2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

Robust Closed-Loop State Predictor for Unstable Systems with Input Delay

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***Abstract*— The paper addresses the problem of the safe implementation of a state predictor used for stabilization of linear time-invariant systems with input delay. To this end, we design an observer generating an estimate of the state prediction error, i.e., the estimate of the difference between the inaccessible true state prediction and the state prediction estimate elaborated by a state predictor. This estimate is used as feedback in the closed-loop state predictor. The robustness of the proposed closed-loop predictor with respect to parametric perturbations is proved and illustrated by simulation results. Robustness with respect to approximation errors caused by limited accuracy of a numerical solver is discussed and illustrated by simulation results.**

I. INTRODUCTION AND PROBLEM STATEMENT

Consider the problem of state-feedback asymptotic stabilization of the following linear time-invariant system with input delay[1]:

$$\dot{x}(t) = Ax(t) + bu(t-h), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state with the initial condition x_0, $u \in \mathbb{R}^1$ is the control ($u(t) \equiv 0$ for $t < h$), $A \in \mathbb{R}^{n\times n}$ is not Hurwitz, $b \in \mathbb{R}^n$, the pair (A, b) is controllable, and h is the known time delay.

Remark 1: We will assume that the states of all considered dynamical systems are zero for $t < 0$, and under initial condition we will assume the state at the instance $t = 0$, e.g., $x_0 = x(0)$. By $x_T = x(T)$ we will denote the state of an integrator which is set/reset at the instance T. ■

One of the known approaches to the problem consists in the use of the following control law [9], [7]:

$$u(t) = -k^\top \left(e^{Ah}x(t) + \int_{t-h}^{t} e^{A(t-\tau)} bu(\tau)d\tau \right), \tag{2}$$

where the vector of feedback gains $k \in \mathbb{R}^n$ is such that the matrix $A_k = A - bk^\top$ is Hurwitz. It is worth noting that the expression in the brackets is exactly predicted value of the state, i.e.,

$$e^{Ah}x(t) + \int_{t-h}^{t} e^{A(t-\tau)} bu(\tau)d\tau = x(t+h), \tag{3}$$

while the control (2) yields the closed-loop characteristic equation $\det\left(sI - A + bk^\top\right) = 0$.

Stabilizing feedback of the form (2) is widely used in many control techniques such as model reduction through state transformation [1], [2], finite spectrum assignment [9], [14], predictor-based stabilization of systems with multiple delays [6], [13], output regulation in systems with delays [4], [5], [11] and so on.

However, this elegant theoretical result has essential drawback related to practical implementation of the integral term called *distributed delay*. In [9], it was proposed to approximate the integral with some numerical quadrature method when the integral is replaced by a discrete sum with a finite number of terms. It was suggested that the closed-loop system should be stable for sufficiently high accuracy (i.e., for sufficiently large number of the terms). However, in [3] it was demonstrated that for some conventional numerical quadrature methods the control law (2) may not stabilize the closed-loop system even for arbitrary large number of the discrete terms in the sum. To overcome this obstacle, in [10] it was proposed to include a low-pass filter in the control loop, while in [15] modified safe discrete-delay implementations of the distributed delay were designed. However, in [10], [15] robustness of the proposed techniques to the plant parametric perturbations was not analyzed.

An alternative approach to the implementation of the integral term is based on its treatment as the solution of the ordinary differential equation

$$\dot{\psi}(t) = A\psi(t) + bu(t) - e^{Ah}bu(t-h), \quad \psi_0 = 0, \tag{4}$$

where $\psi \in \mathbb{R}^n$. In this case,

$$x(t+h) = e^{Ah}x(t) + \psi(t), \tag{5}$$

and the control (2) takes the form

$$u = -k^\top \left(e^{Ah}x + \psi\right). \tag{6}$$

Hereafter, we will consider (4) and (5) as a virtual mathematical model of the *true state prediction* $x(t+h)$, while a solution $\psi(t)$ of (4) as the unmeasured *true integral term*.

At the same time, we will assume that for implementation of a control law the *integral term estimate* $\hat{\psi}(t)$ is generated as a solution (may be approximated by a numerical solver) of the equation

$$\dot{\hat{\psi}}(t) = \hat{A}\hat{\psi}(t) + \hat{b}u(t) - e^{\hat{A}h}\hat{b}u(t-h), \quad \hat{\psi}_0, \tag{7}$$

where $\hat{A}$ and $\hat{b}$ are some constant estimates of A and b, respectively. In this case, *the state prediction estimate* is

$$\hat{x}(t+h) = e^{\hat{A}h}x(t) + \hat{\psi}(t), \tag{8}$$

while the control takes the form

$$u(t) = -\hat{k}^\top \left(e^{\hat{A}h}x(t) + \hat{\psi}(t)\right), \tag{9}$$

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[1]Sometimes the argument t will be omitted if it does not cause misunderstanding.

where the vector of feedback gains $\hat{k}$ is such that the matrix $\hat{A}_k = \hat{A} - \hat{b}\hat{k}^\top$ is Hurwitz.

We will call the prediction estimate (8) as *the open-loop state prediction estimate*, because the value $\hat{x}(t+h)$ is not corrected by a feedback on the state prediction error as it will be proposed in Section II.

