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# Potential Implications of Mixing Perturbations on Robust Stability for Linear Uncertain Systems

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*Abstract*—**This paper is concerned with the stabilization problem of unstable systems with mixed gain and phase perturbations, thence elaborating on the exact computation of optimal robustness margins. We focus on non-minimum phase plants that are stabilized by proportional and proportional-integral (PI) controllers. Specifically, for such systems with mixed perturbations, we first show that the computation of optimal gain margin constitutes a constrained optimization problem. It is proved that the maximum gain margin is attained at zero integral gain, and the boundary value of proportional gain can be determined exactly. Via the Bilherz criterion, we next demonstrate that the maximal phase margin of non-minimum phase systems subject to mixed perturbations is also achieved at zero integral gain. It turns out that the calculation of optimal phase margin amounts to solving a concave optimization problem. Finally, we find that proportional-integral control and proportional control promise the same expressions of optimum robustness margins. Our explicit results clearly characterize the well-established dependence of the maximum robustness margins and/or the optimal controller coefficients on the system's unstable pole, nonminimum phase zero as well as uncertain perturbations.**

## I. INTRODUCTION

Two classical stability margins, gain and phase margins, have been extensively utilized as robustness metrics for the analysis of control systems over the past few decades, which characterize the ability of a control system to preserve stability in the absence of perturbations (see, e.g., [1]–[3]).The study on robustness and stability margins primarily concerns single-agent systems, which consist of single-input-single-output (SISO) systems and multiple-input-multiple-output (MIMO) systems. With a SISO system, phase and gain margins have been determined by using the scalar Nyquist methodology and the Bode analysis method (see the literature in [1], [4]). In terms of their unequivocal physical properties, the well-established results of the SISO phase and gain margins both furnish meaningful insight into the robustness of control systems. Furthermore, many efforts have been devoted to extrapolating robustness analysis from SISO systems to MIMO systems since the 1970s (see [2], [5], [6]). Among them, it is typical that for continuous time feedback systems, linear quadratic state feedback regulators provide desirable stability margins in [2]. Whilst in the work of [5], it has been shown that an extension of the typical Bode analysis method and Nyquist technique to MIMO systems was implemented from the perspective of the transfer function matrix of the feedback system.

Perturbations are constantly experienced in almost all engineered control systems, which have a deleterious impact on the performance of the system and even lead to instability (see, e.g., [7]–[10]).Treatments of the inevitable uncertainties affecting the plant model have long been a central interest in the design of feedback control systems. It is apparently crucial to characterize under what circumstances the closed-loop system remains stable or can be stabilized in the presence of perturbations. In [11], an approach to stabilize a category of linear systems subject to unknown but bounded uncertain parameters was proposed, whose stabilizing controller amounts to dealing with some algebraic Riccati equations. However, it is worth mentioning that this solution is appropriate for particular classes of uncertain systems and requires new considerations for other multiple types of uncertain systems in practice. Afterwards, it has been shown that a robust stability situation was provided in [12] for uncertain discrete-time systems associated with convex polytopic uncertainty, where it is possible to examine the stability condition by utilizing parameter-dependent Lyapunov functions.

Whereas these response- and model-based designs deliver promising performance alluded to above, the plant's unstable pole, non-minimum phase zero, and perturbations impose adverse effects on the robust performance and stability margin of the closed-loop system, which in turn limits the performance of these design approaches. Recently, special attention has been dedicated to the analytical aspects and computation analysis of PID controllers, bringing about tangible perspectives and refined results, wherein the topics range from uncertain SISO systems [13], [14], time-delay systems [15], robot systems [16], to nonlinear systems [17], [18], and multi-agent systems [19], [20].

We have studied the maximal gain and phase margins achieved by PID controllers in previous works for lower-order unstable systems with merely single uncertainty variation [14], [21]. It has been recognized that the PID is the

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most popular controller with high-frequency usage in industrial processes, whereas the majority of industrial systems are modeled by first- and second-order plants. Nevertheless, such exact robust optimization against mixed gain and phase perturbations as well as the analytical results, even for first-order unstable systems, have not been available by far. For this reason, one primary motivation underlying this paper is to characterize the stabilization and robustness problems of unstable systems with respect to mixed perturbations. This consideration stems also partly from the fact that in reality, it is much more prevalent to experience unstable systems afflicted by mixed perturbed variations [22], [23]. Our work in this article proceeds in-depth for the unstable systems suffering from mixed gain and phase perturbations, thereby yielding explicit results of the robustness margins that shall facilitate the understanding and insight of the PID methodology and eventually form an interpretable theory.

