2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

Long-Time Behavior of Stochastic Linear-Quadratic Optimal Control Problems

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***Abstract*—The turnpike property refers to the phenomenon that in many optimal control problems over finite but long-time horizon, optimal trajectories approach to a steady state of the system and stay close to it for the major part of the time horizon. In the past several decades, the turnpike properties have attracted extensive attentions in control theory. Numerous results have been established for deterministic optimal control problems of both finite and infinite dimensions. However, the study of turnpike phenomena for stochastic optimal control is quite lacking in literature. This paper is concerned with the turnpike properties for stochastic linear-quadratic optimal control problems. Under suitable conditions, the strong exponential, the strong integral, and the mean-square turnpike properties are established. The crucial issues are to correctly formulate the corresponding static optimization problem and find the correction processes, which illustrate the essential differences between stochastic and deterministic cases.**

***keywords*—Stochastic optimal control, linear-quadratic, strong turnpike property, static optimization, stabilizability.**

I. Introduction

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space on which a standard one-dimensional Brownian motion $W = \{W(t); t \geqslant 0\}$ is defined. Denote by $\mathbb{F} = \{\mathscr{F}_t\}_{t\geqslant 0}$ the usual augmentation of the natural filtration generated by W. For a stochastic process X, we write $X \in \mathbb{F}$ if it is progressively measurable with respect to the filtration $\mathbb{F}$.

Consider the following controlled linear stochastic differential equation (SDE, for short):

$$\begin{cases} dX(t) = [AX(t) + Bu(t) + b]dt \\ \qquad\qquad + [CX(t) + Du(t) + \sigma]dW(t), \\ X(0) = x, \end{cases} \tag{1}$$

and the following quadratic cost functional:

$$J_T(x; u(\cdot)) = \mathbb{E}\int_0^T \left[\left\langle \begin{pmatrix} Q & S^\top \\ S & R \end{pmatrix}\begin{pmatrix} X(t) \\ u(t)\end{pmatrix}, \begin{pmatrix} X(t) \\ u(t)\end{pmatrix}\right\rangle + 2\left\langle \begin{pmatrix} q \\ r\end{pmatrix}, \begin{pmatrix} X(t) \\ u(t)\end{pmatrix}\right\rangle\right]dt, \tag{2}$$

where $A, C, Q \in \mathbb{R}^{n\times n}$, $B, D \in \mathbb{R}^{n\times m}$, $S \in \mathbb{R}^{m\times n}$, and $R \in \mathbb{R}^{m\times m}$ are constant matrices with Q and R being symmetric; the superscript $\top$ denotes the transpose of matrices; $\langle \cdot\,, \cdot\rangle$ denotes the Frobenius inner product of two matrices; and $b, \sigma, q \in \mathbb{R}^n$ and $r \in \mathbb{R}^m$ are constant vectors. The classical *stochastic linear-quadratic (LQ, for short) optimal control problem* over the finite time horizon $[0, T]$ is to find a control $u_T(\cdot)$ from the space

$$\mathcal{U}[0,T] = \Big\{ u : [0,T]\times\Omega \to \mathbb{R}^m \;\Big|\; u \in \mathbb{F} \text{ and } \mathbb{E}\int_0^T |u(t)|^2 dt < \infty \Big\}$$

such that the cost functional (2) is minimized over $\mathcal{U}[0, T]$ for any given *initial state* $x \in \mathbb{R}^n$. More precisely, the problem can be stated as follows.

Problem (SLQ)$_T$. For any given initial state $x \in \mathbb{R}^n$, find a control $u_T^*(\cdot) \in \mathcal{U}[0, T]$ such that

$$J_T(x; u_T^*(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}[0,T]} J_T(x; u(\cdot)) \equiv V_T(x). \tag{3}$$

The process $u_T^*(\cdot)$, if it exists, is called an *open-loop optimal control* of Problem (SLQ)$_T$ for the initial state x. The corresponding state process $X_T^*(\cdot)$ is called an *open-loop optimal state process*, and $(X_T^*(\cdot), u_T^*(\cdot))$ is called an *open-loop optimal pair*. The function $V_T(\cdot)$ is called the *value function* of Problem (SLQ)$_T$.

It is well-known by now that under proper conditions, Problem (SLQ)$_T$ is uniquely solvable, whose optimal control has a state feedback form and usually depends on the initial state x, as well as the time horizon T. However, in practice it is not convenient to compute the optimal control using the existing results, since this is an infinite dimensional problem without analytical solutions and the optimal control varies with time. The turnpike property of Problem (SLQ)$_T$ implies that for most of the time the optimal control-state pair remains very close to the solution of some static optimization problem. Such a property gives us the essential picture of the optimal pair without having to solve it analytically, which is

The first author is supported by NSFC grant 12271242, Guangdong Basic and Applied Basic Research Foundation 2021A1515010031, Shenzhen Fundamental Research General Program JCYJ20220530112814032, and ZDSYS20210623092007023.

very useful in improving the numerical methods for solving optimal control problems.

The study of turnpike phenomena can be traced back to the work of von Neumann [11] on problems in economics, while the term was first coined by Dorfman, Samuelson, and Solow in 1958 (see [3]), referring to an American English word for a Highway. In the past several decades, the turnpike properties have attracted attentions of many researchers from various areas (see, e.g., [6], [7], [9], [10], [13]) as such a property often gives people an essential picture of the optimal pair without solving it analytically and leads to a significant simplification in numerical methods for solving such kind of optimal control problems. Numerous relevant results have been established for finite and infinite dimensional problems in the context discrete-time and continuous-time systems (see, e.g., [1], [2], [4], [5], [8], [18], [19], [21]–[24] and the references therein).

