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New formal descriptions for timed coloured Petri nets using formal series

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Abstract— **This paper proposes to introduce a dioid of coloured formal series for the description of Timed Coloured Petri Nets (TCPN). These formal series allow us to express TCPN time/event shiftings. As coloured Petri nets can reach a high level of complexity, we first present the coloured formal series for linear systems. Nonetheless, we extend our work to the time and event shiftings in a conflict situation. This example shows the powerful expressiveness of these series and that we can build configurable models for which the transfer function computation can be automated via predictability assumptions.**

I. Introduction

The autors of [1] have presented an important work on Timed Event Graphs (TEGs) and their description using tropical algebra. The use of such framework allows to assess the temporal performances of a large-scaled TEG, without facing combinatory explosions. Besides, these authors explicit a formal series dioid $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. This dioid allows handling simultaneously daters and counters representations for TEGs through dual formal series. Two-dimensional models have many advantages over one-dimensional representations only with (max, +) or only (min, +). They are compactly and powerfully expressing time and event shiftings between transitions. In that sense, [1] and [2] state that this dioid gives greater algebraic representations than (max, +) algebra with compact two-dimensional equations of events and times in TEGs. In the field of control theory, these representations are already used to resolve various problems: [3], [4], [5] and [6]. However, the $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ only works on linear models, ensured by the structure of a TEG, and can not handle common situations as conflicts or choices.

On the other hand, [7] presents Coloured Petri Nets (CPN), a specific class of Petri nets in which tokens are coloured. Colours are *data vectors* defining a particular behaviour when fired by a transition. It allows, for example, to model a shared resource and, more extensively, choices. Likewise, CPNs are a hierarchical and highly compact formalism relevant to model parallelism. Colours having almost no limits to representations and manipulations, CPNs have therefore a high power of expression. In [9] and [10], authors use a natural definition for coloured (max, +) daters to compute temporal performances of a CPN model. However, as we have seen in [8], this definition limits expressiveness, mainly because it is only based on one dimension (no bi-dimensional formal series were used). The (max, +) equations do not naturally integrate conflicts - or choices - resolutions. CPN models are often nonlinear in the (max,+) algebra, and currently, no work exists on dual representations using formal series in the literature.

This paper proposes to introduce the colour notion inside tropical algebra and formal series. Our motivation lies in the possibility to temporally assess large-scaled and compact linear timed CPN, when a conflict or a choice can occur. This will allow to benefit from the compactness of coloured models and their ability to model choices. Section II recalls the basics of tropical linear theories presented in [1]. Section III presents TCPNs and what restrictions we use on this formalism in this paper. Section IV introduces the dioid $\overline{\mathbb{E}}[\![\Gamma,\delta]\!]$. In section V, we present linear applications. Finally, we express in section VI the time/event shiftings in a conflict example (allowing delay computations between inputs and outputs). It shows that coloured formal series can be manipulated to integrate the resolution of a conflict situation.

II. Formal Series Dioid for Daters and Counter

A set $\mathcal{D}$ is a *dioid* (or *idempotent semiring*) for an additive law $\oplus$ and a multiplicative law $\otimes$, if $\oplus$ is associative, commutative, and idempotent, $\otimes$ is associative and distributes over $\oplus$, and if it exists ε and $e \in \mathcal{D}$ respectively the neutral elements of $\oplus$ and $\otimes$ - with ε absorbing for $\otimes$. Moreover, the additive law defines an *order relation* $\preccurlyeq$ between elements such that $a \oplus b = b \iff a \preccurlyeq b \iff \exists c \in \mathcal{D}, a \oplus c = b$. This relation is compatible with $\oplus$ and $\otimes$. If $\forall a,b \in \mathcal{D}, a \preccurlyeq b$ or $b \preccurlyeq a$, then the dioid is said to be *totally ordered*. *Example:* $(\mathbb{N} \cup \{-\infty\}, \max, +)$, $\mathbb{Z}_{\max} = (\mathbb{Z} \cup \{-\infty\}, \max, +)$ and $\mathbb{Z}_{\min} = (\mathbb{Z} \cup \{+\infty\}, \min, +)$ are examples of dioids. A set endowed with $\oplus$ and $\otimes$ but without a neutral element for multiplication e is called a *hemiring* ([11]). The Dorroh extension of an idempotent hemiring $(\mathcal{H}, \oplus, \otimes)$ by the Boolean dioid $(\mathbb{B} = \{0,1\}, \oplus, \otimes)$ is defined by $(\mathcal{S}, +, \star)$ with $S : \mathcal{H} \times \mathbb{B}$, and where $\forall (h,b),(h',b') \in \mathcal{S}, (h,b) + (h',b') = (h \oplus h', b \oplus b')$ and $(h,b) \star (h',b') = (h \otimes b' \oplus h' \otimes b \oplus h \otimes h', b \otimes b')$. $(S, +, \star)$ is a dioid having $(\varepsilon_{\mathcal{H}}, 0)$ for additive neutral element and $(\varepsilon_{\mathcal{H}}, 1)$ for multiplicative neutral element.

$(\mathcal{D}, \oplus, \otimes)$ is said to be complete if it is closed on infinite sum and if $\otimes$ distributes over such sums. In a complete dioid, the Kleene star on an element a is an operator defined by $a^* = \bigoplus_{n \geq 0} a^n$, with $a^0 = e$ and $a^+ = \bigoplus_{n \geq 1} a^n$. Authors of [1] have shown a fundamental theorem: in a complete

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dioid, the equation $x = ax \oplus b$ admits $x = a^*b$ as least solution. The dioid $\overline{\mathbb{Z}}_{\max} = (\mathbb{Z} \cup \{-\infty, +\infty\}, \max, +)$ is complete (and its dual $\overline{\mathbb{Z}}_{\min}$ too).

TEGs are a specific class of Petri nets in which places can only have one upstream and one downstream arc. Figure 1 shows an example. In a TEG, we can associate to each transition a dater d (resp. a counter c) function, $d(k)$ yielding the date of the k^{th} firing (resp. $c(t)$ the number of firings at time t). Let $\{d(k)\}_{k\in\mathbb{Z}}$ be a sequence over $\overline{\mathbb{Z}}_{\max}$. Its γ-transform is a formal power series in γ with coefficients in $\overline{\mathbb{Z}}_{\max}$ and exponents in $\mathbb{Z}$: $d(\gamma) = \bigoplus_{k\in\mathbb{Z}} d(k)\gamma^k$. For a sequence $\{c(t)\}_{t\in\mathbb{Z}}$ over $\overline{\mathbb{Z}}_{\min}$, it also exists its δ-transform with coefficients in $\overline{\mathbb{Z}}_{\min}$ and exponents in $\mathbb{Z}$: $c(\delta) = \bigoplus_{t\in\mathbb{Z}} c(t)\delta^t$. γ and δ are called "shifting operators". For example $d(\gamma) \otimes \gamma^1 = \bigoplus_{k\in\mathbb{Z}} d(k)\gamma^{k+1} = \bigoplus_{k\in\mathbb{Z}} d(k-1)\gamma^k$. We denote by $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ the complete dioid, endowed with $\oplus$ and $\otimes$, of formal power series on γ with coefficients in $\overline{\mathbb{Z}}_{\max}$ and exponents in $\mathbb{Z}$; and by $\overline{\mathbb{Z}}_{\min}[\![\delta]\!]$ on δ with coefficients in $\overline{\mathbb{Z}}_{\min}$ and exponents in $\mathbb{Z}$. $\gamma^*\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ and $(\delta^{-1})^*\overline{\mathbb{Z}}_{\min}[\![\delta]\!]$ are dioids respectively for daters and counters.