We are interested in non-minimum-phase systems. It is important to be aware, however, that in the circumstance of stabilizing a non-minimum phase unstable system robustly, the derivative control will lead to an improper system. Accordingly, in this article, we attempt to give a treatment of the exact computation of optimal robustness margins for first-order unstable plants under proportional controllers and proportional-integral controllers. Such essential measures as PI control and the robust margins against mixed gain and phase perturbations are still of engineering relevance and mathematical importance at present and remain in necessity for in-depth investigations. Furthermore, it is noted that the introduction of integral gain considerably enlarges the order of degree and sophistication of the stabilization problem. Specifically, we perform the stability analysis from the perspective of the frequency domain in two scenarios. One condition is to determine the achievable gain margin under mixed perturbations, and the other is to find the achievable phase margin under mixed perturbations. In other words, we extend the definitions of the gain and phase margins to a more practical circumstance and attempt to rigorously derive explicit expressions of optimal robustness margins, which thus clearly specify the system's robust performance from a computational standpoint. Further to this, our explicit results help us to understand how the proportional and integral controller parameters can be constructed to stabilize unstable systems suffering from mixed perturbations and achieve optimal robustness margins.

Due to space constraints, throughout this paper, we omit all the technical proofs.

## II. Preliminaries and Problem Descriptions

In this section, we first go to tell the characterizations of robustness margins under mixed perturbations. Several mathematical tools are also provided herein to facilitate the analysis throughout this work.

### A. The Robustness Margin Problem

We consider the target plant $P(s)$ depicted in Fig. 1, wherein the unstable system $P(s)$ suffers from mixed perturbations $\Delta = \alpha e^{-j\vartheta}$, with $\alpha$ being the uncertain gain variation and $\vartheta$ being the uncertain phase variation. Given such $P(s)$, we introduce a finite-dimensional LTI controller denoted by $K(s)$ to stabilize this system. We first study the family of plants

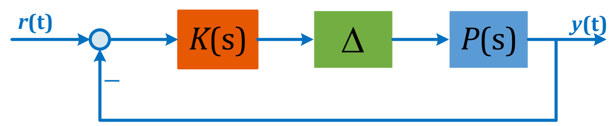


Fig. 1: Feedback system including perturbations.

$$\mathcal{P}_{\mu}^{\vartheta_0} = \{\alpha e^{-j\vartheta_0}P(s) \; : \quad 1 \le \alpha < \mu \; \}, \tag{1}$$

where $\vartheta_0$ denotes a certain phase perturbation. For a given plant $P(s)$, the maximal gain margin under the certain phase perturbation is defined by

$$\kappa_{\vartheta_0}^{M} = \sup\{\mu : \text{There exists some } K(s) \text{ stabilizing } \alpha e^{-j\vartheta_0}P(s), \;\; \forall \alpha \in [1,\mu)\}.$$

We next consider the family of plants

$$\mathcal{Q}_{\nu}^{\alpha_0} = \{e^{-j\vartheta}\alpha_0 P(s) \; : \quad \vartheta \in (-\nu,\nu) \; \}, \tag{2}$$

where $\alpha_0$ represents a certain gain perturbation. The corresponding maximal phase margin for $P(s)$, analogously, is specified as

$$\vartheta_{\alpha_0}^{M} = \sup\{\nu : \text{There exists some } K(s) \text{ stabilizing } e^{-j\vartheta}\alpha_0 P(s), \; \forall \vartheta \in (-\nu,\nu)\}.$$

$\kappa_{\vartheta_0}^{M}$ and $\vartheta_{\alpha_0}^{M}$ delineates, respectively, the largest possible collections $\mathcal{P}_{\mu}^{\vartheta_0}$ and $\mathcal{Q}_{\nu}^{\alpha_0}$ that can be stabilized by a same LTI controller. Particularly, our central curiosity in the article seeks to search for the optimal values of robustness margins under more structured LTI controllers, that is, P controller and PI controller

$$K_{P}(s) = k_p, \quad K_{PI}(s) = k_p + \frac{k_i}{s}. \tag{3}$$

Note, however, that with the stabilization of first-order non-minimum phase unstable systems, the derivative action in a PID controller shall result in an improper system. Hence, we are concerned with P and PI controllers in our subsequent developments. It follows similarly that the optimal robustness margins under mixed perturbations achieved by P control and PI control become