For the deterministic LQ optimal control problem, denoted by Problem (DLQ)$_T$, the exponential turnpike property has been established in [12] and [19], which states that the optimal pair exponentially converges in the transient time (as $T \to \infty$) to the minimum point of a certain static optimization problem. To be precise, let (x^*, u^*) be the solution to the following static optimization problem:

$$\begin{cases} \text{Minimize } F_0(x,u) \triangleq \langle Qx, x\rangle + 2\langle Sx, u\rangle + \langle Ru, u\rangle \\ \qquad\qquad\qquad\qquad + 2\langle q, x\rangle + 2\langle r, u\rangle, \\ \text{subject to } Ax + Bu + b = 0. \end{cases} \tag{4}$$

Then there exist positive constants K and λ, independent of T, such that the optimal pair $(X_T^*(\cdot), u_T^*(\cdot))$ of Problem (DLQ)$_T$ satisfies

$$|X_T^*(t) - x^*| + |u_T^*(t) - u^*| \leqslant K\Big[e^{-\lambda t} + e^{-\lambda(T-t)}\Big]$$

for all $t \in [0, T]$.

For the stochastic LQ optimal control problem (i.e., Problem (SLQ)$_T$), Sun, Wang, and Yong [15] recently found that the corresponding static optimization problem should take the following form:

$$\begin{cases} \text{Minimize } F(x,u) \triangleq \langle P(Cx+Du+\sigma), Cx+Du+\sigma\rangle \\ \qquad\qquad\qquad\qquad + F_0(x,u), \\ \text{subject to } Ax + Bu + b = 0, \end{cases} \tag{5}$$

where $P > 0$ is the solution to the following algebraic Riccati equation (ARE, for short):

$$\mathcal{Q}(P) - \mathcal{S}(P)^\top \mathcal{R}(P)^{-1}\mathcal{S}(P) = 0, \tag{6}$$

where

$$\begin{cases} \mathcal{Q}(P) \triangleq PA + A^\top P + C^\top PC + Q, \\ \mathcal{S}(P) \triangleq B^\top P + D^\top PC + S, \\ \mathcal{R}(P) \triangleq R + D^\top PD. \end{cases} \tag{7}$$

It was shown in [15] that the *expectation* of the optimal pair exhibits similar turnpike properties as the deterministic case, that is, for some constants $K, \lambda > 0$ independent of T,

$$\big|\mathbb{E}[X_T^*(t)] - x^*\big| + \big|\mathbb{E}[u_T^*(t)] - u^*\big| \leqslant K\Big[e^{-\lambda t} + e^{-\lambda(T-t)}\Big]$$

for all $t \in [0, T]$.

In this paper, we establish a stronger turnpike property for Problem (SLQ)$_T$, which gives further information about the paths of the optimal pair. We show that for some stochastic processes $\boldsymbol{X}^*(\cdot)$ and $\boldsymbol{u}^*(\cdot)$ independent of T, the optimal pair $(X_T^*(\cdot), u_T^*(\cdot))$ of Problem (SLQ)$_T$ satisfies the following *strong exponential turnpike property*:

$$\begin{aligned} &\mathbb{E}|X_T^*(t) - \boldsymbol{X}^*(t)|^2 + \mathbb{E}|u_T^*(t) - \boldsymbol{u}^*(t)|^2 \\ &\qquad \leqslant K\Big[e^{-\lambda t} + e^{-\lambda(T-t)}\Big], \quad \forall t \in [0, T], \end{aligned} \tag{8}$$

for some constants $K, \lambda > 0$ independent of T. The pair of processes $(\boldsymbol{X}^*(\cdot), \boldsymbol{u}^*(\cdot))$, which we call the *turnpike limit* of Problem (SLQ)$_T$, can be determined explicitly: the expectation (x^*, u^*) of $(\boldsymbol{X}^*(\cdot), \boldsymbol{u}^*(\cdot))$ is time-invariant, and is exactly the solution to the static optimization problem (5); and the *correction processes* (for the state and control)

$$X^*(\cdot) \triangleq \boldsymbol{X}^*(\cdot) - x^*, \quad u^*(\cdot) \triangleq \boldsymbol{u}^*(\cdot) - u^*$$

are solutions of some linear SDEs that are easy to solve. As consequences of (8), the strong integral and the mean-square turnpike properties are also established for Problem (SLQ)$_T$. Further, we show that the value function of Problem (SLQ)$_T$ converges to the minimum of the static optimization problem (5) in the time-average sense.

The rest of this paper is structured as follows. In section II, we give the preliminaries, introduce the static optimization problem, and collect some relevant results. In section III, we state the main results of the paper and give some brief proofs. In section IV, we conclude the paper.

II. Preliminaries

In this section, we first introduce the basic notation and assumptions that will be used throughout this paper. Then we recall the connection between the differential Riccati equation and the algebraic Riccati equation. Finally, we formulate the static optimization problem associated with Problem (SLQ)$_T$.