$\mathbb{B}[\![\gamma,\delta]\!]$ is a complete dioid formed by the set of formal series with two commutative variables γ and δ with Boolean coefficients in $\{\varepsilon, e\}$ and exponents in $\mathbb{Z}$. The neutral elements are $\varepsilon(\gamma,\delta) = \bigoplus_{k,t\in\mathbb{Z}} \varepsilon\gamma^k\delta^t$ and $e(\gamma,\delta) = \gamma^0\delta^0$. Considering this dioid modulo the relation $\gamma^*(\delta^{-1})^*$ gives a quotient dioid named $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$, in which $\forall x, y \in \mathbb{B}[\![\gamma,\delta]\!], x = y \Leftrightarrow x\gamma^*(\delta^{-1})^* = y\gamma^*(\delta^{-1})^*$. This relation creates equivalence classes $[s]_{\gamma^*(\delta^{-1})^*}$ regrouping elements of $\mathbb{B}[\![\gamma,\delta]\!]$ modulo $\gamma^*(\delta^{-1})^*$. For sake of clarity, we denote after equivalence classes $[s]_{\gamma^*(\delta^{-1})^*}$ of $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ only by s. Finally, the following properties stand for any $k, k', t, t' \in \mathbb{Z}$: $\gamma^k\delta^t \oplus \gamma^k\delta^{t'} = \gamma^k\delta^{\max(t,t')}$, $\gamma^k\delta^t \oplus \gamma^{k'}\delta^t = \gamma^{\min(k,k')}\delta^t$ and $\gamma^k\delta^t \otimes \gamma^{k'}\delta^{t'} = \gamma^{k+k'}\delta^{t+t'}$. The neutral elements are $\varepsilon = \gamma^{+\infty}\delta^{-\infty}$ and $e = \gamma^0\delta^0$. The complete dioid $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ gives compact and powerful series (in terms of simplifications and, by extension, computationally - see [2]) to simultaneously describe the dater and counter functions of a TEG. In these series, a monomial $\gamma^k\delta^t$ means : "the $(k+1)^{th}$ event occurrence happens at earliest at time t" (firings begin at index 0). A *trajectory* is a series expressing n successive firings of a transition and having the form $\gamma^0\delta^{t_0} \oplus \gamma^1\delta^{t_1} \oplus ... \oplus \gamma^{n-1}\delta^{t_{n-1}} \oplus \gamma^n\delta^{+\infty}$ - the $(n+1)^{th}$ event never occurs.

TEGs structure ensures a linear model when expressing the corresponding daters/counters with formal series in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. Their behaviour can be expressed using $x = Ax \oplus Bu$, and $y = Cx$, with $A \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{m\times m}$, $B \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{m\times p}$ and $C \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{q\times m}$, m, p and q being respectively the size of the vectors x (system), u (input) and y (output). The least solution theorem gives: $y = CA^*Bu$ and we call *transfer function* $H = CA^*B$ the matrix expressing the relationship between inputs and outputs. The following equations represent the temporal and event behaviours for the TEG in Figure 1:

$$\begin{cases} x_1(\gamma,\delta) &= \gamma^1\delta^{t_2}x_2(\gamma,\delta) \oplus \gamma^1\delta^0 u(\gamma,\delta) \\ x_2(\gamma,\delta) &= \gamma^0\delta^{t_1}x_1(\gamma,\delta) \\ y(\gamma,\delta) &= \gamma^0\delta^0 x_2(\gamma,\delta) \end{cases} \tag{1}$$

Its transfer function is then: $H = \gamma^0\delta^{t_1}(\gamma^1\delta^{t_1+t_2})^*$. One can see that transfer functions allow to represent the temporal behaviour of events in a TEG through a single linear function, hence their "computational power". Using the same approach, the algebraic representation of any TEG behaviour through formal series is possible, even with multiple inputs and/or outputs. We refer the reader to [1] for a complete introduction to the link between TEGs and formal series.

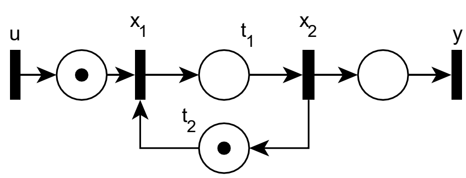

Fig. 1. Example of a simple TEG

III. Timed Coloured Petri Nets

In [7], the authors give a formal definition of coloured Petri nets. Notably, the *bag* concept describes a marking across a CPN, and how markings change over firing transitions. They are also helpful to implicitly use *multiplicities* (or *weights*) in a Petri nets.

Definition 3.1: A multi-set of a set E is a function $\Omega : E \mapsto \mathbb{N}$, with $\Omega(e)$ the number of times e is contained in the set built by the function (the image of E by Ω). $Bag(E)$ is the set of the multi-sets of E.

Example: Let $E = \{a, b, c\}$ be a set of 3 elements, and Ω_E a multi-set of E. If Ω_E is defined by $a \to 3, b \to 1, c \to 2$, the set built is: $\{a, a, a, b, c, c\}$. Ω_E is formally represented by: $\sum_{e\in E} \Omega_E(e)`e = 3`a + 1`b + 2`c$.

Definition 3.2: Let $(\mathcal{P}, \mathcal{T}, C, \mathcal{W}^-, \mathcal{W}^+, \Theta, \Phi)$ be a TCPN: $\mathcal{P}$ is a nonempty set of places, $\mathcal{T}$ a nonempty set of transitions and C is a function from $\mathcal{P} \cup \mathcal{T}$ to a finite set of colours. $\mathcal{W}^-$ and $\mathcal{W}^+$ are respectively *firing* and *filing* functions which associate each $(P, T) \in \mathcal{P} \times \mathcal{T}$ to a colour function from $C(T)$ to $Bag(C(P))$. Θ is a time function from $\mathcal{P} \cup \mathcal{T}$ to $\mathbb{N}$, which associates a time delay to a place or a transition. Finally, Φ associates each transition to a Boolean guard.