$$\kappa_{\vartheta_0}^{PI} = \sup\{\mu : \text{There exists a } K_{PI}(s) \text{ stabilizing } \alpha e^{-j\vartheta_0}P(s) \;\; \forall \alpha \in [1,\mu)\},$$

and

$$\vartheta_{\alpha_0}^{PI} = \sup\{\nu : \text{There exists a } K_{PI}(s) \text{ stabilizing } e^{-j\vartheta}\alpha_0 P(s) \;\; \forall \vartheta \in (-\nu,\nu)\}.$$

$\kappa_{\vartheta_0}^{PI}$ and $\vartheta_{\alpha_0}^{PI}$ alluded to above concretely quantitate the maximal allowable ranges of gain and phase values such that a single PI controller can be found to stabilize all the plants of $\mathcal{P}_{\mu}^{\vartheta_0}$ and $\mathcal{Q}_{\nu}^{\alpha_0}$ undergoing the mixed gain and phase perturbations over the whole scopes of perturbation variations. Likewise, we utilize the notations $\kappa_{\vartheta_0}^{P}$ and $\vartheta_{\alpha_0}^{P}$ to denote the robustness margins attained by proportional controllers (i.e., $k_i = 0$) for comparison in the sequel.

### B. Mathematical Backgrounds

Before proceeding, we gather the following results from the theory of algebraic geometry, which are helpful in the subsequent analysis and shall be utilized repeatedly.

**Lemma 2.1** ( [24], [25]): *Consider the complex polynomial*

$$(a_0+jb_0)s^n+(a_1+jb_1)s^{n-1}+\cdots+(a_n+jb_n),\ \ a_0+jb_0 \neq 0. \tag{4}$$

*Define the associated* $(2i-1)\times(2i-1)$ *Bilherz submatrices*

$$\Delta_i = \begin{bmatrix} a_1 & a_3 & \cdots & a_{2i-1} & -b_2 & -b_4 & \cdots & -b_{2i-2} \\ a_0 & a_2 & \cdots & a_{2i-2} & -b_1 & -b_3 & \cdots & -b_{2i-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_i & 0 & 0 & \cdots & -b_{i-1} \\ 0 & b_2 & \cdots & b_{2i-2} & a_1 & a_3 & \cdots & a_{2i-3} \\ 0 & b_1 & \cdots & b_{2i-3} & a_0 & a_2 & \cdots & a_{2i-4} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & b_i & 0 & 0 & \cdots & a_{i-1} \end{bmatrix}.$$

*Then, the polynomial (4) is stable if and only if* $\det(\Delta_i)$ *are positive for all* $i = 1,\ 2,\ \cdots,\ n$.

**Lemma 2.2** ( [26]): *Consider the complex polynomial (4). If the polynomial (4) is strictly Hurwitz, then* $a_i a_{i+1} + b_i b_{i+1} > 0$, $i = 0, 1, ..., n-1$.

## III. Optimal Gain Margin under Mixed Perturbations

In this section, we seek to find the explicit computation of the optimal gain margin for first-order unstable plants with respect to a certain phase perturbation. We shall consider the nonminimum-phase systems stabilized by P and PI control, for which the characterization of this maximum gain margin can be determined exactly in the perturbed systems.

We focus on the unstable plant specifically stated by

$$P(s) = \frac{s-z}{s-p}, \tag{5}$$

where $p > 0$ denotes the unstable pole, and $z > 0$ the nonminimum-phase zero. This deliberation emanates principally from the fact that a number of industrial systems can be suitably modeled as lower-order unstable plants, while indeed, the majority may be regarded as first-order unstable plants. Pursuant to the unstable systems (5) in [14], [21], it follows that the single gain or phase perturbations $\alpha_0$, $\vartheta_0$ pertain to the bounded intervals, which are featured by the assumptions as follows.