A. Notation and assumptions

Let $\mathbb{R}^{n\times m}$ be the space of $n \times m$ all real matrices equipped with the Frobenius inner product

$$\langle M, N\rangle \triangleq \text{tr}\,(M^\top N), \quad M, N \in \mathbb{R}^{n\times m},$$

where $\text{tr}\,(M^\top N)$ is the trace of the matrix $M^\top N$. The norm induced by the Frobenius inner product is denoted by $|\cdot|$. For a subset $\mathbb{H}$ of $\mathbb{R}^{n\times m}$, we denote by $C([0,T];\mathbb{H})$ the space of all continuous functions from $[0, T]$ into $\mathbb{H}$. The identity matrix of size n is denoted by I_n (or simply by I if no confusion should occur), and a vector always refers to a column vector if not specified. Let $\mathbb{S}^n$ be the subspace of $\mathbb{R}^{n\times n}$ consisting of symmetric matrices. For $\mathbb{S}^n$-valued functions M and N, we write $M \geqslant N$ (respectively, $M > N$) if $M - N$ is positive semidefinite (respectively, positive definite) almost everywhere with respect to the Lebesgue measure.

The following basic assumptions will be imposed throughout the paper.

(A1) The weighting matrices in the cost functional (2) satisfy

$$R > 0, \quad Q - S^\top R^{-1} S > 0.$$

(A2) The controlled linear system

$$dX(t) = [AX(t) + Bu(t)]dt + [CX(t) + Du(t)]dW(t) \quad (9)$$

is L^2-*stabilizable*, i.e., there exists a matrix $\Theta \in \mathbb{R}^{m\times n}$ such that for any initial state x, the solution $X(\cdot)$ of (9) corresponding to the linear state feedback control $u(\cdot) = \Theta X(\cdot)$ satisfies

$$\mathbb{E}\int_0^\infty |X(t)|^2 dt < \infty.$$

In this case, Θ is called a *stabilizer* of the system (9).

B. Differential and algebraic Riccati equations

With the notation (7), the differential Riccati equation associated with Problem (SLQ)$_T$ can be written as

$$\begin{cases} \dot{P}_T(t) + \mathcal{Q}(P_T(t)) \\ \quad - \mathcal{S}(P_T(t))^\top \mathcal{R}(P_T(t))^{-1}\mathcal{S}(P_T(t)) = 0, \\ P_T(T) = 0. \end{cases} \quad (10)$$

Under the assumption (A1), it can be shown that (10) admits a unique positive semidefinite solution $P_T(\cdot) \in C([0,T]; \mathbb{S}^n)$; see, e.g., [14], [20]. So we have the following result.

Lemma 2.1. Let (A1) hold. Then for any $T > 0$, the differential Riccati equation (10) admits a unique solution $P_T(\cdot) \in C([0,T]; \mathbb{S}^n)$ satisfying $P_T(t) \geqslant 0$ for all $t \in [0,T]$.

The following result is concerned with the solvability of the ARE (6), whose proof can be found in [16]; see also the book [17].

Lemma 2.2. Let (A1)–(A2) hold. Then the ARE (6) admits a unique solution $P \in \mathbb{S}^n$ satisfying $P > 0$. Moreover, the matrix

$$\Theta \triangleq -\mathcal{R}(P)^{-1}\mathcal{S}(P) \quad (11)$$

is a stabilizer of the system (9).

Now we present the connection between the solutions to the differential Riccati equation (10) and the ARE (6). Let $P_T(\cdot)$ and P be the unique solutions to (10) and (6), respectively.

Lemma 2.3. Let (A1)–(A2) hold. Then there exist constants $K, \lambda > 0$, independent of T, such that

$$|P_T(t) - P| \leqslant Ke^{-\lambda(T-t)}, \quad \forall t \in [0,T]. \quad (12)$$

The property (12) plays an essential role in establishing the strong turnpike property of Problem (SLQ)$_T$. Please see [15] for a proof of Lemma 2.3.

C. The static optimization problem

We now introduce the static optimization problem associated with Problem (SLQ)$_T$, whose solution serves as part of the strong turnpike limit of Problem (SLQ)$_T$. Let

$$\mathscr{V} \triangleq \big\{(x,u) \in \mathbb{R}^n \times \mathbb{R}^m \mid Ax + Bu + b = 0\big\}, \quad (13)$$

which is nonempty under the assumption (A2) since the stabilizability of the system (9) implies that (A, B) has rank n. Also, define a continuous function $F : \mathscr{V} \to \mathbb{R}$ by

$$\begin{aligned} F(x,u) \triangleq\ & \langle Qx, x\rangle + \langle Ru, u\rangle + 2\langle Sx, u\rangle + 2\langle q, x\rangle + 2\langle r, u\rangle \\ & + \langle P(Cx + Du + \sigma), Cx + Du + \sigma\rangle, \end{aligned} \quad (14)$$

where P is the unique solution of the ARE (6). The static optimization problem associated with Problem (SLQ)$_T$ can be stated as follows.

Problem (O). Find a pair $(x^*, u^*) \in \mathscr{V}$ such that

$$F(x^*, u^*) = \min_{(x,u)\in\mathscr{V}} F(x,u) \equiv V.$$

For the above static optimization problem, we have the following result.

Lemma 2.4. Let (A1)–(A2) hold. Then Problem (O) admits a unique solution. Moreover, $(x^*, u^*) \in \mathscr{V}$ is the solution if and only if for some $\lambda^* \in \mathbb{R}^n$, the following hold:

$$\begin{cases} Ax^* + Bu^* + b = 0 \\ Qx^* + A^\top\lambda^* + C^\top P\sigma^* + S^\top u^* + q = 0, \\ Ru^* + B^\top \lambda^* + D^\top P\sigma^* + Sx^* + r = 0, \end{cases} \quad (15)$$

where $\sigma^* \triangleq Cx^* + Du^* + \sigma$.

We omit the proof of Lemma 2.4 here and refer the reader to [15] for details.