The main drawback of this approach consists in the fact that it is not robust since the model (4) is internally unstable when A is not Hurwitz [9]. It means that the closed-loop system (1), (7), (9) may lose the stability under *arbitrarily small* perturbations. Such perturbations can be caused either by parametric uncertainty of the plant (when some estimates $\hat{A}$ and $\hat{b}$ are used in the control law (7), (9) and $\hat{A} \neq A$, $\hat{b} \neq b$) or by approximation errors.

The main purpose of this paper is to propose a modification of the control law which preserves the boundedness of all the closed-loop signals and asymptotic convergence $x(t) \to 0$ as $t \to \infty$ at least for small parametric and approximation perturbations. The main contributions of the paper are the following:

1) a new stable closed-loop state predictor is proposed;
2) the robustness of the proposed state predictor with respect to the parametric perturbations is proved.

Additionally, robustness with respect to approximation errors caused by a numerical solver is considered as a conjecture and illustrated by simulation results.

The paper is organized as follows. In Section II, for the sake of methodological purposes, we design and analyze the closed-loop predictor under assumption that the parameters A and b of the plant are known exactly and there are no approximation errors. Then, in Section III, for the case of $\hat{A} \neq A$, we prove the robustness of the closed-loop predictor. Simulation results are presented in Section IV.

II. Closed-loop prediction

A. Control law design

To clarify motivation of the design, we start with the ideal case when the control involves the true values of A and b, and the equation (7) is solved absolutely precisely. In this case, instead of equations (7)–(9) we have

$$\dot{\hat{\psi}}(t) = A\hat{\psi}(t) + bu(t) - e^{Ah}bu(t-h), \ \hat{\psi}_0. \tag{10}$$

$$\hat{x}(t+h) = e^{Ah}x(t) + \hat{\psi}(t), \tag{11}$$

$$u(t) = -k^\top \left(e^{Ah}x(t) + \hat{\psi}(t)\right). \tag{12}$$

Here, we set $\hat{\psi}_0 \neq 0$ just to stress the fact that the designed closed-loop predictor is asymptotically stable. Of course, in practical implementation it is reasonable to set $\hat{\psi}_0 = 0$. However, this choice is not crucial due to asymptotic stability of the proposed predictor.

Introduce *the state prediction error*

$$\varepsilon(t) := x(t+h) - \hat{x}(t+h). \tag{13}$$

In view of (5) and (11), we have

$$\varepsilon(t) = \psi(t) - \hat{\psi}(t), \tag{14}$$

and in view of (4) and (10) we obtain:

$$\dot{\varepsilon}(t) = A\varepsilon(t), \quad \varepsilon(0) = -\hat{\psi}_0, \tag{15}$$

where $\varepsilon(t)$ is not measured. However, the value[2]

$$y(t) := \varepsilon(t-h) = x(t) - \hat{x}(t) \tag{16}$$

or (see (11))

$$y(t) = x(t) - e^{Ah}x(t-h) - \hat{\psi}(t-h) \tag{17}$$

is measured and can be considered as the delayed output of the model (15). Then, in view of (15) and (16) we can write:

$$y(t) = e^{-Ah}\varepsilon(t). \tag{18}$$

The key idea of the closed-loop predictor design is to estimate the difference between the inaccessible true state prediction $x(t+h)$ and the state prediction estimate $\hat{x}(t+h)$ described by the model

$$\dot{\varepsilon}(t) = A\varepsilon(t), \quad y(t) = \varepsilon(t-h) \tag{19}$$

with the measured output y, and to use this estimate for online correction of $\hat{x}(t+h)$.

To this end, we use the following observer (which is a particular case of the observer from Chapter 3 of [7]):

$$\dot{\hat{\varepsilon}}(t) = A\hat{\varepsilon}(t) + Le^{Ah}\left(y(t) - e^{-Ah}\hat{\varepsilon}(t)\right), \quad \hat{\varepsilon}_0 = 0, \tag{20}$$

where $\hat{\varepsilon} \in \mathbb{R}^n$ is *the estimate of the state prediction error*, and the designer-chosen matrix $L \in \mathbb{R}^{n\times n}$ is such that $A_L = A - L$ is Hurwitz.

Introducing *the estimate error* $\tilde{\varepsilon} := \varepsilon - \hat{\varepsilon}$ and calculating its time derivative in view of (15), (18), and (20), we obtain

$$\dot{\tilde{\varepsilon}} = A_L\tilde{\varepsilon} \tag{21}$$

and, therefore, $\hat{\varepsilon}(t) \to \varepsilon(t) = x(t+h) - \hat{x}(t+h)$ as $t \to \infty$.

Remark 2: Since the model (18) is valid for $t \geq h$ only (because $x(t-h) \equiv 0$ and $\hat{\psi}(t-h) \equiv 0$ for $0 \leq t < h$), we set $y(t) \equiv 0$ for $0 \leq t < h$. ■

Relation (14) and model (21) motivate us to correct the state prediction estimate according to the expression

$$\bar{x}(t+h) := e^{Ah}x(t) + \hat{\psi}(t) + \hat{\varepsilon}(t). \tag{22}$$

It is easy to see that the following equality holds:

$$\bar{x}(t+h) = x(t+h) - \tilde{\varepsilon}(t), \tag{23}$$

where $\tilde{\varepsilon}(t) \to 0$ as $t \to \infty$. Therefore, instead of (12) we use the control

$$u(t) = -k^\top \left(e^{Ah}x(t) + \hat{\psi}(t) + \hat{\varepsilon}(t)\right). \tag{24}$$

We will call the state prediction estimate (22) as *the closed-loop state prediction estimate* because it is corrected by $\hat{\varepsilon}$ depending on the feedback signal of the state prediction error (see (20) and (16)).

