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Stabilization by 1D Boundary Actuation of Distal 1D Reaction-Diffusion PDE through Heat PDE on a Rectangle

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Abstract

This paper presents a backstepping control design method of stabilization unstable 1D reaction-diffusion system, where the input is a 1D function on an edge of a rectangle, the distal system is a 1D reaction-diffusion PDE on the opposite edge of the rectangle, and the actuator dynamics in between are a 2D heat PDE on the rectangle between the opposite edges. A novel invertible integral transformation is introduced and the resulting controller with feedback of both PDEs' states (the distal 1D and the interior 2D states). We define a new Lyapunov function that contains cosine coefficients to prove the exponential stability in H^2 norm of the closed-loop system. Finally, the theoretical result is illustrated by simulations on a numerical example.

1. INTRODUCTION

The ability to manipulate infinite-dimensional actuator and sensor has also become a question of major technological importance, in which convection governed by hyperbolic partial differential equation (PDE) [1] and/or diffusion governed by parabolic PDE occur [2]. Research on compensating infinite-dimensional actuator dynamics has attracted increasing attention recently. For unstable ODE systems with actuator and sensor dynamics governed by PDEs, controllers have been designed to stabilize the ODE systems by applying the backstepping method, such as the diffusion type actuator in [3] and the string type actuators [4]. [5] extends the work [4] to a general nonlinear ODE with wave actuator dynamics and applies the result to the stabilization of off-shore oil drill. The authors solve the problem of stabilizing a linear ODE having a system of linearly coupled hyperbolic PDEs in the actuating path [1]. For a general nonlinear ODE, a method for stabilization of the PDE-ODE system with quasilinear first-order hyperbolic PDEs actuator dynamics is presented in [6], where a PDE predictor-feedback control law is introduced to compensate the transport actuator dynamics. In [7], a robust output-feedback stabilization controller is designed for linear ODEs with a transport PDE actuator and a heat PDE actuator, respectively. Combining the slide model control with the backstepping approach, [8] proposes a boundary control matched external disturbances to stabilize a cascade PDE-ODE system with Dirichlet/Neumann interconnection.

Input-delay systems can be regarding a particular case of the systems with PDE actuator dynamics, as delays are represented by simple first-order (transport) PDEs [9–11]. Based on [12], [9] uses a first-order hyperbolic equation to describe delay and get an ODE-PDE coupled system. Similarly, the approach is also applied in [10] to reformulate the delay for an ODE system and [11] to capture the delay effect for a reaction-diffusion system. To deal with the distributed delayed input in PDEs, 2-D transport PDEs are introduced in [13–15] for the delay-compensator design.

We advance the efforts on stabilizing infinite-dimensional cascades, where one past example is a 1D heat PDE-ODE cascade [16], to two dimensions, on a rectangle. Rather than the input being a vector and the distal system being an ODE, we consider the problem where the input is a 1D function on the one edge of a

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rectangle, the distal system is a 1D reaction-diffusion PDE on the opposite edge of the rectangle, and the actuator dynamics in between are a heat PDE on the rectangle between the opposite edges. The backstepping transformation and its inverse present challenges, as they are in a 2D domain. We resolve the challenge with explicit kernels and derive a controller with feedback of both PDEs' states (the distal 1D state and the interior 2D state). We show the closed-loop system to be exponentially stable in H^2 norm, which is necessary for having continuity of the state variables in 2D space. To prove the H^2 norm stability of the target system, we introduce a novel Lyapunov function containing cosine coefficients, which is equivalent to the H^2 norm.

This paper is organized as follows. Section 2 presents the problem. Section 3 designs the backstepping control law. Section 4 proves exponential stability in H^2 norm of the closed-loop system, and the numerical simulations are presented in Section 5. The paper ends with concluding remarks in Section 6.