III. MAIN RESULTS

In this section, we state and prove the main results of the paper, including the strong exponential, the strong integral, and the mean-square turnpike properties, as well as the convergence of the value function of Problem (SLQ)$_T$.

Let (x^*, u^*) be the unique solution of Problem (O), and let Θ be the matrix defined by (11). Let $X^*(\cdot)$ be the solution to the SDE (recalling $\sigma^* = Cx^* + Du^* + \sigma$)

$$\begin{cases} dX^*(t) = (A + B\Theta)X^*(t)dt \\ \qquad\qquad + [(C + D\Theta)X^*(t) + \sigma^*]dW(t), \\ X^*(0) = 0, \end{cases} \quad (16)$$

and set

$$\boldsymbol{X}^*(t) \triangleq X^*(t) + x^*, \quad \boldsymbol{u}^*(t) \triangleq \Theta X^*(t) + u^*. \quad (17)$$

Note that $\mathbb{E}[X^*(t)] \equiv 0$. Thus,

$$\mathbb{E}[\boldsymbol{X}^*(t)] = x^*, \quad \mathbb{E}[\boldsymbol{u}^*(t)] = u^*, \quad \forall t \geqslant 0.$$

Proposition 3.1. Let (A1)–(A2) hold. Then there exist a constant $K > 0$, such that

$$\mathbb{E}|X^*(t)|^2 \leqslant K, \quad \forall t \geqslant 0.$$

Proof. Let $P > 0$ be the solution to the ARE (6) and let Θ be defined by (11). It is easy to see that

$$P(A+B\Theta)+(A+B\Theta)^\top P+(C+D\Theta)^\top P(C+D\Theta) = -(Q+\Theta^\top R\Theta+S^\top\Theta+\Theta^\top S).$$

This, together with Itô's rule, implies that

$$\begin{aligned}&\frac{d}{dt}\mathbb{E}\langle PX^*(t),X^*(t)\rangle\\ &=\mathbb{E}\Big\{2\langle PX^*(t),(A+B\Theta)X^*(t)\rangle\\ &\quad+\langle P[(C+D\Theta)X^*(t)+\sigma^*],(C+D\Theta)X^*(t)+\sigma^*\rangle\Big\}\\ &=\mathbb{E}\Big[-\langle(Q+\Theta^\top R\Theta+S^\top\Theta+\Theta^\top S)X^*(t),X^*(t)\rangle\\ &\quad+2\langle P(C+D\Theta)X^*(t),\sigma^*\rangle+\langle P\sigma^*,\sigma^*\rangle\Big].\end{aligned}$$

Note that by the assumption (A1),

$$\begin{aligned}&Q+\Theta^\top R\Theta+S^\top\Theta+\Theta^\top S\\ &=Q-S^\top R^{-1}S+(\Theta+R^{-1}S)^\top R(\Theta+R^{-1}S)>0.\end{aligned}$$

Let $\mu > 0$ and $\nu > 0$ be the largest and the smallest eigenvalues of P and $Q+\Theta^\top R\Theta+S^\top\Theta+\Theta^\top S$, respectively. Then, with $\beta \triangleq |(C+D\Theta)^\top P\sigma^*|$, we have

$$\begin{aligned}&\frac{d}{dt}\mathbb{E}\langle PX^*(t),X^*(t)\rangle\\ &\leqslant\mathbb{E}\Big[-\nu|X^*(t)|^2+2\beta|X^*(t)|+\langle P\sigma^*,\sigma^*\rangle\Big]\\ &=\mathbb{E}\Big\{-\frac{\nu}{2}|X^*(t)|^2-\frac{\nu}{2}\Big[|X^*(t)|-\frac{2}{\nu}\beta\Big]^2\\ &\quad+\frac{2}{\nu}\beta^2+\langle P\sigma^*,\sigma^*\rangle\Big\}\\ &\leqslant\mathbb{E}\Big[-\frac{\nu}{2\mu}\langle PX^*(t),X^*(t)\rangle+\frac{2}{\nu}\beta^2+\langle P\sigma^*,\sigma^*\rangle\Big].\end{aligned}$$

It follows from the Gronwall inequality that

$$\begin{aligned}\mathbb{E}\langle PX^*(t),X^*(t)\rangle&\leqslant\Big[\frac{2}{\nu}\beta^2+\langle P\sigma^*,\sigma^*\rangle\Big]\int_0^t e^{\frac{\nu}{2\mu}(s-t)}ds\\ &\leqslant\frac{2\mu}{\nu}\Big[\frac{2}{\nu}\beta^2+\langle P\sigma^*,\sigma^*\rangle\Big].\end{aligned}$$

The desired result follows, since $P > 0$. □

A. The strong exponential turnpike property

Under (A1), the uniform convexity holds (see [17, Proposition 2.5.1]). Thus, by Proposition 2.5.2 of [17], Problem (SLQ)$_T$ admits a unique optimal control for every initial state x. Further, by Theorem 2.3.2 of [17], a state-control pair $(X_T^*(\cdot),u_T^*(\cdot))$ is optimal for x if and only if the adapted solution $(Y_T^*(\cdot),Z_T^*(\cdot))$ to the backward SDE

$$\begin{cases}dY_T^*(t)=-\big[A^\top Y_T^*(t)+C^\top Z_T^*(t)+QX_T^*(t)\\ \qquad\qquad+S^\top u_T^*(t)+q\big]dt+Z_T^*(t)dW(t),\\ Y_T^*(T)=0\end{cases}\tag{18}$$

satisfies the following stationary condition:

$$B^\top Y_T^*(t)+D^\top Z_T^*(t)+SX_T^*(t)+Ru_T^*(t)+r=0.\tag{19}$$

Let $(X_T^*(\cdot),u_T^*(\cdot))$ be the optimal pair of Problem (SLQ)$_T$ for the initial state x, and let $(Y_T^*(\cdot),Z_T^*(\cdot))$ be the corresponding adapted solution to (18). Let $\lambda^*\in\mathbb{R}^n$ be the Lagrange multiplier in (15) and define

$$\begin{cases}\widetilde X_T(t)=X_T^*(t)-x^*,\\ \tilde u_T(t)=u_T^*(t)-u^*,\\ \widetilde Y_T(t)=Y_T^*(t)-\lambda^*.\end{cases}\tag{20}$$