However, despite of the fact that $\tilde{\varepsilon}(t) \to 0$ as $t \to \infty$, the estimates $\hat{\varepsilon}$ and $\hat{\psi}$ may grow unboundedly since the model

[2] Similar approach was employed in [8], where the state prediction error was used to cope with external disturbances.

(19) is not stable. To prevent unbounded growth of $\hat{\psi}$ and $\hat{\varepsilon}$, we propose to reset the states in the models (10) and (20) at each instance mT (where $m = 1, 2, 3, \ldots$ and T is a sufficiently large time interval to be defined later) according to the rule[3]

$$\hat{\psi}_{mT} = \hat{\psi}(mT) + \hat{\varepsilon}(mT), \qquad \hat{\varepsilon}_{mT} = 0. \tag{25}$$

Taking into account that $\hat{\varepsilon} = \varepsilon - \tilde{\varepsilon} = \psi - \hat{\psi} - \tilde{\varepsilon}$, we obtain

$$\hat{\psi}_{mT} = \psi(mT) - \tilde{\varepsilon}(mT). \tag{26}$$

To provide smooth switching of the feedback signal $y(t) - e^{-Ah}\hat{\varepsilon}(t)$ in the observer (20) depending on the past values $\hat{\psi}(t-h)$, instead of (17) we use the equation

$$y(t) = x(t) - e^{Ah}x(t-h) - \bar{\psi}(t-h), \tag{27}$$

where $\bar{\psi}(t-h)$ is defined as

$$\bar{\psi}(t-h) = \begin{cases} \hat{\psi}(t-h) + \hat{\varepsilon}(t-h) \\ \qquad\qquad \text{if } mT \le t < mT + h, \\ \hat{\psi}(t-h) \text{ if } mT + h \le t < (m+1)T. \end{cases} \tag{28}$$

To provide a decreasing sequence $\tilde{\varepsilon}(mT)$ needed for (26), we choose T according to the following rule. Let a positive-defined matrix P be a solution of the matrix equation $A_L^\top P + PA_L = -I$, while λ_1 and λ_2 are the minimum and maximum eigenvalues of the matrix P, respectively. Then, in view of (21), we derive

$$|\tilde{\varepsilon}(t)|^2 \le \frac{\lambda_2}{\lambda_1} e^{-\frac{t}{\lambda_2}} |\tilde{\varepsilon}(0)|^2.$$

To provide inequality $\tilde{\varepsilon}((m+1)T) < \tilde{\varepsilon}(mT)$, we choose the time interval T as

$$T \ge \max\left\{ -\lambda_2 \ln\left(\frac{\lambda_1}{\lambda_2}\right), h \right\}. \tag{29}$$

Remark 3: Note, since $(\lambda_1/\lambda_2) < 1$, then $\ln(\lambda_1/\lambda_2) < 0$ and $-\lambda_2 \ln\left(\frac{\lambda_1}{\lambda_2}\right) > 0$. ■

Thus, the proposed control law consists of: the closed-loop state predictor (10) and (22); the prediction error observer (20), (27), and (28); the controller (24); the scheme of the states resetting (25).

B. Stability properties in the ideal case

To analyze the stability properties, we derive a model of the closed-loop system. As it will be seen later, it is convenient to derive separately two dynamic subsystems. The first one will be represented in the components $(\tilde{\varepsilon}, x, \psi)$, while the second one will be represented in $(\varepsilon, \hat{\varepsilon}, \hat{\psi})$.

Remark 4: Since we switch the initial conditions for $\hat{\psi}$ and $\hat{\varepsilon}$, the derived dynamic models are valid for the time intervals $mT \le t < (m+1)T$. However, we will use these models for stability analysis on the infinite time interval $t \in [0, \infty)$ taking into account the properties of the sequences $\tilde{\varepsilon}(mT)$, $\hat{\varepsilon}(mT)$ and $\varepsilon(mT)$. ■

[3] An alternative approach to the resetting of the model (10) was proposed in [12]. However, in this paper the predictor remains open-loop and does not provide *asymptotic* behavior of the prediction.

Substituting (23) and (24) into (1), we obtain

$$\dot{x} = Ax - bk^\top x + bk^\top \tilde{\varepsilon}(t-h) = A_k x + bk^\top e^{-A_L h}\tilde{\varepsilon}.$$

Now, substituting (24) into (4), we have

$$\begin{aligned} \dot{\psi} &= A\psi - bk^\top \left(e^{Ah}x + \hat{\psi} + \hat{\varepsilon} \pm \psi\right) \\ &+ e^{Ah}bk^\top x - e^{Ah}bk^\top \tilde{\varepsilon}(t-h) \\ &= A_k\psi + Dx + C\tilde{\varepsilon}, \end{aligned}$$

where $D = e^{Ah}bk^\top - bk^\top e^{Ah}$, $C = bk^\top - e^{Ah}bk^\top e^{-A_L h}$.

