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On Enlarging the Domain of Attraction for Linear Systems Subject to Asymmetric Actuator Saturation

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Abstract— **In this paper, we revisit the problem of enlarging the domain of attraction for linear systems with asymmetric actuator saturation. We partition the state space into several regions according to the sign of each input and rewrite the linear system subject to asymmetric actuator saturation as an equivalent switched system, each subsystem of which is associated with one partition of the state space and is a linear system subject to symmetric actuator saturation. Based on this equivalent representation of the system, we present a Lyapunov function, which is composed of a set of quadratic functions associated with matrices that are not required to be positive definite. We establish sufficient conditions for regional stability and, based on them, formulate optimization problems to enlarge the estimate of the domain of attraction. Simulation results illustrate the effectiveness of the proposed approach.**

I. INTRODUCTION

In practical control systems, actuator saturation is a ubiquitous nonlinearity due to physical limitations and safety requirements. The presence of actuator saturation may degrade the performance of the closed-loop system and may even cause instability. In the past decades, linear systems subject to actuator saturation have become a focus of study in the field of nonlinear control and a large number of interesting results have been reported (see, for example, [1]–[3]).

Among the results available in the literature on linear systems subject to actuator saturation are many on the study of their domain of attraction (see, for example, [4], [8], [9] and the references therein). Since the boundary of the domain of attraction is difficult to obtain, a common practice is to estimate the domain of attraction by using the level sets of a Lyapunov function. The conservativeness of such estimation depends to a large degree on the choice of the Lyapunov function. Many Lyapunov functions have been proposed. For example, in [15], a piece-wise quadratic Lyapunov function is proposed, which effectively incorporates the properties of the dead-zone function to provide a less conservative estimate of the domain of attraction. In [6] and [17], a regional sector condition is introduced to relax the positive definiteness requirement of the matrix that defines the Lyapunov function for a larger estimate of the domain of attraction.

The majority of the existing results, including those mentioned above, pertain to actuator saturation that is symmetric. Asymmetric actuator saturation widely exists in physical systems. There have been some attempts to achieve desirable properties of linear systems with asymmetric actuator saturation. In [10] and [11], a novel state transformation approach is proposed to extending the available results on symmetric saturation to the asymmetric case. In [14], for the enlargement of the domain of attraction, a system with asymmetric actuator saturation is regarded as an equivalent switching model and a set of conditions for determining linear controllers and anti-windup compensators simultaneously are established. Adopting the equivalent system representation given in [14], Reference [7] constructs asymmetric piece-wise quadratic Lyapunov functions, which contribute to a considerable improvement in the estimation of the domain of attraction. By an asymmetric Lyapunov function we mean a Lyapuov function whose level sets are not symmetric with respect to the origin of the state space. A parallel approach for discrete-time systems is presented in [13]. Recently, by shifting the state coordinates, Reference [12] proposes a nonlinear asymmetric stabilizer for linear systems subject to asymmetric saturation, which is demonstrated to have the ability to enlarge the domain of attraction. In addition, it is worth mentioning that, as a mechanism that naturally accounts for constrained inputs, model predictive control also attracts considerable attention in the enlargement of the domain of attraction for linear systems subject to actuator saturation (see, for example, [18], [19]).

In this paper, we aim to construct a generalized asymmetric Lyapunov function for linear systems with asymmetric actuator saturation to further enlarge the estimates of the domain of attraction. Following the idea of [7], we will consider a partition of the state space according to the sign of each input, and decompose the linear system with asymmetric actuator saturation into a set of subsystems with symmetric actuator saturation. For each subsystem, we assign a piece-wise quadratic function that is defined by a positive definite matrix. These quadratic functions are then combined in a switching manner to develop a generalized asymmetric Lyapunov function. We establish sufficient conditions under which a level set of the proposed Lyapunov function is contractively invariant, and hence can be used as an estimate of the domain of attraction. To further reduce the conservativeness, we develop relaxed conditions where the positive definiteness of the associated matrices is not required. Based

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on these conditions, we formulate the optimization problems for maximizing the estimate of the domain of attraction. A numerical example is given to illustrate the effectiveness of our approach.

The remainder of this paper is organized as follows. In Section II, we give the problem statement and present the generalized asymmetric Lyapunov function. In Section III, sufficient conditions are established for regional stability for systems with asymmetric actuator saturation. Based on these conditions, we formulate in Section IV the optimization problems for obtaining the largest estimates of the domain of attraction. Section V illustrates the effectiveness of the proposed approach through a numerical example. Section VI concludes this paper.