Using (15) and (20), we can obtain by a straightforward calculation that

$$\begin{cases}d\widetilde X_T(t)=[A\widetilde X_T(t)+B\tilde u_T(t)]dt\\ \qquad\qquad+[C\widetilde X_T(t)+D\tilde u_T(t)+\sigma^*]dW(t),\\ d\widetilde Y_T(t)=-[A^\top\widetilde Y_T(t)+C^\top Z_T^*(t)+Q\widetilde X_T(t)\\ \qquad\qquad+S^\top\tilde u_T(t)-C^\top P\sigma^*]dt+Z_T^*(t)dW(t),\\ \widetilde X_T(0)=x-x^*,\quad \widetilde Y_T(T)=-\lambda^*,\\ B^\top\widetilde Y_T(t)+D^\top Z_T^*(t)+S\widetilde X_T(t)+R\tilde u_T(t)=D^\top P\sigma^*.\end{cases}\tag{21}$$

Again, by Theorem 2.3.2 of [17], we see from (21) that $(\widetilde X_T(\cdot),\tilde u_T(\cdot))$ is the optimal pair of the stochastic LQ problem with the state equation

$$\begin{cases}dX(t)=[AX(t)+Bu(t)]dt\\ \qquad\qquad+[CX(t)+Du(t)+\sigma^*]dW(t),\\ X(0)=x-x^*,\end{cases}$$

and the cost functional

$$\begin{aligned}J(x;u(\cdot))\triangleq\mathbb{E}\Big\{&-2\langle\lambda^*,X(T)\rangle\\ &+\int_0^T\Big[\Big\langle\begin{pmatrix}Q&S^\top\\ S&R\end{pmatrix}\begin{pmatrix}X(t)\\ u(t)\end{pmatrix},\begin{pmatrix}X(t)\\ u(t)\end{pmatrix}\Big\rangle\\ &-2\Big\langle\begin{pmatrix}C^\top P\sigma^*\\ D^\top P\sigma^*\end{pmatrix},\begin{pmatrix}X(t)\\ u(t)\end{pmatrix}\Big\rangle\Big]dt\Big\}.\end{aligned}$$

Thus, by Corollary 4.7 of [14], we have the following result.

Proposition 3.2. Let (A1)–(A2) hold. Let $P_T(\cdot)$ and P be the solutions to (10) and (6), respectively. Define

$$\Theta_T(t)\triangleq-\mathcal{R}(P_T(t))^{-1}\mathcal{S}(P_T(t)),\tag{22}$$

and let $\varphi_T(\cdot)$ be the solution to the ordinary differential equation (ODE, for short)

$$\begin{cases}\dot\varphi_T(t)+[A+B\Theta_T(t)]^\top\varphi_T(t)\\ \qquad+[C+D\Theta_T(t)]^\top[P_T(t)-P]\sigma^*=0,\\ \varphi_T(T)=-\lambda^*.\end{cases}$$

Then the process $\tilde u_T(\cdot)$ defined in (20) is given by

$$\tilde u_T(t)=\Theta_T(t)\widetilde X_T(t)+\theta_T(t),\tag{23}$$

where

$$\theta_T(t)=-\mathcal{R}(P_T(t))^{-1}[B^\top\varphi_T(t)+D^\top(P_T(t)-P)\sigma^*].$$

Lemma 3.3. Let (A1)–(A2) hold. Then there exist constants $K,\lambda>0$, independent of T, such that

$$|\varphi_T(t)|+|\theta_T(t)|\leqslant Ke^{-\lambda(T-t)},\quad\forall t\in[0,T].$$

Proof. It is shown in Lemma 5.1 of [15] that $\varphi_T(\cdot)$ satisfies

$$|\varphi_T(t)| \leqslant Ke^{-\lambda(T-t)}, \quad \forall t \in [0,T]$$

for some constants $K, \lambda > 0$ independent of T. The desired result then follows from the fact $\mathcal{R}(P_T(t)) \geqslant R > 0$ and Lemma 2.3. □

Substituting (23) into the SDE for $\widetilde{X}_T(\cdot)$ in (21), we see that the process $\widetilde{X}_T(\cdot)$ satisfies the following closed-loop system:

$$\begin{cases} d\widetilde{X}_T(t) = \big\{[A + B\Theta_T(t)]\widetilde{X}_T(t) + B\theta_T(t)\big\}dt \\ \qquad\qquad + \big\{[C + D\Theta_T(t)]\widetilde{X}_T(t) + D\theta_T(t) + \sigma^*\big\}dW(t), \\ \widetilde{X}_T(0) = x - x^* \equiv \tilde{x}. \end{cases}$$

The function $t \to \mathbb{E}[\widetilde{X}_T(t)]$ satisfies the following ODE:

$$\begin{cases} \dfrac{d}{dt}\mathbb{E}[\widetilde{X}_T(t)] = [A + B\Theta_T(t)]\mathbb{E}[\widetilde{X}_T(t)] + B\theta_T(t), \\ \mathbb{E}[\widetilde{X}_T(0)] = \tilde{x}. \end{cases}$$

Now we give an estimate for $\mathbb{E}[\widetilde{X}_T(t)]$. The proof is similar to that of Theorem 5.1 in [15] and is omitted here.