Notation. For functions $f(x) \in H^2[0,1]$ and $g(x,y) \in H^2([0,1]\times[0,1])$, the L^2 norms are defined respectively as

$$\|f\|^2_{L^2_1} = \int_0^1 f^2(x)dx,$$

$$\|g\|^2_{L^2_2} = \int_0^1\int_0^1 g^2(x,y)dxdy.$$

The Sobolev norms $\|\cdot\|_{H^1_1}$ and $\|\cdot\|_{H^1_2}$ are defined as

$$\|f\|^2_{H^1_1} = \|f\|^2_{L^2_1} + \|f_x\|^2_{L^2_1},$$

$$\|g(x,y)\|^2_{H^1_2} = \|g\|^2_{L^2_2} + \|g_x\|^2_{L^2_2} + \|g_y\|^2_{L^2_2}.$$

The Sobolev norms $\|\cdot\|_{H^2_1}$ and $\|\cdot\|_{H^2_2}$ are defined as

$$\|f\|^2_{H^2_1} = \|f\|^2_{H^1_1} + \|f_{xx}\|^2_{L^2_1},$$

$$\|g(x,y)\|^2_{H^2_2} = \|g\|^2_{H^1_2} + \|g_{xx}\|^2_{L^2_2} + 2\|g_{xy}\|^2_{L^2_2} + \|g_{yy}\|^2_{L^2_2}.$$

2. Problem Statement

Consider the following reaction-diffusion PDE with diffusive actuator dynamics governed by a 2D heat equation:

$$u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t) + v(x,0,t), \tag{1}$$

$$u_x(0,t) = 0, \quad u(1,t) = 0, \tag{2}$$

$$u(x,0) = u_0(x), \tag{3}$$

$$v_t(x,y,t) = v_{xx}(x,y,t) + v_{yy}(x,y,t), \tag{4}$$

$$v_x(0,y,t) = 0, \quad v(1,y,t) = 0, \tag{5}$$

$$v_y(x,0,t) = 0, \tag{6}$$

$$v(x,1,t) = U(x,t), \tag{7}$$

$$v(x,y,0) = v_0(x,y), \tag{8}$$

where state u and v evolve in $\{(x,t)|x\in[0,1],\ t>0\}$ and in $\{(x,y,t)|x,\ y\in[0,1],\ t>0\}$, respectively. The system (1)–(3) is unstable as the reaction coefficient $\lambda > \pi^2$. $U(x,t)$ is the control input actuated at the boundary of the 2D heat equation (4)–(8). Extended domain (x,y) is shown in Fig 1, where the distal system is a 1D reaction-diffusion PDE on the opposite edge of the rectangle, and the actuator dynamics in between are a heat PDE on the rectangle.

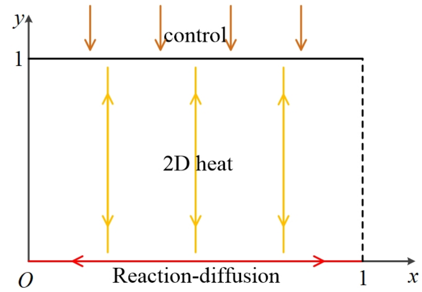


Figure 1: Domain (x,y) of heat equation $v(x,y,t)$.

In the following section, we will apply the PDE backstepping method to design a controller actuating on the boundary of the 2D PDE which stabilizes the unstable 1D reaction-diffusion PDE through $v(x,0,t)$ transverse the 1D domain.

3. Control design

3.1. Backstepping transformation

We introduce the following backstepping transformation:

$$\beta(x,y,t) = v(x,y,t) - \int_0^1 K(s,x,y)u(s,t)ds - \int_0^1\int_0^y G(s,x,y-r)v(s,r,t)drds, \tag{9}$$

where the kernel functions $K(s,x,y)$ and $G(s,x,y-r)$ are defined on $\mathscr{T}_1 = \{(s,x,y)|s,x,y\in[0,1]\}$ and $\mathscr{T}_2 = \{(s,x,y-r)|s,x,y\in[0,1], 0\le r\le y\}$, respectively. This transformation maps the original system (1)–(8) into the following stable target system