Therefore, on the time intervals $mT \le t < (m+1)T$ we have

$$\begin{bmatrix} \dot{\tilde{\varepsilon}} \\ \dot{x} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} A_L & 0 & 0 \\ bk^\top e^{-A_L h} & A_k & 0 \\ C & D & A_k \end{bmatrix} \begin{bmatrix} \tilde{\varepsilon} \\ x \\ \psi \end{bmatrix},$$

where A_k and A_L are Hurwitz. Therefore, this subsystem is asymptotically stable due to its diagonal structure. If T is defined by (29), the resetting does not destroy asymptotic convergence and $\tilde{\varepsilon}(mT) \to 0$ as $m \to \infty$, and $\tilde{\varepsilon}(t) \to 0$ as $t \to \infty$. Therefore, we conclude that $\tilde{\varepsilon}(t), x(t), \psi(t) \to 0$ as $t \to \infty$.

Now, we will derive a subsystem in the components $(\varepsilon, \hat{\varepsilon}, \hat{\psi})$. For $\hat{\varepsilon}$, we have

$$\dot{\hat{\varepsilon}} = A\hat{\varepsilon} + Le^{Ah}\left(e^{-Ah}\varepsilon - e^{-Ah}\hat{\varepsilon}\right) = A_L\hat{\varepsilon} + L\varepsilon.$$

Replacing (24) in (10), we obtain

$$\begin{aligned} \dot{\hat{\psi}} &= A\hat{\psi} - bk^\top \left(e^{Ah}x + \hat{\psi} + \hat{\varepsilon}\right) \\ &+ e^{Ah}bk^\top x - e^{Ah}bk^\top \left(\varepsilon(t-h) - \hat{\varepsilon}(t-h)\right) \\ &= A_k\hat{\psi} + Dx - K\varepsilon + \hat{K}\hat{\varepsilon}, \end{aligned}$$

where $K = e^{Ah}bk^\top e^{-Ah}$ and $\hat{K} = K - bk^\top$.

Therefore, on the time intervals $mT \le t < (m+1)T$ we have

$$\begin{bmatrix} \dot{\varepsilon} \\ \dot{\hat{\varepsilon}} \\ \dot{\hat{\psi}} \end{bmatrix} = \begin{bmatrix} A & 0 & 0 \\ L & A_L & 0 \\ -K & \hat{K} & A_k \end{bmatrix} \begin{bmatrix} \varepsilon \\ \hat{\varepsilon} \\ \hat{\psi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ D \end{bmatrix} x. \tag{30}$$

This subsystem is unstable because of the matrix A. However, due to the states resetting (25), we have $\varepsilon(mT) = \tilde{\varepsilon}(mT)$, where $\tilde{\varepsilon}(mT) \to 0$ as $m \to \infty$ (as proved above). Therefore, $\varepsilon(mT)$ is a decreasing sequence. Since $\varepsilon(t) = e^{At}\varepsilon(mT)$ for $t \in [mT, (m+1)T)$, we conclude that $\varepsilon(t) \to 0$ as $t \to \infty$. Now, taking into account that $x(t), \varepsilon(t) \to 0$ as $t \to \infty$, from (30) we conclude that $\hat{\varepsilon}(t), \hat{\psi}(t) \to 0$ as $t \to \infty$. Thus, we have proved the following statement.

***Proposition** 1: In the ideal case, in the closed-loop system consisting of the plant* (1)*, the closed-loop state predictor* (10) *and* (22)*, prediction error observer* (20)*,* (27)*, and* (28)*, the controller* (24)*, and the scheme of states resetting* (25) *all the signals are bounded and* $\tilde{\varepsilon}(t), \hat{\varepsilon}(t), \varepsilon(t), x(t), \psi(t), \hat{\psi}(t) \to 0$ *as* $t \to \infty$ *for any initial conditions* x_0, $\hat{\psi}_0$. ■

In the next section, we analyze robustness of the proposed control with respect to parametric perturbations.

III. Robustness with Respect to Parametric Perturbations

Now let us assume that in the control law we use the estimate $\hat{A}$ instead of the true value A, and $\hat{A} \neq A$.

Remark 5: All results can be also extended to the estimation of $\hat{b}$. However, we consider the case of $\hat{A}$ only to simplify expressions and the proof. ■