Lemma 3.4. Let (A1)–(A2) hold. Then there exist constants $K, \lambda > 0$, independent of T, such that

$$|\mathbb{E}[\widetilde{X}_T(t)]| \leqslant K\Big[e^{-\lambda t} + e^{-\lambda(T-t)}\Big], \quad \forall t \in [0,T].$$

We are ready for the main result of the paper, which establishes the strong exponential turnpike property of Problem $(\text{SLQ})_T$.

Theorem 3.5. Let (A1)–(A2) hold. Let $(X_T^*(\cdot), u_T^*(\cdot))$ be the optimal pair of Problem $(\text{SLQ})_T$ for the initial state x, and let $\boldsymbol{X}^*(\cdot)$ and $\boldsymbol{u}^*(\cdot)$ be defined by (17). Then there exist constants $K, \lambda > 0$, independent of T, such that

$$\begin{aligned} &\mathbb{E}|X_T^*(t) - \boldsymbol{X}^*(t)|^2 + \mathbb{E}|u_T^*(t) - \boldsymbol{u}^*(t)|^2 \\ &\quad \leqslant K\Big[e^{-\lambda t} + e^{-\lambda(T-t)}\Big], \quad \forall t \in [0,T]. \end{aligned} \tag{24}$$

Proof. Let Θ and $\Theta_T(\cdot)$ be as in (11) and (22), respectively. For notational simplicity, we write

$$\mathcal{A} = A + B\Theta, \quad \mathcal{A}_T(t) = A + B\Theta_T(t),$$
$$\mathcal{C} = C + D\Theta, \quad \mathcal{C}_T(t) = C + D\Theta_T(t).$$

Then the process

$$\widetilde{V}_T(t) \triangleq \widetilde{X}_T(t) - \mathbb{E}[\widetilde{X}_T(t)] - X^*(t)$$

satisfies $\widetilde{V}_T(0) = 0$ and

$$\begin{aligned} d\widetilde{V}_T &= [\mathcal{A}_T\widetilde{V}_T + (\mathcal{A}_T - \mathcal{A})X^*]dt \\ &\quad + [\mathcal{C}_T\widetilde{V}_T + (\mathcal{C}_T - \mathcal{C})X^* + h_T]dW(t), \end{aligned}$$

where we have suppressed t, and

$$h_T(t) \triangleq \mathcal{C}_T(t)\mathbb{E}[\widetilde{X}_T(t)] + D\theta_T(t).$$

Let $P > 0$ be the solution to the ARE (6). By Itô's rule,

$$\begin{aligned} \frac{d}{dt}\mathbb{E}\langle P\widetilde{V}_T(t), \widetilde{V}_T(t)\rangle &= \mathbb{E}\Big\{\big\langle(P\mathcal{A}_T + \mathcal{A}_T^\top P + \mathcal{C}_T^\top P\mathcal{C}_T)\widetilde{V}_T, \widetilde{V}_T\big\rangle \\ &\quad + 2\big\langle[P(\mathcal{A}_T - \mathcal{A}) + \mathcal{C}_T^\top P(\mathcal{C}_T - \mathcal{C})]X^*, \widetilde{V}_T\big\rangle \\ &\quad + \big\langle\mathcal{C}_T^\top P\mathcal{C}_T\mathbb{E}[\widetilde{X}_T], \mathbb{E}[\widetilde{X}_T]\big\rangle + k_T\Big\}, \end{aligned} \tag{25}$$

where

$$\begin{aligned} k_T(t) &\triangleq \big\langle P[\mathcal{C}_T(t) - \mathcal{C}]X^*(t), [\mathcal{C}_T(t) - \mathcal{C}]X^*(t)\big\rangle \\ &\quad + \big\langle 2P\mathcal{C}_T(t)\mathbb{E}[\widetilde{X}_T(t)] + PD\theta_T(t), D\theta_T(t)\big\rangle. \end{aligned}$$

We observe the following facts:

$$\begin{aligned} &P\mathcal{A}_T + \mathcal{A}_T^\top P + \mathcal{C}_T^\top P\mathcal{C}_T \\ &\quad = P\mathcal{A} + \mathcal{A}^\top P + \mathcal{C}^\top P\mathcal{C} + P(\mathcal{A}_T - \mathcal{A}) + (\mathcal{A}_T - \mathcal{A})^\top P \\ &\qquad + (\mathcal{C}_T - \mathcal{C})^\top P\mathcal{C} + \mathcal{C}_T^\top P(\mathcal{C}_T - \mathcal{C}), \\ &P\mathcal{A} + \mathcal{A}^\top P + \mathcal{C}^\top P\mathcal{C} \\ &\quad = -(Q + \Theta^\top R\Theta + S^\top\Theta + \Theta^\top S) < 0. \end{aligned}$$

Also, observe that by Lemma 2.3, for some positive constants K and λ,

$$|\mathcal{A}_T(t) - \mathcal{A}| + |\mathcal{C}_T(t) - \mathcal{C}| \leqslant Ke^{-\lambda(T-t)}, \quad \forall 0 \leqslant t \leqslant T < \infty.$$