C associates each transition and each place with its own colours set. As we will not address WTEGs in this first work, we suppose that $\mathcal{W}^-$ always yields a multi-set W having for associated function $\Omega_W : W \mapsto \{0, 1\}$ (the consumption when a transition fires) and $\mathcal{W}^+$ a multi-set composed of a unique token (the production after a firing). After having fired, a transition can still drop different tokens in several places, as long as there is only one token added in each place. A coloured marking M is a function which associates each $P \in \mathcal{P}$ to an element of $Bag(C(P))$. M describes a distribution of coloured token across a CPN. A transition T fires for $c_T \in C(T)$ and a coloured marking M as soon as $\forall P \in \mathcal{P}$, $\mathcal{W}^-(P, T)(c_T) \leq M(P)$. Hence,

$\mathcal{W}^-$ can also be seen as the condition to fire a transition in terms of resource distribution. When a transition fires, the marking is updated according to its $\mathcal{W}^-$ and $\mathcal{W}^+$. The function Θ corresponds to delays on transitions and places, and the guard $\Phi(T)$ must be true to allow T to fire. In [7], the time aspect in CPNs is based on timestamps given to each token. Here we give a new definition based on a time consistent with the classical (max,+) description of TEGs. The TCPN in Figure 2 presents a simple jobshop with two types of pieces $\langle a\rangle$ and $\langle b\rangle$. Delay t_1 is a first processing. There are two machines: $\langle m_1\rangle$ and $\langle m_2\rangle$. The transition T_1 can fire colours $\langle m_1, a\rangle$ or $\langle m_2, b\rangle$, standing for a job beginning with machine m_i, of duration t_2. Transition T_2 stands for the end of the job. Place P_3 corresponds to the minimum delay t_3 between one machine two successive jobs. Delay t_4 is the assembling time of two processed pieces. As help to understand TCPNs dynamics, here are some of the $\mathcal{W}$ functions: $\mathcal{W}^-(P_1, T_1)(\langle m_1, a\rangle) = 1`\langle a\rangle$, $\mathcal{W}^+(P_3, T_1)(\langle m_1, a\rangle) = 1`\langle m_1\rangle$ and $\mathcal{W}^-(P_4, y)(\langle c\rangle) = 1`\langle m_1, a\rangle + 1`\langle m_2, b\rangle$.

IV. Tropical Algebra for Colours

A. Dioid of Colours

Let $\mathbb{E}$ be a set of *colours*. As colours are data vectors and have no predefined structure, we consider here and for the rest of the paper $\mathbb{E} = \{\langle\emptyset\rangle, \langle 1\rangle, \langle 2\rangle, ...\}$; with $\langle\emptyset\rangle \in \mathbb{E}$ the colour standing for the *absence of colour*. Hereafter, $\langle.\rangle \in \mathbb{E}$ stands for a generic notation of one colour. One main difficulty to be solved for colours in tropical algebra is the nonexistence of a natural definition for addition and product. We assume the following order for $\mathbb{E}$ elements:

Definition 4.1 (Order in $\mathbb{E}$): Let $\preccurlyeq$ be an order relation such that: $\forall\langle.\rangle \in \mathbb{E}, \langle.\rangle \preccurlyeq \langle\emptyset\rangle$

The relation $\preccurlyeq$ *partially* orders $\mathbb{E}$. A possible law $\oplus$ is then associated to this order relation and defined in $\mathbb{E}$ by:

$$\forall\langle i\rangle, \langle j\rangle \in \mathbb{E}, \langle i\rangle \oplus \langle j\rangle = \begin{cases} \langle i\rangle \text{ if } \langle i\rangle = \langle j\rangle \\ \langle\emptyset\rangle \text{ otherwise} \end{cases} \tag{2}$$

This rule reads: the maximum (according to the order $\preccurlyeq$) between two colours naturally leads to a colour absence, and $\oplus$ is idempotent. The neutral element is an added colour $\langle o\rangle$ defined by: $\forall\langle.\rangle \in \mathbb{E}, \langle o\rangle \oplus \langle.\rangle = \langle.\rangle \oplus \langle o\rangle = \langle.\rangle$.

With respect to the multiplication $\otimes$, as we aim to express in our equations how colours evolve in a model, the multiplication $\otimes$ is given by:

$$\forall\langle i\rangle, \langle j\rangle \in \mathbb{E}, \langle i\rangle \otimes \langle j\rangle = \begin{cases} \langle o\rangle \text{ if } \langle i\rangle = \langle o\rangle \text{ or } \langle j\rangle = \langle o\rangle \\ \langle i\rangle \text{ otherwise} \end{cases} \tag{3}$$

As it stands, the algebraic structure $(\mathbb{E} \cup \{\langle o\rangle\}, \oplus, \otimes)$ is not a dioid (because it does not have a neutral element for $\otimes$) but only an additively-idempotent hemiring.

Remark 1: As colours are abstract objects, we could define a neutral element e for $\otimes$. However, $(\mathbb{E}\cup\{\langle o\rangle, e\}, \oplus, \otimes)$ would not be a dioid since $\forall\langle.\rangle \in \mathbb{E}, (e \oplus \langle.\rangle) \otimes \langle.\rangle \neq e \otimes \langle.\rangle \oplus \langle.\rangle \otimes \langle.\rangle$.

In [11], the author proposes the *Dorroh extension* to build a dioid structure from an idempotent hemiring. We denote by $\breve{\mathbb{E}} : \mathbb{E}\cup\{\langle o\rangle\}\times\mathbb{B}$ the set of pairs $(\langle.\rangle, b)$, where $b \in \{0, 1\}$. As $(\mathbb{B}, \oplus, \otimes)$ is the dioid of Booleans, we recall that $1 \oplus 1 = 1$ in it (by idempotency of $\oplus$). In addition, we specify that $\langle.\rangle \otimes 0 = \langle o\rangle$ and $\langle.\rangle \otimes 1 = \langle.\rangle$.

Proposition 4.1: The set $\breve{\mathbb{E}} : \mathbb{E}\cup\{\langle o\rangle\}\times\mathbb{B}$, endowed with $\oplus$ and $\otimes$, is a complete dioid, having for neutral elements $\varepsilon = (\langle o\rangle, 0)$ and $e = (\langle o\rangle, 1)$. In this dioid, the addition and the multiplication of two elements belonging to $\mathbb{E}\cup\{\langle o\rangle\}$ are respectively given by (2) and (3). Lastly, $(\langle i\rangle, b) \oplus (\langle j\rangle, b') = (\langle i\rangle \oplus \langle j\rangle, b \oplus b')$ and $(\langle i\rangle, b) \otimes (\langle j\rangle, b') = (\langle i\rangle b' \oplus \langle j\rangle b \oplus \langle i\rangle \otimes \langle j\rangle, b \otimes b')$.

Proof: $\breve{\mathbb{E}}$ verifies the dioid properties. Particularly, $\otimes$ distributes over $\oplus$: $(\langle i\rangle, b)\otimes((\langle j\rangle, b')\oplus(\langle k\rangle, b'')) = (\langle i\rangle, b)\otimes(\langle j\rangle, b') \oplus (\langle i\rangle, b) \otimes (\langle k\rangle, b'')$ and $((\langle j\rangle, b') \oplus (\langle k\rangle, b'')) \otimes (\langle i\rangle, b) = (\langle j\rangle, b') \otimes (\langle i\rangle, b) \oplus (\langle k\rangle, b'') \otimes (\langle i\rangle, b)$.

Lemma 4.2: Let $\overline{\mathbb{E}} \subset \breve{\mathbb{E}}$ such that $\overline{\mathbb{E}} = \{(\langle.\rangle, 0) \mid \langle.\rangle \in \mathbb{E}\cup\{\langle o\rangle\}\}$. Then, $\forall\overline{a}_1, \overline{a}_2 \in \overline{\mathbb{E}}, \overline{a}_1\otimes\overline{a}_2 = \overline{a}_1$, and $\overline{a}_1\oplus\overline{a}_2 \in \overline{\mathbb{E}}$. In addition, $\forall\overline{a} \in \overline{\mathbb{E}}$, $\overline{a} \oplus e \notin \overline{\mathbb{E}}$, but $\overline{a} \otimes e \in \overline{\mathbb{E}}$.