$$u_t(x,t) = u_{xx}(x,t) - cu(x,t) + \beta(x,0,t), \tag{10}$$

$$u_x(0,t) = 0, \quad u(1,t) = 0, \tag{11}$$

$$u(x,0) = u_0(x), \tag{12}$$

$$\beta_t(x,y,t) = \beta_{xx}(x,y,t) + \beta_{yy}(x,y,t), \tag{13}$$

$$\beta_x(0,y,t) = 0, \quad \beta(1,y,t) = 0, \tag{14}$$

$$\beta_y(x,0,t) = 0, \tag{15}$$

$$\beta(x,1,t) = 0, \tag{16}$$

$$\beta(x,y,0) = \beta_0(x,y), \tag{17}$$

where $c > 0$ is the adjustable converge rate. According to the equivalence between the original system (1)–(8) and the target system (10)–(17), we find the kernel equation of $K(s,x,y)$:

$$K_{ss}(s,x,y) = K_{xx}(s,x,y) + K_{yy}(s,x,y) - \lambda K(s,x,y), \tag{18}$$

$$K_s(0,x,y) = 0, \quad K(1,x,y) = 0, \tag{19}$$

$$K_x(s,0,y) = 0, \quad K(s,1,y) = 0, \tag{20}$$

$$K_y(s,x,0) = 0, \tag{21}$$

$$K(s,x,0) = -(\lambda + c)\delta(x-s); \tag{22}$$

and $G(s,x,y)$

$$G_{ss}(s,x,y-r) = G_{xx}(s,x,y-r), \tag{23}$$

$$G_x(s,0,y-r) = 0, \quad G(s,1,y-r) = 0, \tag{24}$$

$$G(1,x,y-r) = 0, \quad G_s(0,x,y-r) = 0, \tag{25}$$

$$G(s,x,0) = 0, \quad G_y(s,x,y) = K(s,x,y). \tag{26}$$

Applying the method of separation of variables yields

$$K(s,x,y) = -2(\lambda + c)\cosh(\sqrt{\lambda}y)\cdot \sum_{k=0}^{\infty} \cos\left[\left(k\pi + \frac{\pi}{2}\right)s\right]\cos\left[\left(k\pi + \frac{\pi}{2}\right)x\right], \tag{27}$$

$$G(s,x,y-r) = -2\frac{(\lambda + c)}{\sqrt{\lambda}}\sinh(\sqrt{\lambda}(y-r))\cdot \sum_{k=0}^{\infty} \cos\left[\left(k\pi + \frac{\pi}{2}\right)s\right]\cos\left[\left(k\pi + \frac{\pi}{2}\right)x\right]. \tag{28}$$

It is known that the Dirac Delta function can be expressed by the Fourier series as $\delta(x-s) = 2\sum_{k=0}^{\infty}\cos\left[\left(k\pi + \frac{\pi}{2}\right)s\right]\cos\left[\left(k\pi + \frac{\pi}{2}\right)x\right]$, which verifies the initial condition (22) by letting $y = 0$ in (27). As a result, we have

$$K(x,y,s) = -(\lambda + c)\cosh(\sqrt{\lambda}y)\delta(x-s), \tag{29}$$

$$G(x,s,y-r) = -\frac{(\lambda + c)}{\sqrt{\lambda}}\sinh(\sqrt{\lambda}(y-r))\delta(x-s). \tag{30}$$

Substituting (29) and (30) into (9), one can rewrite the transformation (9) as

$$\beta(x,y,t) = v(x,y,t) + (\lambda + c)\cosh(\sqrt{\lambda}y)u(x,t) + \frac{(\lambda + c)}{\sqrt{\lambda}}\int_0^y \sinh(\sqrt{\lambda}(y-r))v(x,r,t)dr, \tag{31}$$

and let (16) hold, the controller should satisfy

$$U(x,t) = -\frac{(\lambda + c)}{\sqrt{\lambda}}\int_0^1 \sinh(\sqrt{\lambda}(1-r))v(x,r,t)dr - (\lambda + c)\cosh(\sqrt{\lambda})u(x,t). \tag{32}$$

3.2. Inverse transformation

The transformation (31) is proved to be invertible, that is, the inverse transformation exists and maps (u,β) back to (u,v). The inverse transformation is given by

$$v(x,y,t) = \beta(x,y,t) - (\lambda + c)\cos(\sqrt{c}y)u(x,t) - \frac{(\lambda + c)}{\sqrt{c}}\int_0^y \sin(\sqrt{c}(y-r))\beta(x,r,t)dr, \tag{33}$$

from which we can get $\beta(x,0,t) = v(x,0,t) + (\lambda + c)u(x,t)$. In addition, it is easy to prove that the transformation (33) can transform the target system (10)–(17) into the original system (1)–(8).