For simplicity, in the following proof we shall denote by K and λ two generic positive constants, which do not depend on T and may vary from line to line. Then it follows that for some constant $\alpha > 0$,

$$\begin{aligned} &\mathbb{E}\big\langle[P\mathcal{A}_T(t) + \mathcal{A}_T(t)^\top P + \mathcal{C}_T(t)^\top P\mathcal{C}_T(t)]\widetilde{V}_T(t), \widetilde{V}_T(t)\big\rangle \\ &\quad \leqslant K\Big[-2\alpha + e^{-\lambda(T-t)}\Big]\mathbb{E}|\widetilde{V}_T(t)|^2. \end{aligned} \tag{26}$$

Moreover, by the Cauchy–Schwarz inequality, Proposition 3.1, and Lemmas 3.3 and 3.4, we have

$$\begin{aligned} &2\mathbb{E}\big\langle[P(\mathcal{A}_T(t) - \mathcal{A}) + \mathcal{C}_T(t)^\top P(\mathcal{C}_T(t) - \mathcal{C})]X^*(t), \widetilde{V}_T(t)\big\rangle \\ &\quad \leqslant K\Big[\alpha\mathbb{E}|\widetilde{V}_T(t)|^2 + e^{-\lambda(T-t)}\Big], \end{aligned} \tag{27}$$

$$\begin{aligned} &\big\langle\mathcal{C}_T(t)^\top P\mathcal{C}_T(t)\mathbb{E}[\widetilde{X}_T(t)], \mathbb{E}[\widetilde{X}_T(t)]\big\rangle + \mathbb{E}|k_T(t)| \\ &\quad \leqslant K\Big[e^{-\lambda t} + e^{-\lambda(T-t)}\Big]. \end{aligned} \tag{28}$$

Substitution of (26)–(28) into (25) yields

$$\begin{aligned} \frac{d}{dt}\mathbb{E}\langle P\widetilde{V}_T(t), \widetilde{V}_T(t)\rangle &\leqslant K\Big[\Big(-\alpha + e^{-\lambda(T-t)}\Big)\mathbb{E}|\widetilde{V}_T(t)|^2 \\ &\quad + e^{-\lambda t} + e^{-\lambda(T-t)}\Big]. \end{aligned}$$

Proceeding similarly to the proof of Proposition 3.1, we obtain

$$\mathbb{E}|\widetilde{V}_T(t)|^2 \leqslant K\Big[e^{-\lambda t} + e^{-\lambda(T-t)}\Big], \quad \forall t \in [0,T].$$

By (17), (20), (23), and the definition of $\widetilde{V}_T(\cdot)$, we have

$$\begin{aligned} X_T^*(t) - \boldsymbol{X}^*(t) &= \widetilde{V}_T(t) + \mathbb{E}[\widetilde{X}_T(t)], \\ u_T^*(t) - \boldsymbol{u}^*(t) &= \Theta_T(t)[X_T^*(t) - \boldsymbol{X}^*(t)] \\ &\quad + [\Theta_T(t) - \Theta]X^*(t) + \theta_T(t), \end{aligned}$$

which, together with Lemma 3.4, implies (24). □

B. Other turnpike properties

The following result, which is a direct consequence of Theorem 3.5, shows that the strong integral and the mean-square turnpike properties hold for Problem (SLQ)$_T$.

Theorem 3.6. Let (A1)–(A2) hold. Then the following strong integral turnpike property holds:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \mathbb{E}\Big[|X_T^*(t)-\boldsymbol{X}^*(t)|^2+|u_T^*(t)-\boldsymbol{u}^*(t)|^2\Big]dt=0.$$

Consequently, the mean-square turnpike property also holds:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \Big[|\mathbb{E}[X_T^*(t)]-x^*|^2+|\mathbb{E}[u_T^*(t)]-u^*|^2\Big]dt=0.$$

The next result shows that for any initial state x, the value $V_T(x)$ of Problem (SLQ)$_T$ converges to the minimum of Problem (O) in the time-average sense.

Theorem 3.7. Let (A1)–(A2) hold. Then

$$\lim_{T\to\infty}\frac{1}{T}V_T(x)=V,\quad \forall x\in\mathbb{R}^n,$$

where V is the minimum of Problem (O).

Proof. By Proposition 3.1, $\mathbb{E}|\boldsymbol{X}^*(\cdot)|^2$ and $\mathbb{E}|\boldsymbol{u}^*(\cdot)|^2$ are bounded. Thus, by Theorem 3.5,

$$\mathbb{E}|X_T^*(t)|^2+\mathbb{E}|u_T^*(t)|^2\leqslant K,\quad \forall 0\leqslant t\leqslant T<\infty,$$

for some $K>0$. The above, together with Theorem 3.6, implies that

$$\begin{aligned}&\lim_{T\to\infty}\frac{1}{T}V_T(x)=\lim_{T\to\infty}\frac{1}{T}J_T(x;u_T^*(\cdot))\\&=\lim_{T\to\infty}\frac{1}{T}\int_0^T\Big\{\mathbb{E}\big[\langle QX_T^*(t),X_T^*(t)\rangle+2\langle SX_T^*(t),u_T^*(t)\rangle\big]\\&\quad+\langle Ru_T^*(t),u_T^*(t)\rangle+2\langle q,\mathbb{E}[X_T^*(t)]\rangle+2\langle r,\mathbb{E}[u_T^*(t)]\rangle\Big\}dt\\&=\lim_{T\to\infty}\frac{1}{T}\int_0^T\Big\{\mathbb{E}\big[\langle Q\boldsymbol{X}^*(t),\boldsymbol{X}^*(t)\rangle+2\langle S\boldsymbol{X}^*(t),\boldsymbol{u}^*(t)\rangle\\&\quad+\langle R\boldsymbol{u}^*(t),\boldsymbol{u}^*(t)\rangle\big]+2\langle q,\mathbb{E}[\boldsymbol{X}^*(t)]\rangle+2\langle r,\mathbb{E}[\boldsymbol{u}^*(t)]\rangle\Big\}dt.\end{aligned}$$