The subset of colours $\overline{\mathbb{E}}$ allows us to use the $\otimes$ law as define in (3) to model the manipulation of colours in a TCPN while keeping a structure of dioid - for example when computing the Kleene Star of matrix A. Lastly, we use the notation $e = \langle\mathbb{1}\rangle = (\langle o\rangle, 1) \notin \overline{\mathbb{E}}$ to avoid any ambiguity later on. As $\forall\overline{a} \in \overline{\mathbb{E}}, \langle\mathbb{1}\rangle \oplus \overline{a} \notin \overline{\mathbb{E}}$, the $\oplus$ operation involving $\langle\mathbb{1}\rangle$ will be forbidden. If not, we could not ensure the colours absorption on left when computing some A^*.

B. Coloured Daters

Let $\overline{\mathbb{E}}$ be the colour set of a specific TCPN. Each TCPN model can have a different set $\overline{\mathbb{E}}$ associated, but at least equal to $\{\langle o\rangle, \langle\emptyset\rangle\}$. We define hereby mono-coloured daters in the dioid theory:

Definition 4.2: A *mono-coloured dater* x is a dater endowed with a colour $\langle.\rangle$, defining a new function $x : (\mathbb{Z} \times \{\langle.\rangle\}) \mapsto \mathbb{Z}_{\max}$. We note it $x(k, \langle.\rangle)$.

Following this definition, a classical TEG dater $x(k)$ is equivalent to a mono-coloured dater $x(k, \langle\emptyset\rangle)$. A pair (transition, colour) defines a unique mono-coloured dater function. This definition fits well for linear model, as each dater has one unique colour associated. It reads "*the date of the $(k+1)^{th}$ firing of the transition x with the colour $\langle.\rangle$*". The dioid $\overline{\mathbb{Z}}_{\max,\langle.\rangle}$ is the dioid of daters coloured by $\langle.\rangle$, and is equivalent to $\overline{\mathbb{Z}}_{\max}$ as in a TEG we can colour

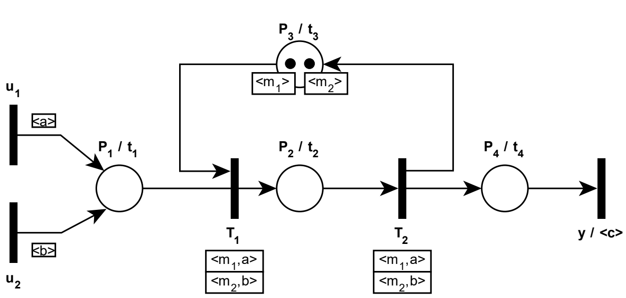

Fig. 2. Example of a TCPN modeling a jobshop

each dater with one unique colour without changing the model. All the $\overline{\mathbb{Z}}_{\max,\langle . \rangle}$ are *mono-coloured dioids*. A dater $x(k, \langle . \rangle)$ having the value $+\infty$ means "*the* $(k+1)^{th}$ *x-firing with the colour* $\langle . \rangle$ *never occurs*". Hence, over infinity, we assume that colours do not matter: if $x(k, \langle . \rangle) = +\infty$, then $x(k, \langle . \rangle) \equiv x(k, \langle \emptyset \rangle), \forall \langle . \rangle \in \overline{\mathbb{E}}$.

What follows is to define the relation existing between two daters. For all $\langle i \rangle, \langle j \rangle, \langle a \rangle$ in $\overline{\mathbb{E}}$, a shifting of k_0 firings between two daters x_1 and x_2 gives:

$$x_1(k, \langle i \rangle) = x_2(k - k_0, \langle j \rangle) \tag{4}$$

and the synchronisation between two daters yields:

$$x_3(k, \langle a \rangle) = x_1(k, \langle i \rangle) \oplus x_2(k, \langle j \rangle) \tag{5}$$

As the index is shifted by k_0 in (4), the change of colours in (4) and (5) state the necessary existence of an operation to shift them. This operations is implicitly given when defining a dater - for example in (4) the colour $\langle j \rangle$ is shifted to $\langle i \rangle$. For the convolution, if a dater x_3 is given by the convolution $x_1(k, \langle i \rangle) \otimes x_2(k, \langle j \rangle) = \bigoplus_{i \in \mathbb{Z}} x_1(i, \langle i \rangle) \otimes x_2(k-i, \langle j \rangle)$, then its colour is $\langle i \rangle \otimes \langle j \rangle$. These manipulations are, from a TCPN point of view, carried out by the functions $\mathcal{W}^-(P,T)(c_T)$ and $\mathcal{W}^+(P,T)(c_T)$. As it stands for mono-coloured dater, these functions are not represented and, as we attempt to express the behaviour of a TCPN with nonlinear operations on time and event based on colours, the inherent representations power of bi-dimensional formal series will be useful.

C. Coloured Formal Series

For any $(\langle . \rangle, b) \in \breve{\mathbb{E}}$, we define the undetermined operator $\gamma_{(\langle . \rangle, b)}$ and the transformation on it for a formal series s with exponents in $\mathbb{Z}_{\max}$ as:

$$s(\gamma_{(\langle . \rangle, b)}) = \bigoplus_{k \in \mathbb{Z}} s(k, (\langle . \rangle, b)) \gamma^k_{(\langle . \rangle, b)} \tag{6}$$

with $s(k, (\langle . \rangle, b))$ a mono-coloured dater. Let $\breve{\mathbb{E}}[\![\gamma_{(\langle . \rangle, b)}]\!]$ be the set of all formal series on $\gamma_{(\langle . \rangle, b)}$: $(\breve{\mathbb{E}}[\![\gamma_{(\langle . \rangle, b)}]\!], \oplus, \otimes)$ is a complete dioid - the laws $(\oplus, \otimes)$ being the same as in the dioid $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$. Let $\Gamma = \{\gamma_a \mid a \in \breve{\mathbb{E}}\}$ be the set of all undetermined operator coloured by a pair of $\breve{\mathbb{E}}$. We define the operations $\oplus$ and $\otimes$ on two elements of Γ as $\gamma^k_{a_1} \oplus \gamma^k_{a_2} = \gamma^k_{a_1 \oplus a_2}$ and $\gamma^k_{a_1} \otimes \gamma^{k'}_{a_2} = \gamma^{k+k'}_{a_1 \otimes a_2}$.

Remark 2: The set $\breve{\mathbb{E}}[\![\gamma_{(\langle . \rangle, b)}]\!]$ of formal series mono-coloured by $(\langle . \rangle, b)$ is a dioid completely analogous and isomorphic to $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$, and $\breve{\mathbb{E}}[\![\gamma_{\langle \mathbb{1} \rangle}]\!] \equiv \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$.

Proposition 4.3: The set $\breve{\mathbb{E}}[\![\Gamma]\!]$ of all formal series coloured by a pair of $\breve{\mathbb{E}}$ endowed with $\oplus$ and $\otimes$, is a complete dioid. Its neutral elements are $\varepsilon(\gamma_{(\langle o \rangle, 0)}) = \bigoplus_{k \in \mathbb{Z}} \varepsilon \gamma^k_{(\langle o \rangle, 0)}$ and $e(\gamma_{\langle \mathbb{1} \rangle}) = e\gamma^0_{\langle \mathbb{1} \rangle}$, with $\varepsilon = -\infty$, $e = 0$ the classical neutral elements of $\overline{\mathbb{Z}}_{\max}$.