**Assumption 3.1**: *The phase perturbation* $\vartheta_0$ *falls into the interval* $0 \le |\vartheta_0| \le \bar{\vartheta}$, *where*

$$\bar{\vartheta} = \cos^{-1}\frac{2\sqrt{pz}}{p+z}.$$

**Assumption 3.2**: *The gain perturbation* $\alpha_0$ *falls into the interval* $1 \le \alpha_0 \le \bar{\alpha}$, *where*

$$\bar{\alpha} = \max\left\{\frac{z}{p}, \frac{p}{z}\right\}.$$

*1) Achieved by P control:* Under Assumption 3.1, we first seek to search for the maximal gain margin of the system (5) associated with a certain phase perturbation achieved by a proportional controller. Toward this end, we begin by defining the indexes

$$\underline{L} = \frac{-(p+z)\cos\vartheta_0 - \sqrt{(p+z)^2\cos^2\vartheta_0 - 4pz}}{2z}, \tag{6}$$

and

$$\overline{U} = \frac{-(p+z)\cos\vartheta_0 + \sqrt{(p+z)^2\cos^2\vartheta_0 - 4pz}}{2z}. \tag{7}$$

We present the following result by tackling the constrained optimization problem explicitly.

**Theorem 3.1**: *Let* $P(s)$ *be given by (5). Under Assumption 3.1, the following statements are correct.*

*(i) The feasible proportional coefficient* $k_p \in (\underline{L},\ \overline{U})$.

*(ii) The optimum gain margin under certain phase perturbation attained by P controllers* $\vartheta_0$ *is exactly given by*

$$\kappa_{\vartheta_0}^{P} = \frac{\cos\vartheta_0 + \sqrt{\left|\frac{p-z}{p+z}\right|^2 + \cos^2\vartheta_0 - 1}}{\cos\vartheta_0 - \sqrt{\left|\frac{p-z}{p+z}\right|^2 + \cos^2\vartheta_0 - 1}}, \tag{8}$$

*where the optimal proportional coefficient* $k_p^* = \overline{U}$.

*(iii) The gain margin* $\kappa_{\vartheta_0}^{P}$ *is monotonically decreasing with* $\vartheta_0$ *for* $0 \le \vartheta_0 \le \bar{\vartheta}$ *and increasing with* $\vartheta_0$ *for* $-\bar{\vartheta} \le \vartheta_0 < 0$.

*(iv)*

$$\underline{\kappa}_{\pm\bar{\vartheta}}^{P} \le \kappa_{\vartheta_0}^{P} \le \overline{\kappa}_0^{P} \tag{9}$$

*with*

$$\kappa_{\pm\bar{\vartheta}}^{P} = 1, \qquad \overline{\kappa}_0^{P} = \frac{1 + \left|\frac{p-z}{p+z}\right|}{1 - \left|\frac{p-z}{p+z}\right|}. \tag{10}$$

**Remark 3.1**: Theorem 3.1 reveals that as a function of phase perturbation $\vartheta_0$, the gain margin $\kappa_{\vartheta_0}^{P}$ is monotonically decreasing with $|\vartheta_0|$. This points to the observation that the maximal gain margin of the unstable system (5) is intrinsically curtailed by the phase perturbation $\vartheta_0$. In the scenario of $\vartheta_0 = 0$, the gain margin is reduced to $\kappa_0^{P}$ in (10), or alternatively

$$\kappa_0^{P} = \bar{\alpha} = \max\left\{\frac{z}{p}, \frac{p}{z}\right\}.$$

which is exactly the gain margin implemented by P or PI controllers in [14], [21].

**Remark 3.2**: A further inspection of Theorem 3.1 leads us to the fact that the gain margin $\kappa_{\vartheta_0}^{P}$ increases gradually with $|p-z|$, i.e., the relative distance of the unstable pole $p$ and nonminimum phase zero $z$. In a limiting case of $p = z$, the gain margin $\kappa_0^{P} = 1$ since it follows from Assumption 3.1 that $\vartheta_0 = 0$. Moreover, in view of (8), we note that to render the radical expression feasible in (8), it suffices to

$$\left|\frac{p-z}{p+z}\right|^2 + \cos^2\vartheta_0 - 1 \geq 0.$$

By solving the equality above, it follows that

$$\vartheta_0 \leq \cos^{-1}\frac{2\sqrt{pz}}{p+z},$$

which in turn shows that this explicit upper bound is the same as $\bar{\vartheta}$ in Assumption 3.1. The determination of this exact upper bound $\bar{\vartheta}$ herein furnishes a fresh insight to explore the properties of phases margin in [14], [21].

*2) Achieved by PI control:* Also of interest in this work is to explore the gain margin achieved by PI controllers under certain phase perturbations. The computation of optimal gain margin in this scenario, however, brings about a more intricate question due to the increasing order generated by the integral control action than its counterpart. We ultimately arrive at the explicit description of the maximum gain margin in the following theorem.