Therefore, the control law consists of:

the closed-loop state predictor

$$\dot{\hat{\psi}}(t) = \hat{A}\hat{\psi}(t) + bu(t) - e^{\hat{A}h}bu(t-h), \quad (31)$$

$$\bar{x}(t+h) := e^{\hat{A}h}x(t) + \hat{\psi}(t) + \hat{\varepsilon}(t); \quad (32)$$

the prediction error observer

$$\begin{aligned} \dot{\hat{\varepsilon}}(t) &= \hat{A}\hat{\varepsilon}(t) + \hat{L}e^{\hat{A}h}\left(y(t) - e^{-\hat{A}h}\hat{\varepsilon}(t)\right), \quad (33) \\ y(t) &= x(t) - e^{\hat{A}h}x(t-h) - \bar{\psi}(t-h), \quad (34) \end{aligned}$$

where $\hat{L}$ is a designer-chosen matrix such that $\hat{A}_L = \hat{A} - \hat{L}$ is Hurwitz and $\bar{\psi}(t-h)$ is defined by (28);

the controller

$$\begin{aligned} u(t) &= -\hat{k}^\top \bar{x}(t+h) \quad (35) \\ &= -\hat{k}^\top\left(e^{\hat{A}h}x(t) + \hat{\psi}(t) + \hat{\varepsilon}(t)\right), \end{aligned}$$

where a designer-chosen vector $\hat{k}$ is such that the matrix $\hat{A}_k = \hat{A} - b\hat{k}^\top$ is Hurwitz;

the scheme of the states resetting (25), where T is chosen as

$$T \geq \max\left\{-\hat{\lambda}_2 \ln\left(\gamma\frac{\hat{\lambda}_1}{\hat{\lambda}_2}\right), h\right\}, \quad (36)$$

$\hat{\lambda}_1$ and $\hat{\lambda}_2$ are the minimum and maximum eigenvalues of the matrix $\hat{P}$, respectively, while $\hat{P}$ is a solution of the matrix equation $\hat{A}_L^\top\hat{P} + \hat{P}\hat{A}_L = -I$.

As in Section II-B, we first derive a model of the closed-loop system in the components $(\tilde{\varepsilon}, x, \psi)$.

For methodological purposes, we define the open-loop state prediction estimate $\hat{x}(t+h) := e^{\hat{A}h}x(t) + \hat{\psi}(t)$. Then, for the state prediction error we have

$$\varepsilon(t) = x(t+h) - \hat{x}(t+h) = \tilde{E}x(t) + \psi(t) - \hat{\psi}(t), \quad (37)$$

where $\tilde{E} = e^{Ah} - e^{\hat{A}h}$.

Therefore,

$$\hat{\varepsilon} = \tilde{E}x + \psi - \hat{\psi} - \tilde{\varepsilon} \quad (38)$$

and (see (32))

$$\begin{aligned} \bar{x}(t+h) &= e^{\hat{A}h}x + \hat{\psi} + e^{Ah}x - e^{\hat{A}h}x + \psi - \hat{\psi} - \tilde{\varepsilon} \\ &= e^{Ah}x + \psi - \tilde{\varepsilon} = x(t+h) - \tilde{\varepsilon}. \quad (39) \end{aligned}$$

Substituting (39) into (35) and (1), we obtain

$$\dot{x} = \bar{A}_k x + b\hat{k}^\top\tilde{\varepsilon}(t-h),$$

where $\bar{A}_k = A - b\hat{k}^\top$.

Now consider $y(t)$ defined by (34):

$$\begin{aligned} y(t) &= e^{Ah}x(t-h) + \psi(t-h) - e^{\hat{A}h}x(t-h) - \bar{\psi}(t-h) \\ &= \tilde{E}x(t-h) + \psi(t-h) - \bar{\psi}(t-h) = \varepsilon(t-h). \end{aligned}$$

In view of (37), (4), and (31), we have

$$\begin{aligned} \dot{\varepsilon} &= \tilde{E}\dot{x} + \dot{\psi} - \dot{\hat{\psi}} = \tilde{E}Ax + \tilde{E}bu(t-h) \\ &+ A\psi + bu - e^{Ah}bu(t-h) \\ &- \hat{A}\hat{\psi} - bu + e^{\hat{A}h}bu(t-h) \\ &= \hat{A}\varepsilon + \tilde{E}_A x + \tilde{A}\psi, \quad (40) \end{aligned}$$

where $\tilde{A} = A - \hat{A}$ and $\tilde{E}_A = \tilde{A}e^{Ah} - e^{\hat{A}h}\tilde{A}$.

In view of (40), we have

$$\varepsilon(t-h) = e^{-\hat{A}h}\varepsilon(t) - \int_{t-h}^{t} e^{\hat{A}(t-h-\tau)}\left(\tilde{E}_A x(\tau) + \tilde{A}\psi(\tau)\right)d\tau.$$

Define

$$\eta(x,\psi) = -\int_{t-h}^{t} e^{\hat{A}(t-h-\tau)}\left(\tilde{E}_A x(\tau) + \tilde{A}\psi(\tau)\right)d\tau. \quad (41)$$

Then

$$y(t) = \varepsilon(t-h) = e^{-\hat{A}h}\varepsilon(t) + \eta(x,\psi) \quad (42)$$

and (in view of (40) and (33)) we have

$$\begin{aligned} \dot{\tilde{\varepsilon}} &= \dot{\varepsilon} - \dot{\hat{\varepsilon}} = \hat{A}\varepsilon + \tilde{E}_A x + \tilde{A}\psi \\ &- \hat{A}\hat{\varepsilon} + \hat{L}e^{\hat{A}h}\left(e^{-\hat{A}h}\varepsilon + \eta(x,\psi) - e^{-\hat{A}h}\hat{\varepsilon}\right) \\ &= \hat{A}_L\tilde{\varepsilon} + \tilde{E}x + \tilde{A}\psi + \hat{L}e^{\hat{A}h}\eta(x,\psi). \quad (43) \end{aligned}$$