Because daters are monotonous, their formal series need to be filtered. The exact dioid corresponding to formal series for daters (i.e., in which the series are well-ordered) is therefore $\gamma^*_{\langle \mathbb{1} \rangle} \breve{\mathbb{E}}[\![\Gamma]\!]$. This filtering is equivalent to the one existing in $\gamma^* \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$.

As in the lemma 4.2, we reduce the use of colours for daters to $\overline{\mathbb{E}}$ elements. From here, we denote $(\langle . \rangle, 0)$ by $\langle . \rangle$. This reduction enables the representations of mono-coloured daters through formal series equations, knowing that these equations evolve in a dioid structure. The transformation on $\gamma_{\langle . \rangle}$, for a mono-coloured dater x, is:

$$x(\gamma_{\langle . \rangle}) = \bigoplus_{k \in \mathbb{Z}} x(k, \langle . \rangle) \gamma^k_{\langle . \rangle} \tag{7}$$

The operator $\gamma_{\langle \emptyset \rangle} = \gamma$ is used for non coloured series. The undetermined operator bears two data: the index of firing and the event colour. We denote by $\overline{\mathbb{E}}[\![\Gamma]\!] \subset (\gamma^*_{\langle \mathbb{1} \rangle} \breve{\mathbb{E}}[\![\Gamma]\!])$ the set of all formal series representing mono-coloured daters. In consistency with the defined algebraic laws, the following properties hold :

$$\gamma^k_{\langle i \rangle} \oplus \gamma^{k'}_{\langle j \rangle} = \gamma^{\min(k,k')}_{\langle i \rangle \oplus \langle j \rangle} \text{ and } \gamma^k_{\langle i \rangle} \otimes \gamma^{k'}_{\langle j \rangle} = \gamma^{k+k'}_{\langle i \rangle \otimes \langle j \rangle}$$

Also, it yields $\gamma^k_{\langle i \rangle} = \gamma^k_{\langle i \rangle \otimes \langle j \rangle} = \gamma^0_{\langle i \rangle} \otimes \gamma^k_{\langle j \rangle}$, on the understanding that $\langle j \rangle \neq \langle o \rangle$. Elements of $\overline{\mathbb{E}}[\![\Gamma]\!]$ can describe linear TCPNs. For two mono-coloured daters such that $x_1(k, \langle i \rangle) = x_2(k-k_0, \langle j \rangle)$, switching to formal series gives:

$$\begin{aligned} x_1(\gamma_{\langle i \rangle}) &= \bigoplus_{k \in \mathbb{Z}} x_1(k, \langle j \rangle) \gamma^k_{\langle i \rangle} \\ &= \bigoplus_{k \in \mathbb{Z}} x_2(k - k_0, \langle j \rangle) \gamma^{k_0}_{\langle i \rangle} \gamma^{k-k_0}_{\langle j \rangle} \\ &= \gamma^{k_0}_{\langle i \rangle} x_2(\gamma_{\langle j \rangle}) \end{aligned} \tag{8}$$

If x_3 is defined by $x_3(k, \langle a \rangle) = x_1(k, \langle i \rangle) \oplus x_2(k, \langle j \rangle)$, then:

$$\begin{aligned} x_3(\gamma_{\langle a \rangle}) &= \bigoplus_{k \in \mathbb{Z}} [x_1(k, \langle i \rangle) \oplus x_2(k, \langle j \rangle)] \gamma^k_{\langle a \rangle} \\ &= \bigoplus_{k \in \mathbb{Z}} x_1(k, \langle i \rangle) \gamma^0_{\langle a \rangle} \gamma^k_{\langle i \rangle} \oplus \bigoplus_{k \in \mathbb{Z}} x_2(k, \langle j \rangle) \gamma^0_{\langle a \rangle} \gamma^k_{\langle j \rangle} \\ &= \gamma^0_{\langle a \rangle} x_1(\gamma_{\langle i \rangle}) \oplus \gamma^0_{\langle a \rangle} x_2(\gamma_{\langle j \rangle}) \end{aligned} \tag{9}$$

Remark 3: Note that $\gamma^k_{\langle . \rangle}$ is commutative with the dater functions, but not with other operators γ.

The operator $\gamma^{k_0}_{\langle i \rangle}$ stands simultaneously for event shifting and colour shifting. This notion of shifting colours as event or time are being shifted is fundamental. In the tropical algebra, daters are meant to link event with time, like counters are meant to link time with event. Coloured formal series allow us to integrate $\mathcal{W}^-(P,T)(c_T)$ and $\mathcal{W}^+(P,T)(c_T)$ functions to tropical equations. For a mono-coloured dater, it exists its dual as a coloured counter $c_{\langle . \rangle}(t) : \mathbb{Z} \mapsto \mathbb{Z}_{\min}$, which associates to a timing t a number of firings n. For a transition, n is the number of firing at time t with the colour $\langle . \rangle$.

Let $\mathbb{B}[\![\gamma, \delta]\!]$ be the complete dioid of mono-coloured formal series with Boolean coefficients in $\{\varepsilon, e\}$ and exponents in $\mathbb{Z}$. $\mathbb{B}[\![\gamma, \delta]\!]$ elements are the series $s(\gamma_{(\langle . \rangle, b)}, \delta) = \bigoplus_{k,t \in \mathbb{Z}} s(k, t, (\langle . \rangle, b)) \gamma^k_{(\langle . \rangle, b)} \delta^t$, with $s(k, t, (\langle . \rangle, b)) \in \{\varepsilon, e\}$ a Boolean. In the same way that $\mathcal{M}^{ax}_{in}[\![\gamma, \delta]\!]$ is built in the tropical algebra, the complete dioid $\mathbb{B}[\![\gamma, \delta]\!]$ modulo the relation $\gamma^*_{\langle \mathbb{1} \rangle} (\delta^{-1})^*$ gives a quotient dioid named $\breve{\mathbb{E}}[\![\Gamma, \delta]\!]$. This dioid gives new algebraic bi-dimensional representations. Like previously, we reduce our use of $\breve{\mathbb{E}}[\![\Gamma, \delta]\!]$ elements to a subset $\overline{\mathbb{E}}[\![\Gamma,\delta]\!]$ with the following properties holding in it:

$$\gamma_{\langle i\rangle}^{k}\delta^{t}\otimes\gamma_{\langle j\rangle}^{k'}\delta^{t'}=\gamma_{\langle i\rangle}^{k+k'}\delta^{t+t'},\ \gamma_{\langle i\rangle}^{k}\delta^{t}\oplus\gamma_{\langle j\rangle}^{k'}\delta^{t}=\gamma_{\langle i\rangle\oplus\langle j\rangle}^{\min(k,k')}\delta^{t}$$

$$\gamma_{\langle i\rangle}^{k}\delta^{t}\oplus\gamma_{\langle j\rangle}^{k}\delta^{t'}=\gamma_{\langle i\rangle\oplus\langle j\rangle}^{k}\delta^{\max(t,t')} \tag{10}$$

From now on, ε and e respectively refer to $\gamma_{\langle o\rangle}^{-\infty}\delta^{+\infty}$ and $\gamma_{\langle\mathbb{1}\rangle}^{0}\delta^{0}$, the $\breve{\mathbb{E}}[\![\Gamma,\delta]\!]$ neutral elements.