**Theorem 3.2**: *Let $P(s)$ be given by (5). Under Assumption 3.1, the following statements are correct.*

*(i) The optimal integral coefficient for maximizing gain margin lies in $k_i^* = 0$.*

*(ii) The exact gain margin under certain phase perturbation $\vartheta_0$ is found as*

$$\kappa_{\vartheta_0}^{PI} = \kappa_{\vartheta_0}^{P} = \frac{\cos\vartheta_0 + \sqrt{\left|\frac{p-z}{p+z}\right|^2 + \cos^2\vartheta_0 - 1}}{\cos\vartheta_0 - \sqrt{\left|\frac{p-z}{p+z}\right|^2 + \cos^2\vartheta_0 - 1}}, \quad (11)$$

*and the optimal proportional coefficient achieving optimal gain margin is $k_p^* = \overline{U}$.*

**Remark 3.3**: It is noted that the intervention of integral control action increases the order of the closed-loop control system, thereby bringing more intricate challenges than Theorem 3.1. Towards this end, built on the Bilherz criterion (Lemma 2.1) and necessary stability condition (Lemma 2.2), the gain margin optimization problem under certain phase perturbation can be then determined by solving a constrained optimization problem over a fixed feasible set of the proportional and integral control parameters. With the analysis of the monotonicity and concavity of this optimization problem as well as several mathematical operation techniques, we arrive at the exactly computable expression of optimal gain margin (11) associated with phase perturbation.

**Remark 3.4**: From Theorem 3.2, we are further availed to a couple of salutary insights. First, in accordance with the above design of the PI coefficients, it is clear that only the proportional gain contributes to robust stability performance, thus indicating that integral gain does not exert any active power in enlarging the gain margin. In view of the proof in Theorem 3.2, actually, we note that the gain margin tends to be lessened provided that the integral parameter is nonzero. Second, we find that the gain margin $\kappa_{\vartheta_0}^{PI}$ decrease with $|\vartheta_0|$. Similar to that in Theorem 3.1, this insight displays that the similar property in (10) also suitable to (11), that is

$$\underline{\kappa}_{\pm\bar{\vartheta}}^{PI} \leq \kappa_{\vartheta_0}^{PI} \leq \overline{\kappa}_0^{PI}$$

with $\kappa_{\pm\bar{\vartheta}}^{PI} = \kappa_{\pm\bar{\vartheta}}^{P}$ and $\overline{\kappa}_0^{PI} = \overline{\kappa}_0^{P}$.

In what follows, we shall examine a few special cases in depth to show our observations.

**Corollary 3.1**: *Let $P(s)$ be given by (5).*

*(i) If $\cos\vartheta_0 = 2\cos\bar{\vartheta}$, then*

$$\kappa_{\vartheta_0}^{PI} = \kappa_{\vartheta_0}^{P} = \frac{2+\sqrt{3}}{2-\sqrt{3}}. \quad (12)$$

*(ii) If $\cos\vartheta_0 = \sqrt{2}\cos\bar{\vartheta}$, then*

$$\kappa_{\vartheta_0}^{PI} = \kappa_{\vartheta_0}^{P} = \frac{\sqrt{2}+1}{\sqrt{2}-1}. \quad (13)$$

The statements (i) and (ii) follow immediately from Theorems 3.1 and 3.2. In turn, Corollary 3.1 exhibits clearly that the relatively larger the value of phase perturbation, the tighter the resultant gain margin is achieved. ■

## IV. Optimal Phase Margin under Mixed Perturbations

This section is devoted to studying the optimal phase margin of the unstable plant in the presence of certain gain perturbation. In other words, we also attempt to find a computationally analytical result of the maximal phase variation, which specifies the robust stability performance for a type of perturbed system.

We dedicate ourselves to examining the unstable systems including nonminimum phase zero. To proceed, we define the sets

$$\Omega_+ = \{(k_p, k_i) : -1/\alpha_0 < k_p < (\alpha_0 k_i - p)/(\alpha_0 z), k_i < 0\},$$

$$\Omega_- = \{(k_p, k_i) : (\alpha_0 k_i - p)/(\alpha_0 z) < k_p < -1/\alpha_0, k_i > 0\},$$

and

$$\Omega_+^0 = \{k_p : -1/\alpha_0 < k_p < -p/(\alpha_0 z)\},$$

$$\Omega_-^0 = \{k_p : -p/(\alpha_0 z) < k_p < -1/\alpha_0\}.$$

Denote the function $\Psi_{\alpha_0}(k_p)$ by

$$\Psi_{\alpha_0}(k_p) = \frac{z\alpha_0^2 k_p^2 + p}{-(p+z)\alpha_0 k_p}. \quad (14)$$

Our main results are stated in the following theorem.