4. Stability Analysis

In this section, we present the main result of the paper which states that system (1)–(8) under the boundary control (32) is exponentially stable in H^2 norm.

Theorem 1. Consider the original system (1)–(8) with the controller (32). If the initial conditions $u_0(x) \in H_1^2[0,1]$, $v_0(x,y) \in H_2^2([0,1]\times[0,1])$ are compatible such that $u_{0x}(0) = u_0(1) = 0$, $v_{0x}(0,y) = v_0(1,y) = v_{0y}(x,0) = 0$, $v_0(x,1) = U(x,0)$, then the system admits a unique solution and it is exponentially stable at the zero-equilibrium, i.e., there exist positive constants α_1 and β_1, such that

$$V_1(t) \le \alpha_1 e^{-\beta_1 t}V_1(0), \tag{34}$$

where

$$V_1(t) = \|u\|_{H_1^2}^2 + \|v\|_{H_2^2}^2. \tag{35}$$

The proof of stability consists of two prior propositions. First, we prove the exponential stability of the target system (10)–(17) in H^2 sense which is necessary for having continuous state variables in 2D space. Second, we establish the norm equivalence between the original system (1)–(8) and the target system (10)–(17) through the invertible transformation (31).

Proposition 1. Consider the system (10)–(17) with the initial conditions $u_0(x) \in H_1^2[0,1]$, $\beta_0(x,y) \in H_2^2([0,1]\times[0,1])$ being compatible such that $u_{0x}(0) = u_0(1) = 0$, $\beta_{0x}(0,y) = \beta_0(1,y) = \beta_{0y}(x,0) = \beta_0(x,1) = 0$, then the system has a unique solution and it is exponentially stable, i.e., there exist positive constants α_2 and β_2, such that

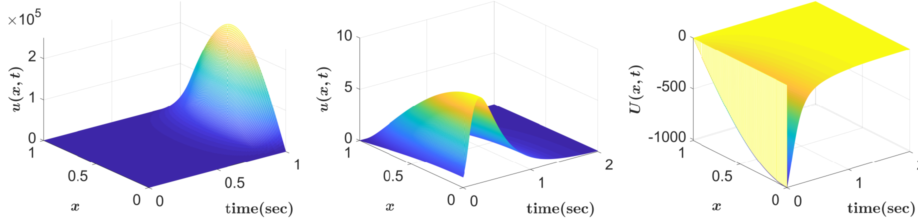
$$V_2 \le \alpha_2 e^{-\beta_2 t}V_2(0), \tag{36}$$

where

$$V_2(t) = \|u\|_{H_1^2}^2 + \|\beta\|_{H_2^2}^2. \tag{37}$$

Proof. The well-posedness result is standard because (13)–(17) is a heat equation which can be solved explicitly. Once function $\beta(x,y,t)$ is known, (10)–(12) can also be determined. To analyze the stability, we define a new Lyapunov function as follows:

$$V_3(t) = \frac{1}{2}\int_0^1 u^2(x,t)dx + \frac{1}{2}\int_0^1 u_x^2(x,t)dx$$



(a) (b) (c)

Figure 2: The dynamics of the state $u(x,t)$: (a) open-loop system, (b) closed-loop system. (c) The controller effort $U(x,t)$.