In view of (4), (35), and (39), we obtain

$$\begin{aligned} \dot{\psi} &= A\psi - b\hat{k}^\top\left(e^{\hat{A}h}x + \hat{\psi} + \hat{\varepsilon} \pm \psi\right) \\ &+ e^{Ah}b\hat{k}^\top x - e^{Ah}b\hat{k}^\top\tilde{\varepsilon}(t-h) \\ &= \bar{A}_k\psi + \hat{D}x + b\hat{k}^\top\tilde{\varepsilon} - e^{Ah}b\hat{k}^\top\tilde{\varepsilon}(t-h), \end{aligned}$$

where $\hat{D} = e^{Ah}b\hat{k}^\top - b\hat{k}^\top e^{\hat{A}h}$.

Therefore, on the time intervals $mT \leq t < (m+1)T$ we have

$$\begin{aligned} \begin{bmatrix}\dot{\tilde{\varepsilon}} \\ \dot{x} \\ \dot{\psi}\end{bmatrix} &= \begin{bmatrix}\hat{A}_L & \tilde{E} & \tilde{A} \\ 0 & \bar{A}_k & 0 \\ b\hat{k}^\top & \hat{D} & \bar{A}_k\end{bmatrix}\begin{bmatrix}\tilde{\varepsilon} \\ x \\ \psi\end{bmatrix} \\ &+ \begin{bmatrix}0 \\ b\hat{k}^\top \\ -e^{Ah}b\hat{k}^\top\end{bmatrix}\tilde{\varepsilon}(t-h) + \begin{bmatrix}\hat{L}e^{\hat{A}h}\tilde{A} \\ 0 \\ 0\end{bmatrix}\eta(x,\psi). \quad (44) \end{aligned}$$

In view of Corollary 1 from Appendix, we conclude that the matrix $\bar{A}_k$ is Hurwitz for sufficiently small $\|\tilde{A}\|$. Further, since the matrices $\hat{A}_L$, $\bar{A}_k$ are Hurwitz, then the matrix

$$\begin{bmatrix}\hat{A}_L & 0 & 0 \\ 0 & \bar{A}_k & 0 \\ e^{Ah}b\hat{k}^\top & \hat{D} & \bar{A}_k\end{bmatrix}$$

is Hurwitz due to its diagonal structure, and by virtue of Lemma 1 the matrix

$$\begin{bmatrix}\hat{A}_L & \tilde{E} & \tilde{A} \\ 0 & \bar{A}_k & 0 \\ b\hat{k}^\top & \hat{D} & \bar{A}_k\end{bmatrix}$$

is Hurwitz for sufficiently small $\|\tilde{A}\|$ (and, as a result, for sufficiently small $\|\tilde{E}\|$). By Lemmas 2 and 3 from Appendix, we conclude that the origin of the system (44) is asymptotically stable on time intervals $mT \le t < (m+1)T$. Further, we can prove that the sequence $\tilde{\varepsilon}(mT)$ is decreasing for sufficiently small $\|\tilde{A}\|$ if T is defined by (36). As a result, we have that $\tilde{\varepsilon}(t), x(t), \psi(t), \eta(t) \to 0$ as $t \to \infty$ for sufficiently small $\|\tilde{A}\|$.

Now, we will derive subsystem in the coordinates $(\varepsilon, \hat{\varepsilon}, \hat{\psi})$. The model for ε is given by (40). In view of (33) and (42), for $\hat{\varepsilon}$ we obtain

$$\begin{aligned}\dot{\hat{\varepsilon}} &= \hat{A}\hat{\varepsilon} + \hat{L}e^{\hat{A}h}\left(e^{-\hat{A}h}\varepsilon + \tilde{A}\eta - e^{-\hat{A}h}\hat{\varepsilon}\right) \\ &= \hat{A}_L\hat{\varepsilon} + \hat{L}\varepsilon + \hat{L}e^{\hat{A}h}\tilde{A}\eta.\end{aligned}$$

Further, substituting (35) in (31) instead of $u(t)$, and (35), (39) instead of $u(t-h)$, we obtain

$$\begin{aligned}\dot{\hat{\psi}} &= \hat{A}\hat{\psi} - b\hat{k}^\top\left(e^{\hat{A}h}x + \hat{\psi} + \hat{\varepsilon}\right) \\ &+ e^{\hat{A}h}b\hat{k}^\top x - e^{\hat{A}h}b\hat{k}^\top\tilde{\varepsilon}(t-h) \\ &= \hat{A}_k\hat{\psi} - b\hat{k}^\top\hat{\varepsilon} + \bar{D}x - e^{\hat{A}h}b\hat{k}^\top\tilde{\varepsilon}(t-h),\end{aligned}$$

where $\bar{D} = e^{\hat{A}h}b\hat{k}^\top - b\hat{k}^\top e^{\hat{A}h}$.