**Theorem 4.1**: *Let $P(s)$ be given by (5). Under Assumption 3.2, the following statements are correct.*

*(i) The feasible PI coefficients $(k_p, k_i) \in \Omega$ , where*

$$\Omega = \Omega_+ \cup \Omega_- .$$

*(ii) The optimal integral coefficient for maximizing phase margin lies in $k_i^* = 0$.*

*(iii) The exact computation of the optimal phase margin is characterized by*

$$\vartheta_{\alpha_0}^{PI} = \vartheta_{\alpha_0}^{P} = \sup\left\{\vartheta : \vartheta = \cos^{-1}\Psi_{\alpha_0}(k_p), \left| k_p \in \Omega^0 \right.\right\}, \tag{15}$$

*with $\Omega^0 = \Omega_+^0 \cup \Omega_-^0$.*

*(iv) The phase function $\vartheta = \cos^{-1}\Psi_{\alpha_0}(k_p)$ is concave over the interval $k_p \in \Omega^0$.*

*(v) The optimal phase margin under certain gain perturbation is found as*

$$\vartheta_{\alpha_0}^{PI} = \vartheta_{\alpha_0}^{P} = \cos^{-1}\sqrt{1 - \left|\frac{p-z}{p+z}\right|^2}. \tag{16}$$

*The optimal proportional coefficient is $k_p^* = -\sqrt{p/z}/\alpha_0$.*

**Remark 4.1**: It is also observed from Theorem 4.1 that the phase margin engendered by PI controllers is the same as that by P controllers. This fact thus confirms the fact that integral control action does not contribute to raising the phase margin, while the proportional control gain undertakes a preponderant position in robust stabilization. Similar to Theorem 3.2, the proof in Theorem 4.1 also shows that a nonzero integral parameter generally elicits a diminution of the phase margin. Additionally, once the optimal integral gain $k_i^* = 0$, it is seen that the phase maximization matter over fixed feasible control parameters amounts to solving a concave optimization problem, which is actually unimodal over its feasible proportional interval.

**Remark 4.2**: On account of the statement (v), we note that $\vartheta_{\alpha_0}^{PI}$, $\vartheta_{\alpha_0}^{P}$ are both monotonically increasing with the relative distance of the unstable pole and nonminimum phase zero. It thus follows that $\vartheta_{\alpha_0}^{PI} = \vartheta_{\alpha_0}^{P} = 0$ in the extreme condition $p = z$, which implies that the system is not stabilizable under this circumstance. Furthermore, in contrast to Theorems 3.1 and 3.2 where the phase perturbation imposes a restriction on enlarging the gain margin, the gain perturbation does not commit a restriction on phase margin improvement. Indeed, it can be seen from our proof and results that the gain perturbation performs a similar behavior as the proportional control operation, which impacts the selection of the optimal proportional coefficient.

## V. Illustrative Examples

We now use the numerical examples to show our results.

**Example 5.1** Consider the unstable plant (5). We first let the unstable pole $p = 2$, and allow the nonminimum phase zero $z$ to vary in the interval $[4, 12]$. For the purpose of comparison, the phase perturbation $\vartheta_0$ is selected as $-6$, 10, 15 and 15 deg, respectively. This example is intended to show how the nonminimum phase zero as well as the different phase perturbation may impinge on the achievable gain margin. As depicted in Fig. 2, it can be seen clearly that the gain margin $\kappa_{\vartheta_0}^{PI}$ (drawn as $20\log_{10}\kappa_{\vartheta_0}^{PI}$ in dB) is progressively increasing as the nonminimum phase zero $z$ steps away from the unstable pole $p$.

