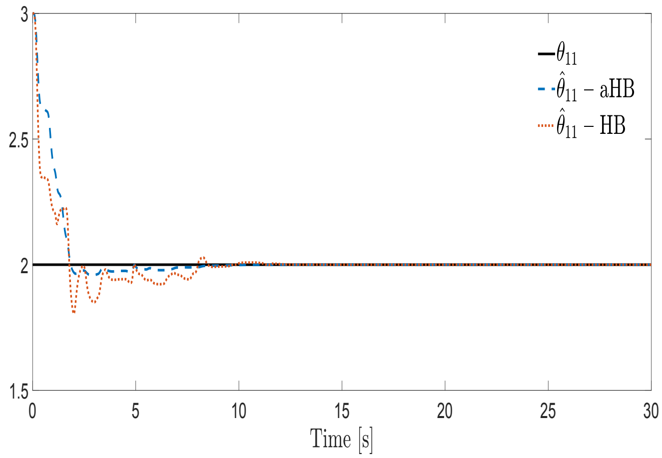

Figure 1. Parameter Identification θ_{11}

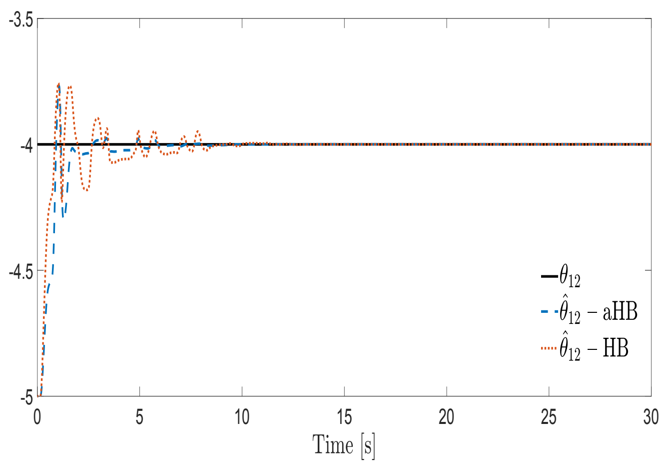

Figure 2. Parameter Identification θ_{12}

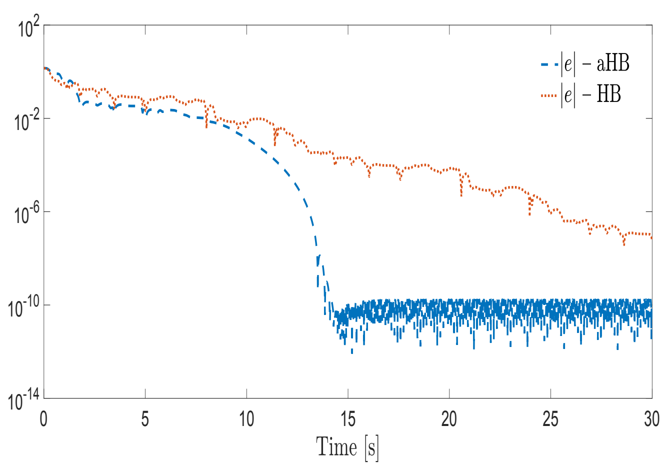

Figure 3. Parameter Identification Error

Figs. 1 and 2 show the parameter identification provided for each of the algorithms. We can see that both algorithms provide a fast identification of the unknown parameters. However, in Fig. 3, which depicts the norm of the parameter identification error e, we can see that the proposed heavy–ball–based algorithm possesses finite–time decay, so a faster convergence rate, and also considerably higher precision than the classic heavy–ball–based algorithm.

VII. Conclusions

In this paper, a novel parameter identification algorithm was proposed for linear regression systems with constant unknown parameters. Our algorithm is based on an accelerated version of the heavy–ball method, which uses a nonlinear version of Kreisselmeier's regressor extension. This accelerated version of the heavy–ball method can identify constant unknown parameters in finite time under a persistence of excitation condition. The local finite–time stability of the parameter identification error dynamics was proven using a Lyapunov function approach. The applicability and effectiveness of the proposed parameter identification algorithm were illustrated through some simulation results.

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