$$+\frac{1}{2}\int_0^1 u_{xx}^2(x,t)dx+\frac{a}{2}\int_0^1\int_0^1 \cos\left(\frac{\pi}{4}(y+\frac{1}{2})\right)\beta^2 dydx$$
$$+\frac{1}{2}\int_0^1\int_0^1(\beta_x^2+\beta_y^2)dydx+\int_0^1\int_0^1(\beta_{yy}^2+\frac{1}{2}\beta_{xy}^2)dydx$$
$$+\frac{b}{2}\int_0^1\int_0^1 \cos\left(\frac{\pi}{4}(y+\frac{1}{2})\right)\beta_{xx}^2 dydx, \quad (38)$$

where $a>0$, $b>0$. It is easy to know that V_3 is equivalent to V_2, i.e., there exist positive constants α_3 and β_3, such that

$$\alpha_3 V_2 \le V_3 \le \beta_3 V_2. \quad (39)$$

Taking derivative first two terms of (38) with respect to t, we get,

$$\frac{1}{2}\frac{d}{dt}\left(\int_0^1 u^2 dx+\int_0^1 u_x^2 dx\right)=-\int_0^1(u_x^2+cu^2)dx$$
$$-\int_0^1(u_{xx}^2+cu_x^2)dx+\int_0^1(u-u_{xx})\beta(x,0,t)dx. \quad (40)$$

Recalling (10), (11) and (14), we infer $u_{xx}(1,t)=u_{xxx}(0,t)=0$. Therefore, it gets

$$\frac{1}{2}\frac{d}{dt}\int_0^1 u_{xx}^2 dx=-\int_0^1 u_{xxx}^2 dx-c\int_0^1 u_{xx}^2 dx$$
$$+\int_0^1 u_{xx}\beta_{xx}(x,0,t)dx. \quad (41)$$

Next, take the time derivative of the fourth term of (38), which yields

$$\frac{d}{dt}\frac{a}{2}\int_0^1\int_0^1 \cos\left(\frac{\pi}{4}(y+\frac{1}{2})\right)\beta^2 dydx$$
$$=-a\int_0^1\int_0^1 \cos\left(\frac{\pi}{4}(y+\frac{1}{2})\right)(\beta_x^2+\beta_y^2)dydx$$
$$-\frac{\pi^2 a}{32}\int_0^1\int_0^1 \cos\left(\frac{\pi}{4}(y+\frac{1}{2})\right)\beta^2 dxdy$$
$$-\frac{\pi a}{8}\sin\frac{\pi}{8}\int_0^1 \beta^2(x,0,t)dx. \quad (42)$$

Similarly,

$$\frac{1}{2}\frac{d}{dt}\int_0^1\int_0^1(\beta_x^2+\beta_y^2)dydx=-\int_0^1\int_0^1(\beta_{xx}+\beta_{yy})^2 dxdy. \quad (43)$$

Applying (13)–(16), it infers that $\beta_{xx}(1,y,t)=\beta_{xxx}(0,y,t)=\beta_{yy}(x,1,t)=\beta_{yyy}(x,0,t)=0$, so

$$\frac{d}{dt}\frac{b}{2}\int_0^1\int_0^1 \cos\left(\frac{\pi}{4}(y+\frac{1}{2})\right)\beta_{xx}^2 dydx$$
$$=-b\int_0^1\int_0^1 \cos\left(\frac{\pi}{4}(y+\frac{1}{2})\right)\beta_{xxx}^2 dydx$$
$$-b\int_0^1\int_0^1 \cos\left(\frac{\pi}{4}(y+\frac{1}{2})\right)\beta_{xxy}^2 dydx$$
$$-\frac{\pi^2 b}{32}\int_0^1\int_0^1 \cos\left(\frac{\pi}{4}(y+\frac{1}{2})\right)\beta_{xx}^2 dydx$$
$$-\frac{\pi b}{8}\sin\frac{\pi}{8}\int_0^1 \beta_{xx}^2(x,0,t)dx, \quad (44)$$

and

$$\frac{d}{dt}\int_0^1\int_0^1(\beta_{yy}^2+\frac{1}{2}\beta_{xy}^2)dydx \quad (45)$$
$$=-\int_0^1\int_0^1 \beta_{yyy}^2 dydx$$
$$-\int_0^1\int_0^1 \beta_{yyy}\beta_{xxy}dydx-\int_0^1\int_0^1(\beta_{yyy}+\beta_{xxy})^2 dydx.$$