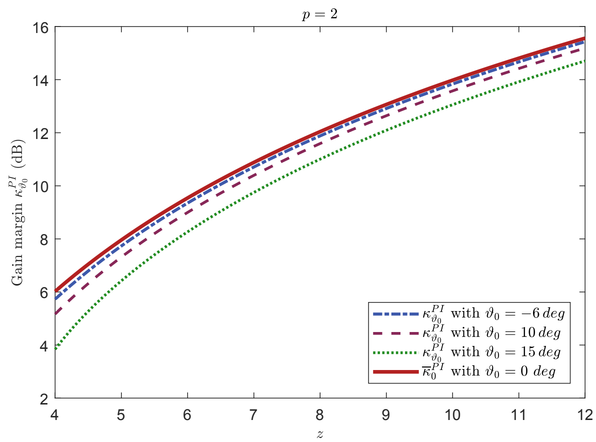


Fig. 2: Nonminimum zero effect on gain margin $\kappa_{\vartheta_0}^{PI}$.

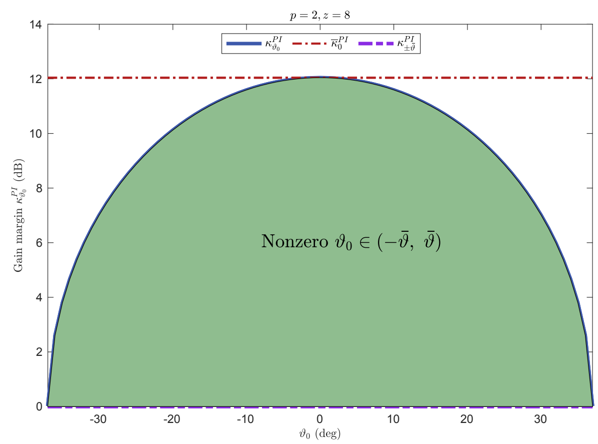


Fig. 3: Gain margin $\kappa_{\vartheta_0}^{PI}$ w.r.t the perturbation $\vartheta_0$.

We next set the pole $p = 2$, zero $z = 8$ and allow $\vartheta_0$ vary in the interval $[-\bar{\vartheta}, \bar{\vartheta}]$ with $\bar{\vartheta} = 36.8699$ deg. Fig 3 plots the property of the gain margin $\kappa_{\vartheta_0}^{PI}$ along with the perturbation $\vartheta_0$. Fig 3 then confirms the concavity of the gain margin $\kappa_{\vartheta_0}^{PI}$, where more specifically, $\kappa_{\vartheta_0}^{PI}$ decreases monotonically with positive $\vartheta_0$ and increases with negative $\vartheta_0$, whose maximum and minimum are exactly attained at $\vartheta_0 = 0$ and $\vartheta_0 = \pm\bar{\vartheta}$ respectively. This observation is consistent with Theorems 3.1 and 3.2. Indeed, Fig. 2 also exhibits the same property for the gain margin $\kappa_{\vartheta_0}^{PI}$.

## VI. Conclusions

In this paper, we have investigated the stabilization problems and derived the optimal robustness margins for first-order unstable systems subject to mixed gain/phase perturbations, both under P and PI control actions. For non-minimum phase plants, the optimum gain margin under phase perturbation was obtained with an exact expression at first, then showing the intrinsic limit required for achieving robust stability. We next proved that the maximal phase margin under gain perturbation constitutes a concave optimization problem, whose unique expression can be determined exactly. All the analytical results derived in this paper reveal that the integral gain serves no role in robust stability improvement. Additionally, the explicit expressions of optimal robustness margins provide clear metrics, which is helpful for us to recognize how the unstable pole, the non-minimum phase zero of the plants, and the extra perturbations jointly determine the achievable robustness margins and the optimal controller parameters.

## REFERENCES


[1] I. M. Horowitz, *Synthesis of Feedback Systems*. New York: Academic., 1963.

[2] M. Safonov and M. Athans, "Gain and phase margin for multiloop LQG regulators," *IEEE Transactions on Automatic Control*, vol. 22, no. 2, pp. 173–179, 1977.

[3] J. C. Doyle, B. A. Francis, and A. R. Tannenbaum, *Feedback Control Theory*. Courier Corporation, 2013.

[4] H. W. Bode, "Network analysis and feedback amplifier design," 1945.

[5] J. Doyle, "Robustness of multiloop linear feedback systems," in *1978 IEEE Conference on Decision and Control including the 17th Symposium on Adaptive Processes*. IEEE, 1979, pp. 12–18.

[6] Q.-G. Wang, Y. He, Z. Ye, C. Lin, and C. C. Hang, "On loop phase margins of multivariable control systems," *Journal of Process Control*, vol. 18, no. 2, pp. 202–211, 2008.