Remark 4: It must be noted that in $\overline{\mathbb{E}}[\![\Gamma,\delta]\!]$, $(\gamma_{\langle .\rangle}^{k}\delta^{t})^{*}\neq\gamma_{\langle .\rangle}^{0}\delta^{0}\oplus(\gamma_{\langle .\rangle}^{k}\delta^{t})^{+}$ but $(\gamma_{\langle .\rangle}^{k}\delta^{t})^{*}=e\oplus(\gamma_{\langle .\rangle}^{k}\delta^{t})^{+}=\gamma_{\langle\mathbb{1}\rangle}^{0}\delta^{0}\oplus(\gamma_{\langle .\rangle}^{k}\delta^{t})^{+}$. Therefore, in consistency with the fact that $\langle\mathbb{1}\rangle\notin\overline{\mathbb{E}}$, a t-periodic series s is expressed by : $s=\gamma_{\langle .\rangle}^{0}\delta^{0}\oplus(\gamma_{\langle .\rangle}^{k}\delta^{t})^{+}$

V. Application on paralleled tasks

Figure 3 shows an example of a TCPN with n paralleled tasks. Each task has a different processing duration: $t_1\neq t_2\neq ...\neq t_n$. Let us consider a colour per task, that is n colours. The colour set of this TCPN is $\mathbb{E}=\{\langle\emptyset\rangle,\langle 1\rangle ...,\langle n\rangle\}\cup\{\langle o\rangle\}$. The transitions u, x_0, x_1, and y each define n daters, one for each colour. This TCPN can be represented with the following linear system of equations:

$$\left\{\begin{array}{lcl} x_0(\gamma_{\langle i\rangle},\delta) & = & \gamma_{\langle i\rangle}^{0}\delta^{0}u(\gamma_{\langle i\rangle},\delta)\\ x_1(\gamma_{\langle i\rangle},\delta) & = & \gamma_{\langle i\rangle}^{0}\delta^{t_i}x_0(\gamma_{\langle i\rangle},\delta)\\ y(\gamma_{\langle i\rangle},\delta) & = & \gamma_{\langle i\rangle}^{0}\delta^{0}x_1(\gamma_{\langle i\rangle},\delta)\end{array}\right. \tag{11}$$

Each colour $\langle i\rangle$ defines a linear TCPN, with $A_{\langle i\rangle}=\begin{pmatrix}\varepsilon & \varepsilon\\ \gamma_{\langle i\rangle}^{0}\delta^{t_i} & \varepsilon\end{pmatrix}$, $B_{\langle i\rangle}=\begin{pmatrix}\gamma_{\langle i\rangle}^{0}\delta^{0}\\ \varepsilon\end{pmatrix}$ and $C_{\langle i\rangle}=\begin{pmatrix}\varepsilon & \gamma_{\langle i\rangle}^{0}\delta^{0}\end{pmatrix}$. These systems give n transfer function : $H_{\langle i\rangle}=C_{\langle i\rangle}A_{\langle i\rangle}^{*}B_{\langle i\rangle}=\begin{pmatrix}\gamma_{\langle i\rangle}^{0}\delta^{t_i}\end{pmatrix}$ and $y(\gamma_{\langle i\rangle},\delta)=H_{\langle i\rangle}u(\gamma_{\langle i\rangle},\delta)=\gamma_{\langle i\rangle}^{0}\delta^{t_i}u(\gamma_{\langle i\rangle},\delta)$. The compactness aspect can still be improved by considering a hierarchical model. Hierarchy is a concept that already exists in the CPN theory. By replacing a time delay in a place P_i by the transfer function of a linear Single-Input Single-Output (SISO) TEG, we can express a hierarchical model (for example, the one presented in Figure 1). The daters of a classical TEG are expressed using the dioid $\overline{\mathbb{E}}[\![\gamma_{\langle .\rangle}]\!]$ (cf Remark 2). Each SISO TEG has a transfer function H_i expressing a linear time/event shifting, with $H_1\neq ...\neq H_n$. That said, we could obviously replace them by TCPNs having a linear transfer function. This time/event shiftings of our new TCPN are given by the following equations:

$$\left\{\begin{array}{lcl} x_0(\gamma_{\langle i\rangle},\delta) & = & \gamma_{\langle i\rangle}^{0}\delta^{0}u(\gamma_{\langle i\rangle},\delta)\\ x_1(\gamma_{\langle i\rangle},\delta) & = & \gamma_{\langle i\rangle}^{0}\delta^{0}\otimes H_i\otimes x_0(\gamma_{\langle i\rangle},\delta)\\ y(\gamma_{\langle i\rangle},\delta) & = & \gamma_{\langle i\rangle}^{0}\delta^{0}x_1(\gamma_{\langle i\rangle},\delta)\end{array}\right. \tag{12}$$

Computing the new matrices gives finally a transfer function $H_{\langle i\rangle}$ defined by: $H_{\langle i\rangle}=C_{\langle i\rangle}A_{\langle i\rangle}^{*}B_{\langle i\rangle}=\begin{pmatrix}\gamma_{\langle i\rangle}^{0}\delta^{0}H_i\end{pmatrix}$ and $y(\gamma_{\langle i\rangle},\delta)=H_i\otimes u(\gamma_{\langle i\rangle},\delta)$. The $A_{\langle i\rangle}^{*}$ computation follows from results given in ([1]). These equations shows that mono-coloured formal series can handle parallel tasks in a TCPN, and a fortiori, can also express the behaviour of any linear TCPN in the tropical algebra since we know how to shift colours. Hence, the expressive power of formal series (that is the reduction of a linear system to a single transfer function) can be applied on TCPNs. It is now possible to describe a compact linear TCPN model using formal series, without using TEGs. Nonetheless, coloured nets also model operations that can not be represented in the classical tropical algebra. We aim to present a study case where the given representations of coloured formal series express more complex behaviours. For instance, one can define laws conditioning the times and events on the value of the colours.