Then, combining (40)–(45), and utilizing the Cauchy-Schwarz inequality and Poincare's inequality, we finally obtain

$$\dot{V}_3(t)\le-\left(c-\frac{\gamma_1}{2}\right)\int_0^1 u^2 dx-(1+c)\int_0^1 u_x^2 dx$$
$$-\left(\frac{5}{4}+c-\frac{\gamma_2}{2}-\frac{\gamma_3}{2}\right)\int_0^1 u_{xx}^2 dx$$
$$-\frac{\pi^2 a}{32}\int_0^1\int_0^1 \cos\left(\frac{\pi}{4}(y+\frac{1}{2})\right)\beta^2 dydx$$
$$-a\int_0^1\int_0^1 \cos\left(\frac{\pi}{4}(y+\frac{1}{2})\right)(\beta_x^2+\beta_y^2)dydx$$
$$-\left(\frac{b}{4}+\frac{\pi^2 b}{32}\right)\int_0^1\int_0^1 \cos\left(\frac{\pi}{4}(y+\frac{1}{2})\right)\beta_{xx}^2 dydx$$
$$-\left(b-\frac{1}{2\gamma_4}\right)\int_0^1\int_0^1 \cos\left(\frac{\pi}{4}(y+\frac{1}{2})\right)\beta_{xxy}^2 dydx$$
$$-\left(1-\frac{\gamma_4}{2}\right)\int_0^1\int_0^1 \cos\left(\frac{\pi}{4}(y+\frac{1}{2})\right)\beta_{yyy}^2 dydx$$

$$
-\left(\frac{\pi a}{8}\sin\frac{\pi}{8}-\frac{1}{2\gamma_1}-\frac{1}{2\gamma_2}\right)\int_0^1 \beta^2(x,0,t)dx
$$
$$
-\left(\frac{\pi b}{8}\sin\frac{\pi}{8}-\frac{1}{2\gamma_3}\right)\int_0^1 \beta_{xx}^2(x,0,t)dx
$$
$$
-\int_0^1\int_0^1(\beta_{xx}+\beta_{yy})^2dydx-\int_0^1\int_0^1(\beta_{xxy}+\beta_{yyy})^2dydx.
$$

Let $0<\gamma_1<2c$, $\frac{4c}{ac\pi\sin\frac{\pi}{8}-2}<\gamma_2<\frac{5}{2}+2c-\frac{4}{\pi b\sin\frac{\pi}{8}}$, $\frac{4}{\pi b\sin\frac{\pi}{8}}<\gamma_3$, $\frac{1}{2b}<\gamma_4<2$, which requires $\frac{8}{(5+4c)\pi\sin\frac{\pi}{8}}<b$, $\frac{2}{c\pi\sin\frac{\pi}{8}}<a$, so we can choose

$$
\alpha_4=\min\left\{2(c-\frac{\gamma_1}{2}),2(1+c),\frac{5}{2}+2c-\gamma_2-\gamma_3,\frac{1}{2}+\frac{\pi^2}{16},\right.
$$
$$
\left.2a\cos\frac{3\pi}{8},\frac{1}{2}(b-\frac{1}{2\gamma_4})\cos\frac{3\pi}{8},\frac{1}{4}(1-\frac{\gamma_4}{2})\cos\frac{3\pi}{8}\right\},
$$

such that $\dot{V}_3\leq-\alpha_4V_3(t)$. Recalling V_3 and V_2 are equivalent, there exists a positive constant α_2, such that (36) holds, this proposition gets proven.

The following proposition states that the original system (1)–(8) is equivalent to the target system (10)–(17) in the sense of norm.