On the time intervals $mT \le t < (m+1)T$ we have

$$\begin{bmatrix}\dot{\varepsilon}\\ \dot{\hat{\varepsilon}}\\ \dot{\hat{\psi}}\end{bmatrix} = \begin{bmatrix}\hat{A} & 0 & 0\\ \hat{L} & \hat{A}_L & 0\\ 0 & -b\hat{k}^\top & \hat{A}_k\end{bmatrix}\begin{bmatrix}\varepsilon\\ \hat{\varepsilon}\\ \hat{\psi}\end{bmatrix} + \begin{bmatrix}\tilde{E}_A\\ 0\\ \bar{D}\end{bmatrix}x$$
$$+\begin{bmatrix}\tilde{A}\\ 0\\ 0\end{bmatrix}\psi + \begin{bmatrix}0\\ \hat{L}e^{\hat{A}h}\tilde{A}\\ 0\end{bmatrix}\eta - \begin{bmatrix}0\\ 0\\ e^{\hat{A}h}b\hat{k}^\top\end{bmatrix}\tilde{\varepsilon}(t-h).$$

As it was proved above, $x(t), \tilde{\varepsilon}(t), \psi(t), \eta(t) \to 0$ as $t \to \infty$. However, this subsystem is unstable through the matrix $\hat{A}$. At the same time, due to the states resetting (25) at the instants (36) we have $\varepsilon(mT) = \tilde{\varepsilon}(mT)$, where $\tilde{\varepsilon}(mT) \to 0$. As a result, $\varepsilon(mT)$ is the decreasing sequence. Taking into account that $\hat{A}_L$ and $\hat{A}_k$ are Hurwitz, we can conclude that $\varepsilon(t), \hat{\varepsilon}(t), \hat{\psi}(t) \to 0$ as $t \to \infty$. Thus, we have proved the following statement.

***Proposition** 2: The closed loop system consisting of the plant* (1)*, the closed-loop state predictor* (31) *and* (32)*, the prediction error observer* (33)*,* (34)*, and* (28)*, the controller* (35)*, and the scheme of states resetting* (25)*,* (36) *is robust with respect to parametric disturbances in the sense that for sufficiently small parametric disturbances $\|\tilde{A}\|$ all the closed-loop signals are bounded and $\tilde{\varepsilon}(t), \hat{\varepsilon}(t), \varepsilon(t), x(t), \psi(t), \hat{\psi}(t) \to 0$ as $t \to \infty$ for any initial conditions x_0, $\hat{\psi}_0$.* ■

IV. Simulation Results

Consider simulation results illustrating robustness of the proposed controller with the closed-loop state prediction.

Example 1. Robustness with respect to parametric disturbances. Consider the plant (1) with $h = 1sec$,

$$A = \begin{bmatrix}0.2 & 1\\ 0 & 0\end{bmatrix}, \quad b = \begin{bmatrix}0\\ 1\end{bmatrix}, \quad x_0 = \begin{bmatrix}0\\ 0.5\end{bmatrix},$$

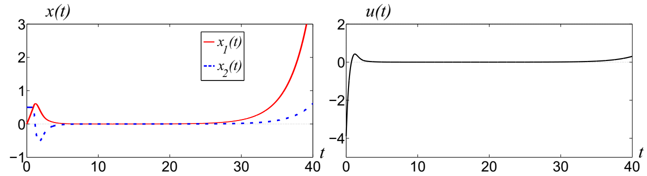

Fig. 1. Transients in the system stabilized by the controller with the open-loop prediction estimate of the state

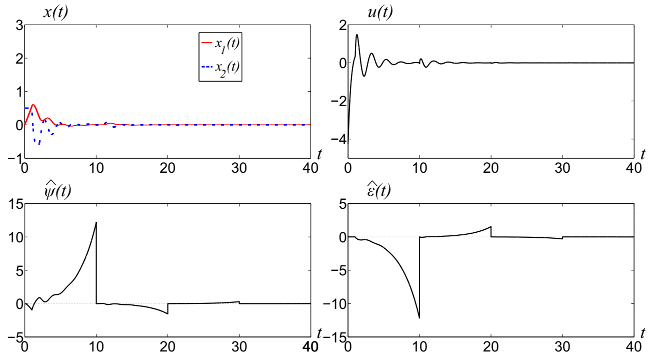

Fig. 2. Transients in the system stabilized by the controller with the proposed closed-loop prediction estimate of the state

and

$$\hat{A} = \begin{bmatrix}0.35 & 1\\ 0 & 0.1\end{bmatrix}.$$

We set $\hat{k} = [3.17,\ 3.45]^\top$. Simulations are carried out in MATLAB/ Simulink with the use of `ode8 (Dormand -Prince)` solver and fixed sample time 0.00001.

Simulation results obtained for the controller with the open-loop state prediction estimate with $\hat{\psi}_0 = 0$ are presented in Fig.1 and demonstrate unbounded growth of $x(t)$. Fig.2 presents simulation results obtained for the proposed controller with the closed-loop state prediction and the states resetting. In this simulation, we set

$$\hat{L} = \begin{bmatrix}5.35 & 0\\ 6 & 0.1\end{bmatrix}, \quad T = 10sec.$$

As seen from the plots, in spite of the parametric perturbations the proposed controller provides asymptotic zeroing of $x(t)$, $\hat{\psi}(t)$, and $\hat{\varepsilon}(t)$ with all the closed-loop signals bounded.