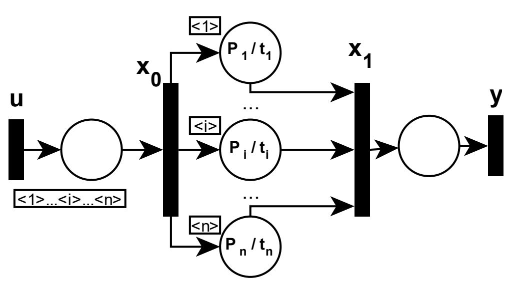

Fig. 3. Example of a TCPN with n paralleled tasks

VI. Application on shared resource

Conflict resolution relies on a choice mechanism, naturally expressed through colours. It is one of the main advantages of CPNs. We present hereafter a model with a shared resource. This example proves that our new coloured formal series offers greater algebraic expressions, allowing us to handle a mechanism we could not express with linear tropical algebra before. Figure 4 shows a conflict situation between the $x_0(k,\langle 1\rangle)$ and the $x_1(k,\langle 2\rangle)$ firings. They both need a $\langle o\rangle$ coloured token in P_3 to fire, corresponding to the shared resource. The firing functions are $\mathcal{W}^{-}(P_3,x_0)(\langle 1\rangle)=1`\langle 1\rangle+1`\langle o\rangle$ and $\mathcal{W}^{-}(P_3,x_1)(\langle 2\rangle)=1`\langle 2\rangle+1`\langle o\rangle$. In Figure 4, the token in place P_3 stands for the presence at time $t=0$ of a resource $\langle o\rangle$. The transition u_1 generates the three types of tokens : $\langle 1\rangle$, $\langle 2\rangle$ and $\langle o\rangle$. Their arrival dates are differentiated using three distinguished mono-coloured daters (one for each colour), all three associated to u_1 and denoted $u_{\langle 1\rangle}$, $u_{\langle 2\rangle}$ and $u_{\langle o\rangle}$. In this TCPN, the conflict resolution relies on the following hypothesis: the tokens in place P_3 are processed with a FIFO order. However, if two tokens of different colour arrive at P_3 at the same date, the colour $\langle 1\rangle$ has priority over $\langle 2\rangle$. This hypothesis is modeled in the TCPN by the guard $\Phi(x_1)=$ "The place P_3 contains no $\langle 1\rangle$ tokens". Thus, the priority between colours applies on tokens arrival date, and not when firing x_1 or x_2. Also, we suppose that tokens $\langle o\rangle$ arrive with a periodicity of t_3 time units, i.e., $u_{\langle o\rangle}(\Gamma,\delta)=\gamma_{\langle o\rangle}^{0}\delta^{0}\oplus(\gamma_{\langle o\rangle}^{0}\delta^{t_3})^{+}$. However, we could also choose to express the resource $\langle o\rangle$ availability through a configurable trajectory, i.e., through arrival dates chosen before the computation but not necessarily periodic. To handle this situation, suppose that it exists a relation order $\prec_p$ different from the $\mathbb{E}$ natural order $\preccurlyeq$ (associated to $\oplus$). This new order relation $\prec_p$ defines a *choice rule* between two colours. If $\langle i\rangle \prec_p \langle j\rangle$, we can associate two laws $\boxplus$ and $\boxminus$ defined by: $\langle i\rangle \boxplus \langle j\rangle = \langle j\rangle$ and $\langle i\rangle \boxminus \langle j\rangle = \langle i\rangle$. $\boxminus$ (resp. $\boxplus$) implies the FIFO-choice of the highest priority (resp. the lowest) and to keep the associated date and index in a trajectory. For $\langle 1\rangle \prec_p \langle 2\rangle$, it yields:

$$\gamma^k_{\langle 1\rangle}\delta^t \boxplus \gamma^k_{\langle 2\rangle}\delta^{t'} = \begin{cases} \gamma^k_{\langle 1\rangle}\delta^t \text{ if } t > t' \\ \gamma^k_{\langle 2\rangle}\delta^{t'} \text{ if } t \leq t' \end{cases} \tag{13}$$

and

$$\gamma^k_{\langle 1\rangle}\delta^t \boxminus \gamma^{k'}_{\langle 2\rangle}\delta^{t'} = \begin{cases} \gamma^{k+k'}_{\langle 1\rangle}\delta^t \text{ if } t \leq t' \\ \gamma^{k+k'}_{\langle 2\rangle}\delta^{t'} \text{ if } t > t' \end{cases} \tag{14}$$

Between two trajectories $s_1, s_2 \in \overline{\mathbb{E}}[\![\Gamma,\delta]\!]$, respectively mono-coloured by $\langle 1\rangle$ and $\langle 2\rangle$, the operation $\boxplus$ is undefined, whereas $\boxminus$ defines a new convolution (in the same way as $\otimes$):

$$s_1 \boxminus s_2 = \bigoplus_{k\in\mathbb{Z}} \boxplus_{i\in\mathbb{Z}} \gamma^i_{\langle 1\rangle}\delta^t \boxminus \gamma^{k-i}_{\langle 2\rangle}\delta^{t'} \tag{15}$$

It means that the $(k+1)^{th}$ firing date is equal to the lowest dates for x_1 and x_2 firings, knowing that the k lowest dates of x_1 and x_2 firings have already been sorted by the convolution. The priority hypothesis between $\langle 1\rangle$ and $\langle 2\rangle$ solves the equality case in (15), that is if two tokens arrive simultaneously. Moreover, such a convolution between two trajectories yields a trajectory having events coloured differently. We define now daters yielding more than one colour:

Definition 6.1: Let $\omega : \mathbb{Z}_{\max} \mapsto \overline{\mathbb{E}}$ be a *colouration function*. A multi-coloured dater $x(k, \omega(k))$ is a dater endowed with a colouration function, $\omega(k)$ being the colour of the $(k+1)^{th}$ firing.

Using coloured formal series, if x_3 is computed by $x_1 \boxminus x_2$:

$$x_3(\gamma_{\omega_3}) = \bigoplus_{k\in\mathbb{Z}} \left[\boxplus_{i\in\mathbb{Z}} [x_1(i, \langle 1\rangle) \boxminus x_2(k-i, \langle 2\rangle)] \right] \gamma^k_{\omega_3(k)} \tag{16}$$

$\omega_3(k)$ being the colour resulting from (13) and (14). This operation orders the firings of x_1 and x_2 by creating a third dater, multi-coloured. The law $\boxminus$ can be seen as the *conflict resolution rule* defined as a parameter in the description of the TCPN model.

Example: Let $x_1(\gamma_{\langle 1\rangle}, \delta) = \gamma^0_{\langle 1\rangle}\delta^4 \oplus \gamma^1_{\langle 1\rangle}\delta^5 \oplus \gamma^2_{\langle 1\rangle}\delta^{+\infty}$ and $x_2(\gamma_{\langle 2\rangle}, \delta) = \gamma^0_{\langle 2\rangle}\delta^3 \oplus \gamma^1_{\langle 2\rangle}\delta^5 \oplus \gamma^2_{\langle 2\rangle}\delta^{+\infty}$ be two coloured trajectories. Operating with $\boxminus$ gives:

$$x_3(\gamma_{\omega_3}, \delta) = \gamma^0_{\langle 2\rangle}\delta^3 \oplus \gamma^1_{\langle 1\rangle}\delta^4 \oplus \gamma^2_{\langle 1\rangle}\delta^5 \oplus \gamma^3_{\langle 2\rangle}\delta^5 \oplus \gamma^4_{\langle \emptyset\rangle}\delta^{+\infty}$$

We also define a law $\boxdot$ conditioned on colours and yielding a mono-coloured trajectory from a multi-coloured one :

$$\langle i\rangle \boxdot x(\Gamma, \delta) = x(\gamma_{\langle i\rangle}, \delta) \tag{17}$$

$\boxdot$ returns ε if there is no event coloured with $\langle i\rangle$ in $x(\Gamma, \delta)$ and a well-ordered mono-coloured trajectory otherwise. If $x(\Gamma, \delta) = \gamma^0_{\langle 1\rangle}\delta^1 \oplus \gamma^1_{\langle 2\rangle}\delta^3 \oplus \gamma^2_{\langle 1\rangle}\delta^4 \oplus \gamma^3\delta^{+\infty}$ is a multi-coloured dater, $\langle 1\rangle \boxdot x(\Gamma, \delta) = \gamma^0_{\langle 1\rangle}\delta^1 \oplus \gamma^1_{\langle 1\rangle}\delta^4 \oplus \gamma^2\delta^{+\infty}$. This law allows us to return to mono-coloured trajectories after a convolution $\boxminus$. We can now express the time/event shiftings of the system presented in Figure 4. As x_0 and x_1 are in conflicts, we note $\tilde{x}$ the multi-coloured dater coming from their composition.