Proposition 2. There exist positive constants α_5 and β_5, such that

$$
\begin{aligned}
\alpha_5(\|u\|_{H_1^2}^2+\|v\|_{H_2^2}^2)\leq&\|u\|_{H_1^2}^2+\|\beta\|_{H_2^2}^2\\
\leq&\beta_5(\|u\|_{H_1^2}^2+\|v\|_{H_2^2}^2).
\end{aligned}\tag{46}
$$

Proof. According to the transformation (31), using the Cauchy-Schwarz inequality, we can get

$$
\begin{aligned}
\|\beta\|_{L_2^2}^2\leq&3\|v\|_{L_2^2}^2+3\int_0^1\int_0^1\left((\lambda+c)\cosh(\sqrt{\lambda}y)u(x,t)\right)^2\\
&dxdy+3\int_0^1\int_0^1\left(\frac{\lambda+c}{\sqrt{\lambda}}\int_0^y\sinh(\sqrt{\lambda}(y-r))\right.\\
&v(x,r,t)dr)^2dxdy\\
\leq&3A_1\|u\|_{L_1^2}^2+3A_2\|v\|_{L_2^2}^2,
\end{aligned}\tag{47}
$$

where $A_1=\frac{(\lambda+c)^2(e^{2\sqrt{\lambda}}+2\sqrt{\lambda})}{4\sqrt{\lambda}}$, $A_2=\left(1+\frac{(\lambda+c)^2e^{2\sqrt{\lambda}}}{\lambda}\right)$.

Applying a similar approach, we get

$$
\|\beta_x\|_{L_2^2}^2\leq3A_1\|u_x\|_{L_1^2}^2+3A_2\|v_x\|_{L_2^2}^2,\tag{48}
$$
$$
\|\beta_{xx}\|_{L_2^2}^2\leq3A_1\|u_{xx}\|_{L_1^2}^2+3A_2\|v_{xx}\|_{L_2^2}^2.\tag{49}
$$

With respect to y, using the Cauchy-Schwarz again, one gets

$$
\begin{aligned}
\|\beta_y\|_{L_2^2}^2\leq&3\|v_y\|_{L_2^2}^2+3(\lambda+c)^2\lambda e^{2\sqrt{\lambda}}\|u\|_{L_1^2}^2\\
&+3(\lambda+c)^2e^{2\sqrt{\lambda}}\|v\|_{L_2^2}^2\\
\leq&3\|v_y\|_{L_2^2}^2+3A_3\|u\|_{L_1^2}^2+3A_4\|v\|_{L_2^2}^2,
\end{aligned}\tag{50}
$$

where $A_3=(\lambda+c)^2\lambda e^{2\sqrt{\lambda}}$, $A_4=(\lambda+c)^2e^{2\sqrt{\lambda}}$, and

$$
\|\beta_{xy}\|_{L_2^2}^2\leq3\|v_{xy}\|_{L_2^2}^2+3A_3\|u_x\|_{L_1^2}^2+3A_4\|v_x\|_{L_2^2}^2,\tag{51}
$$
$$
\begin{aligned}
\|\beta_{yy}\|_{L_2^2}^2\leq&4\|v_{yy}\|_{L_2^2}^2+4\lambda^2A_1\|u\|_{L_1^2}^2\\
&+4((\lambda+c)^2+A_3)\|v\|_{L_2^2}^2.
\end{aligned}\tag{52}
$$

Utilizing the inequalities (47)–(52) and the definitions of the norms in H_1^2 and H_2^2 space, we get that

$$
\begin{aligned}
\|u\|_{H_1^2}^2+\|\beta\|_{H_2^2}^2\leq&(1+3A_1+3A_3+4\lambda^2A_1)\|u\|_{L_1^2}^2+(1+\\
&3A_1+6A_3)\|u_x\|_{L_1^2}^2+(1+3A_1)\|u_{xx}\|_{L_1^2}^2\\
&+(3A_2+3A_4+4((\lambda+c)^2+A_3))\cdot\\
&\|v\|_{L_2^2}^2+(3A_2+6A_4)\|v_x\|_{L_2^2}^2+3\|v_y\|_{L_2^2}^2\\
&+3A_2\|v_{xx}\|_{L_2^2}^2+6\|v_{xy}\|_{L_2^2}^2+4\|v_{yy}\|_{L_2^2}^2\\
\leq&\beta_5(\|u\|_{H_1^2}^2+\|v\|_{H_2^2}^2),
\end{aligned}\tag{53}
$$

where $\beta_5=\max\{1+3A_1+3A_3+4\lambda^2A_1,1+3A_1+6A_3,3A_2+3A_4+4((\lambda+c)^2+A_3),3A_2+6A_4,6\}$. The right part of the inequality (46) gets proven. Using a similar approach and combining the inverse transformation (33), we can prove the left part of the inequality (46).