Noting that

$$\mathbb{E}[X^*(t)]\equiv 0,\quad \mathbb{E}[\boldsymbol{X}^*(t)]\equiv x^*,\quad \mathbb{E}[\boldsymbol{u}^*(t)]\equiv u^*,$$

and letting

$$\Sigma\triangleq Q+S^\top\Theta+\Theta^\top S+\Theta^\top R\Theta,$$

we further have

$$\begin{aligned}&\lim_{T\to\infty}\frac{1}{T}V_T(x)=\lim_{T\to\infty}\frac{1}{T}\int_0^T\Big\{\mathbb{E}\big[\langle QX^*(t),X^*(t)\rangle\\&\quad+\langle Qx^*,x^*\rangle+2\langle SX^*(t),\Theta X^*(t)\rangle+2\langle Sx^*,u^*\rangle\\&\quad+\langle R\Theta X^*(t),\Theta X^*(t)\rangle+\langle Ru^*,u^*\rangle\big]\\&\quad+2\langle q,x^*\rangle+2\langle r,u^*\rangle\Big\}dt\\&=\lim_{T\to\infty}\frac{1}{T}\int_0^T\mathbb{E}\langle\Sigma X^*(t),X^*(t)\rangle dt\\&\quad+\langle Qx^*,x^*\rangle+2\langle Sx^*,u^*\rangle+\langle Ru^*,u^*\rangle\\&\quad+2\langle q,x^*\rangle+2\langle r,u^*\rangle.\end{aligned}\tag{29}$$

On the other hand, letting $P>0$ be the solution to the ARE (6) and noting that $\mathbb{E}[X^*(t)]=0$, we have

$$\begin{aligned}&\mathbb{E}\langle PX^*(T),X^*(T)\rangle\\&=\mathbb{E}\int_0^T\Big\{\langle[P(A+B\Theta)+(A+B\Theta)^\top P\\&\quad+(C+D\Theta)^\top P(C+D\Theta)]X^*(t),X^*(t)\rangle+\langle P\sigma^*,\sigma^*\rangle\Big\}dt\\&=\mathbb{E}\int_0^T\Big[-\langle\Sigma X^*(t),X^*(t)\rangle+\langle P\sigma^*,\sigma^*\rangle\Big]dt.\end{aligned}$$

Since $\mathbb{E}|X^*(T)|^2$ is bounded in T, we have

$$\begin{aligned}&\lim_{T\to\infty}\frac{1}{T}\int_0^T\mathbb{E}\langle\Sigma X^*(t),X^*(t)\rangle dt\\&=\lim_{T\to\infty}\frac{1}{T}\Big[-\mathbb{E}\langle PX^*(T),X^*(T)\rangle+\int_0^T\langle P\sigma^*,\sigma^*\rangle dt\Big]\\&=\langle P\sigma^*,\sigma^*\rangle.\end{aligned}\tag{30}$$

Combining (29) and (30), we get the desired result. □

Example 3.8. Consider the one-dimensional state equation

$$\begin{cases}dX(t)=[u(t)-1]dt+[X(t)+u(t)]dW(t),\\ X(0)=x,\end{cases}$$

and the cost functional

$$J_T(x;u(\cdot))\triangleq\mathbb{E}\int_0^T\Big[|X(t)|^2+|u(t)|^2+2X(t)+2u(t)\Big]dt.$$

The corresponding ARE (6) reads

$$P+1-\frac{4P^2}{1+P}=0,$$

whose positive solution is $P=1$. Thus, the Θ defined by (11) is equal to -1. Now, (15) becomes

$$\begin{cases}u^*-1=0\\ x^*+(x^*+u^*)+1=0,\\ u^*+\lambda^*+(x^*+u^*)+1=0,\end{cases}$$

from which we get

$$u^*=1,\quad x^*=-1,\quad \lambda^*=-2.$$

Moreover, the SDE (16) becomes

$$\begin{cases}dX^*(t)=-X^*(t)dt,\\ X^*(0)=0,\end{cases}$$

whose solution is identically zero. So the turnpike limit defined by (17) is

$$(\boldsymbol{X}^*(t),\boldsymbol{u}^*(t))=(x^*,u^*)=(-1,1).$$

In light of Theorems 3.5 and 3.7, we could regard $(-1,1)$ as an approximation of the optimal pair of Problem (SLQ)$_T$ when T is sufficiently large.

IV. CONCLUSIONS

In this paper, we establish the strong exponential, the strong integral, and the mean-square turnpike properties for a class of stochastic LQ optimal control problems. We also show that the value function of the stochastic LQ optimal control problem converges to the minimum of the corresponding static optimization problem in the time-average sense. The keys are to correctly formulate the corresponding static optimization problem and find the equations determining the correction processes. The idea and methods of the paper might be applied to more general problems, such as mean-field LQ optimal control, periodic LQ optimal control, and LQ optimal control with partial information, etc. We will report some further results along this line in our future publications.

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