Example 2. Robustness with respect to approximation errors. We make conjecture that the proposed predictor is also robust with respect to numerical solver errors. To illustrate this conjecture, consider the plant (1) with $h = 1sec$,

$$\hat{A} = A = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \quad \hat{b} = b = \begin{bmatrix}0\\ 1\end{bmatrix}, \quad x_0 = \begin{bmatrix}0\\ 0.5\end{bmatrix}.$$

We set $\hat{k} = [2,\ 3]^\top$ and

$$\hat{L} = \begin{bmatrix}5.2 & 0\\ 6 & 0\end{bmatrix}.$$

Let us first consider simulation results for the controller with the open-loop state prediction obtained with the use of `ode8 (Dormand-Prince)` solver and fixed sample time 0.00001 (see Fig.3.a). As seen from the plots, the controller

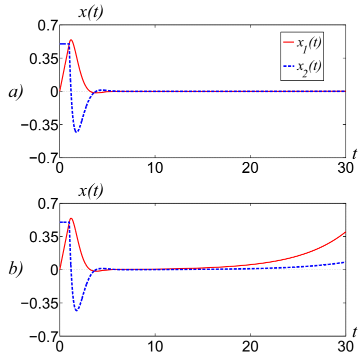

Fig. 3. Evolutions of $x(t)$ in the system stabilized by the controller with the open-loop prediction estimate of the state. Calculation using: a) `ode8 (Dormand-Prince)` solver and fixed sample time 0.00001; b) `ode1 (Euler)` solver and fixed sample time 0.0001.

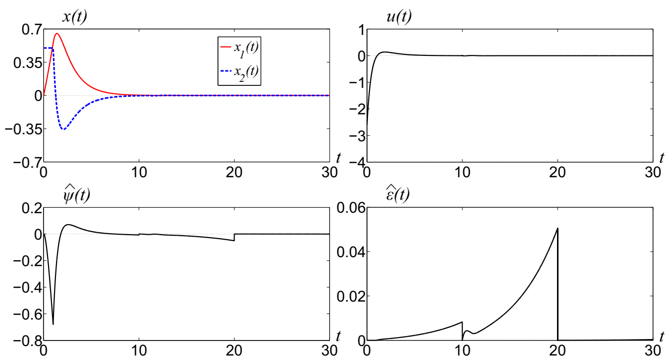

Fig. 4. Transients in the system stabilized by the controller with the proposed closed-loop prediction estimate of the state obtained using `ode1 (Euler)` solver and fixed sample time 0.0001.

provides zeroing of the state $x(t)$ visible at least for the given zoom and $t \leq 30sec$. However, for the less accurate `ode1 (Euler)` solver with fixed sample time 0.0001 we obtain fast diverging of $x(t)$ (see Fig.3.b). The proposed controller with the closed-loop state prediction estimate provides for `ode1 (Euler)` solver with fixed sample time 0.0001 the boundedness of all the closed-loop signals and asymptotic zeroing of $x(t)$, $\hat{\psi}(t)$, and $\hat{\varepsilon}(t)$ (see Fig.4).

V. CONCLUSION

The closed-loop state predictor is designed and its robustness with respect to parametric perturbation is proved. As a conjecture, its robustness with respect to approximation errors of the differential equation solver is considered. The further research directions will be focused on the rigorous proof of the robustness with respect to approximation errors as well as with respect to the plant external disturbances.

APPENDIX

In this appendix, we summarize some mathematical tools used in the proof of the main result in Section III. All these lemmas can be proved via the properties of linear systems.

Lemma 1: If a matrix $A \in \mathbb{R}^{n\times n}$ is Hurwitz, then a matrix $\hat{A} \in \mathbb{R}^{n\times n}$ is also Hurwitz for sufficiently small $\|\tilde{A}\|$, where $\tilde{A} = A - \hat{A}$ and $\|A\|$ is the spectral norm of the matrix. ■

Corollary 1: If a matrix $A_k = A - bk^\top$ is Hurwitz, where $A \in \mathbb{R}^{n\times n}$ and $b, k \in \mathbb{R}^n$, then a matrix $\hat{A}_k = \hat{A} - bk^\top$ is also Hurwitz for sufficiently small $\|\tilde{A}\|$. ■

Lemma 2: If the LTI system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is asymtotically stable, then the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & \tilde{A} \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} x_1(t-h)$$

is asymptotically stable for sufficiently small $\|\tilde{A}\|$, where $A, A_1, A_2, B \in \mathbb{R}^{n\times n}$ and $x_1, x_2 \in \mathbb{R}^n$. ■

Lemma 3: If a matrix $A_1 \in \mathbb{R}^{n\times n}$ is Hurwitz, then the origin of the system

$$\dot{x} = A_1 x + B_1 \eta(x),$$

where $x \in \mathbb{R}^n$, $B_1 \in \mathbb{R}^{n\times m}$,

$$\eta(x) = \int_{t-h}^{t} e^{A_2(t-\tau)} B_2 x(\tau) d\tau, \qquad \eta \in \mathbb{R}^n,$$

$A_2 \in \mathbb{R}^{m\times m}$, $B_2 \in \mathbb{R}^{m\times m}$, and $h > 0$ is a constant, is asymptotically stable for sufficiently small $\|B_2\|$. ■

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