Combining Proposition 1 and Proposition 2, one can prove Theorem 1.

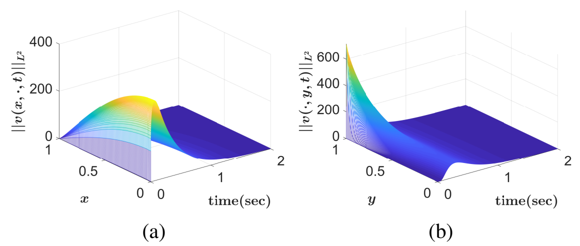


Figure 3: The dynamics of state (a) $||v(x,\cdot,t)||_{L^2}$, (b) $||v(\cdot,y,t)||_{L^2}$.

5. Simulation

In this section, we provide an example to demonstrate the theoretical results. The parameters are set as $\lambda=20$ and $c=1$. The initial conditions are selected as $u_0(x)=\sin(\frac{\pi}{2}(x+1))$ and $v_0(x,y)=\cos(\frac{\pi}{2}x)(\cos(\frac{\pi}{2}y)+1)$, respectively. We use the finite difference method to numerically simulate the system (1)–(8) by discretizing $x,y\in[0,1]$ into $M_x\times M_y$ subregions and choose $M_x=M_y=101$. The three-point central difference approximation and the Crank-Nicolson method are employed in the simulation. For the boundary control, we use the Simpson's integration rule which needs M_x and M_y being odd numbers. The step size for discretization of time is set as $\Delta t=0.001s$.

The dynamics for open-loop and closed-loop systems are shown in Fig.2 (a) and (b), respectively, which

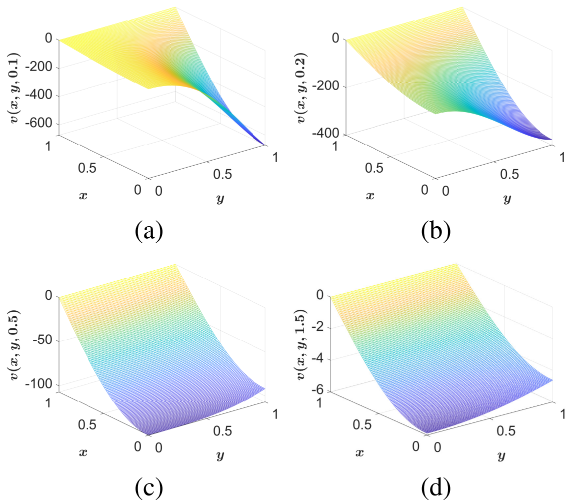


Figure 4: The snapshots of state $v(x,y,t)$ at (a) $t=0.1s$, (b) $t=0.2s$, (c) $t=0.5s$ and (d) $t=1.5s$.

illustrate that the open-loop system is unstable and the state of the closed-loop 1D PDE system converges to zero after about $2s$. The boundary control effort is shown in Fig.2 (c), which indicates that more control effort is required through diffusion actuator dynamics than through direct boundary control acting on the plant (see p. 40 in [17]). The dynamics of the 2D heat PDE are also shown in Fig. 3. It can be observed from Fig. 3 (b) that the control effort from the $y=1$ diffusing to the distal boundary $y=0$ is attenuated greatly. Fig. 4 depicts the snapshot of the 2D heat PDE's state at different times.

6. Conclusion

This paper proposes a control design approach for the unstable reaction-diffusion system whose actuator states are governed by a 2D heat equation. To get the controller, we construct a new backstepping transformation with explicit kernels. Since this heat equation is defined in 2D, we design a new Lyapunov function that includes cosine coefficients to prove the stability of this PDE-PDE coupled system. Future works will consider the observer design to obtain the state feedback for this coupled system.

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