$$\begin{cases} \tilde{x}(\Gamma, \delta) = & \gamma^0_{\langle \mathbb{1}\rangle}\delta^0[u_{\langle 1\rangle}(\Gamma, \delta) \boxminus u_{\langle 2\rangle}(\Gamma, \delta)] \oplus u_{\langle o\rangle}(\Gamma, \delta) \\ x_2(\Gamma, \delta) = & [\gamma^0_{\langle 1\rangle}\delta^{t_1}(\langle 1\rangle \boxdot \tilde{x}(\Gamma, \delta))] \boxminus \\ & \qquad [\gamma^0_{\langle 2\rangle}\delta^{t_2}(\langle 2\rangle \boxdot \tilde{x}(\Gamma, \delta))] \\ y(\Gamma, \delta) = & \gamma^0_{\langle \mathbb{1}\rangle}\delta^0 x_2(\Gamma, \delta) \end{cases} \tag{18}$$

For outputs y, we ultimately have:

$$y(\Gamma, \delta) = [\gamma^0_{\langle 1\rangle}\delta^{t_1}(\langle 1\rangle \boxdot \tilde{x}(\Gamma, \delta))] \boxminus [\gamma^0_{\langle 2\rangle}\delta^{t_2}(\langle 2\rangle \boxdot \tilde{x}(\Gamma, \delta))]$$

$y(\Gamma, \delta)$ yields the output trajectory where $\langle 1\rangle$ and $\langle 2\rangle$ events are well-sorted, taking account of the delays induced by the arriving of resources from $u_2(\Gamma, \delta)$.

To tackle the complexity problem of non-linear models, we propose, as a method, to divide a TCPN model into linear and non-linear parts. The temporal behaviour of linear parts can then be represented using the transfer functions given by a mono-coloured description. This allows to keep the computational power of tropical formal series. The possibility to easily transform a transfer function into a n-sized trajectory ensures the calculus consistency. For example, if we replaced the time delays at P_1 and P_2 by two transfer functions H_1 and H_2, respectively only coloured by $\langle 1\rangle$ and $\langle 2\rangle$, the output trajectory would be:

$$y(\Gamma, \delta) = [[H_1(\langle 1\rangle \boxdot \tilde{x}(\Gamma, \delta))] \boxminus [H_2(\langle 2\rangle \boxdot \tilde{x}(\Gamma, \delta))]]$$

Hence, it is possible to handle the temporal assessment of large-scaled TCPNs by separating the linear parts from the conflict mechanisms.

VII. Conclusion

This paper introduced new formal series and algebraic laws to express colour manipulations, conflict resolution, and choices. One can see that these results are extendable to n tasks with different processing duration or n hierarchical submodels, like the models in section V. As limits, it remains to analyze the computation complexity of the proposed method on a large-scaled TCPN, built from a real system. In addition, we must consolidate our approach to bypass the assumptions made on the TCPN modelling formalism.

Working with mono-coloured daters is more natural and less complex. However, it reduces the advantage of TCPN

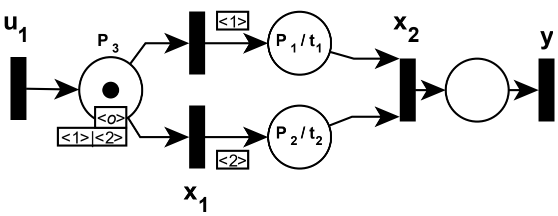

Fig. 4. A conflict for two tasks with different processing duration.

to compact parallel and linear models. On the other hand, multi-coloured daters are much more interesting, even if they quickly lead to nonlinear descriptions.

In any case, coloured formal series are representations that increase expressiveness. We can now integrate colour shiftings into our tropical equations and express nonlinear mechanisms. In the same way that $\oplus$ gives linearity to the operation $\max$ between two elements, $\boxdot$, $\boxplus$ and $\boxminus$ have corresponding operations in natural algebra. However, they could be delicate to define. The resulting equations allow us to automate time delay computation through configurable models. It is a valuable feature in control theory.

Adding a third dimension to daters/counters dual representations opens many possibilities. The conflict resolution law used in this paper is only one of many. We can apply our generic approach to represent TCPN with nonlinear behaviours in transport, logistic or control network areas, where classical tropical representations are limited.

REFERENCES

[1] Baccelli, F., Cohen, G., Olsder, G.-J., and Quadrat, J.-P. (1992). *Synchronization and Linearity: An Algebra for Discrete Event Systems.* New York: Wiley, 1992.

[2] Cohen, G., Gaubert, S., and Quadrat, J.-P. (1999). *Max-plus algebra and system theory: Where we are and where to go now*. Annual Reviews in Control (IFAC, Elsevier), 23 (1), 207-219, 1999.

[3] Trunk, J., Schafaschek, G., Cottenceau, B., Hardouin, L., J. Raisch (2020) *Observer for Weighted Timed Event Graphs*. IFAC-PapersOnLine, Volume 53, Issue 4, 2020, Pages 501-507, ISSN 2405-8963.

[4] Hardouin, L., Cottenceau, B., Shang, Y., and Raisch, J. (2018). *Control and State Estimation for Max-Plus Linear Systems*. Systems and Control, vol. 6, no. 1, pp. 1–116, 2018.

[5] Velasquez, I., Le Corronc, E., and Pencolé, Y. (2022) *Active Diagnosis Algorithm for the Localization of Time Failures in (Max,+)-Linear Systems*. IFAC-PapersOnLine Volume 55, Issue 28, 2022, Pages 276-283.

[6] Trunk, J., Cottenceau, B., and Hardouin, L. (2020) *Modelling and control of periodic time-variant event graphs in dioids*. Discrete Event Dyn Syst 30, 269–300 (2020).

[7] Jensen, K., and L., Kristensen. (2009). *Coloured Petri Nets*. Springer.

[8] Bal dit Sollier, L., Ourghanlian, A., and Amari., S. (2022). *Coloured Petri Nets for Temporal Performance Evaluation of Distributed Control Systems—Application to a FIFO Queue*. IEEE Robotics and Automation Letters (Volume: 7, Issue: 4, October 2022).

[9] Mahjoub,Y.I., Chakir-El-Alaoui, EH., and Nait-Sidi-Moh., A. (2017). *Modeling a bus network for passengers transportation management using coloured Petri nets and (max,+) algebra.* Procedia Computer Science vol. 109, pp. 576-583, 2017.

[10] Outafraou, K., Nait-Sidi-Moh, A., and Chakir-El-Alaoui., EH. (2020). *A Control Approach Based on Coloured Hybrid Petri Nets and (Max, +) Algebra: Application to Multimodal Transportation Systems.* IEEE Transactions on Automation Science and Engineering, vol. 17(3), pp. 1208 - 1220. 2020.

[11] Golan, J., S. (1999). *Semirings and their Applications*. Springer Dordrecht, 1999. DOI:https://doi.org/10.1007/978-94-015-9333-5. ISBN: 978-0-7923-5786-5.