[7] I. R. Petersen and R. Tempo, "Robust control of uncertain systems: Classical results and recent developments," *Automatica*, vol. 50, no. 5, pp. 1315–1335, 2014.

[8] M. J. Lacerda and P. Seiler, "Stability of uncertain systems using Lyapunov functions with non-monotonic terms," *Automatica*, vol. 82, pp. 187–193, 2017.

[9] Q. Mao, Y. Xu, J. Chen, J. Chen, and T. T. Georgiou, "Maximization of gain/phase margins by PID control," *IEEE Transactions on Automatic Control*, 2024, Early Access.

[10] J. Chen, T. Qi, Y. Ding, H. Peng, J. Chen, and S. Hara, "Mean-square stability radii for stochastic robustness analysis: A frequency-domain approach," *IEEE Transactions on Automatic Control*, vol. 69, no. 9, pp. 5915–5930, 2024.

[11] I. R. Petersen, "A stabilization algorithm for a class of uncertain linear systems," *Systems & Control Letters*, vol. 8, no. 4, pp. 351–357, 1987.

[12] M. C. De Oliveira, J. Bernussou, and J. C. Geromel, "A new discrete-time robust stability condition," *Systems & Control Letters*, vol. 37, no. 4, pp. 261–265, 1999.

[13] D.-J. Wang, "Further results on the synthesis of PID controllers," *IEEE Transactions on Automatic Control*, vol. 52, no. 6, pp. 1127–1132, 2007.

[14] Q. Mao, Y. Xu, J. Chen, and T. T. Georgiou, "Implementation-oriented filtered PID control: Optimization of robustness margins," *Automatica*, vol. 152, p. 110974, 2023.

[15] J. Chen, D. Ma, Y. Xu, and J. Chen, "Delay robustness of PID control of second-order systems: Pseudoconcavity, exact delay margin, and performance tradeoff," *IEEE Transactions on Automatic Control*, vol. 67, no. 3, pp. 1194–1209, 2021.

[16] W. E. Dixon, "Adaptive regulation of amplitude limited robot manipulators with uncertain kinematics and dynamics," *IEEE Transactions on Automatic Control*, vol. 52, no. 3, pp. 488–493, 2007.

[17] C. Zhao and L. Guo, "Control of nonlinear uncertain systems by extended PID," *IEEE Transactions on Automatic Control*, vol. 66, no. 8, pp. 3840–3847, 2020.

[18] ——, "Towards a theoretical foundation of PID control for uncertain nonlinear systems," *Automatica*, vol. 142, p. 110360, 2022.

[19] A. Shariati and M. Tavakoli, "A descriptor approach to robust leader-following output consensus of uncertain multi-agent systems with delay," *IEEE Transactions on Automatic Control*, vol. 62, no. 10, pp. 5310–5317, 2016.

[20] Y. Ding, D. Ma, Q. Mao, H. Peng, and J. Chen, "Consensus robustness margins of first-order agents over undirected graphs," *IEEE Transactions on Automatic Control*, vol. 69, no. 5, pp. 3262–3269, 2024.

[21] Q. Mao, Y. Xu, J. Chen, J. Chen, and T. Georgiou, "Maximal gain and phase margins attainable by PID control," in *2021 60th IEEE Conference on Decision and Control (CDC)*. IEEE, 2021, pp. 1820–1825.

[22] G. Froyland, C. González-Tokman, and T. M. Watson, "Optimal mixing enhancement by local perturbation," *SIAM REVIEW*, vol. 58, no. 3, pp. 494–513, 2016.

[23] C. Heffernan and C.-c. P. Caulfield, "Robust and efficient identification of optimal mixing perturbations using proxy multiscale measures," *Philosophical Transactions of the Royal Society A*, vol. 380, no. 2225, p. 20210026, 2022.

[24] E. Frank, "On the zeros of polynomials with complex coefficients," *Bulletin of the American Mathematical Society*, vol. 52, no. 2, pp. 144–157, 1946.

[25] M. Marden, *Geometry of Polynomials*. American Mathematical Soc., 1949.

[26] B. T. Polyak, Y. Z. Tsypkin, Y. Shi, K. Yen, and C. Chen, "Comments on" two necessary conditions for a complex polynomial to be strictly hurwitz and their applications in robust stability analysis"[with reply]," *IEEE Transactions on Automatic Control*, vol. 39, no. 5, pp. 1147–1148, 1994.