Notation. For a square matrix A, $\mathrm{He}(A) := A + A^{\mathrm{T}}$. For two integers l_1 and l_2, $l_2 \geq l_1$, $I[l_1, l_2]$ denotes the set of integers $\{l_1, l_1+1, \cdots, l_2\}$. The asymmetric saturation function $\mathrm{sat}_{\underline{\mu},\overline{\mu}} : \mathbf{R}^m \to \mathbf{R}^m$ is defined as $\mathrm{sat}_{\underline{\mu},\overline{\mu}}(u) = [\mathrm{sat}_{\underline{\mu}_1,\overline{\mu}_1}(u_1)\ \mathrm{sat}_{\underline{\mu}_2,\overline{\mu}_2}(u_2) \cdots \mathrm{sat}_{\underline{\mu}_m,\overline{\mu}_m}(u_m)]^{\mathrm{T}}$, where $u = [u_1\ \ u_2 \cdots u_m]^{\mathrm{T}}$ and for each $i \in I[1,m]$, $\mathrm{sat}_{\underline{\mu}_i,\overline{\mu}_i}(u_i)$ is defined as follows,

$$\mathrm{sat}_{\underline{\mu}_i,\overline{\mu}_i}(u_i) = \begin{cases} \overline{\mu}_i, \ u_i > \overline{\mu}_i, \\ u_i, \ u_i \in [-\underline{\mu}_i, \overline{\mu}_i], \\ -\underline{\mu}_i, \ u_i < -\underline{\mu}_i, \end{cases} \tag{1}$$

where $\overline{\mu}_i > 0$ and $\underline{\mu}_i > 0$ denote the magnitudes of the positive and negative saturation levels, respectively, and $\overline{\mu} = [\overline{\mu}_1\ \overline{\mu}_2 \cdots \overline{\mu}_m]^{\mathrm{T}}$, $\underline{\mu} = [\underline{\mu}_1\ \ \underline{\mu}_2 \cdots \underline{\mu}_m]^{\mathrm{T}}$. For simplicity, we use $\mathrm{sat}_\mu(u)$ to denote the symmetric case, that is, $\mu = \underline{\mu} = \overline{\mu}$. The dead-zone function is defined as $\mathrm{dz}_{\underline{\mu},\overline{\mu}}(u) = u - \mathrm{sat}_{\underline{\mu},\overline{\mu}}(u)$, and for the symmetric case, $\mathrm{dz}_\mu(u) = u - \mathrm{sat}_\mu(u)$.

II. Preliminaries

A. Problem Statement

Consider the following linear system with asymmetric actuator saturation,

$$\dot{x} = Ax + B\mathrm{sat}_{\underline{\mu},\overline{\mu}}(Fx), \tag{2}$$

where $x \in \mathbf{R}^n$ is the state and $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$ and $F \in \mathbf{R}^{m\times n}$ are constant matrices of appropriate dimensions. The closed-loop system is assumed to be asymptotically stable in the absence of saturation. However, it is well known that such asymptotic stability is in general not global. We are thus interested in characterizing the domain of attraction of the closed-loop system, the set of all the initial states from which the state trajectory converges to the origin. However, in general, it is difficult to accurately describe the boundary of the domain of attraction. A common practice is to estimate the domain of attraction with a level set of Lyapunov function. This involves the construction of a Lyapunov function that would result in a less conservative estimate of the domain of attraction. We will focus on a new Lyapunov function, whose construction exploits the special property of the asymmetric saturation.

Motivated by [7], [13] and [14], we partition the state space into 2^m regions according to the sign of each input. Each region is defined as

$$\Gamma_i = \{x \in \mathbf{R}^n : \Lambda_i F x \geq 0\}, \ \ i \in I[1, 2^m], \tag{3}$$

where $\Lambda_i = \mathrm{diag}\{\lambda_{i1}, \lambda_{i2}, \cdots, \lambda_{im}\}$ with $\lambda_{ij} = \mathrm{sgn}(F_j x)$, and F_j, $j \in I[1,m]$, is the j^{th} row of F. System (2) can then be represented equivalently as the following switched system,

$$\dot{x} = Ax + B\mathrm{sat}_{\mu_i}(Fx), \ x \in \Gamma_i, \ i \in I[1, 2^m], \tag{4}$$

where $\mu_i = \frac{1}{2}((\Lambda_i + I)\overline{\mu} + (I - \Lambda_i)\underline{\mu})$. Within each region Γ_i, the associated subsystem (4) is a linear system subject to symmetric actuator saturation.

B. A Generalized Asymmetric Lyapunov Function

An asymmetric Lyapunov function, whose level sets are not symmetric with respect to the origin, has been presented in [7]. To further reduce the conservativeness of this asymmetric Lyapunov function, in this subsection, we will present a generalized asymmetric Lyapunov function based on the partition (3). To this end, we denote, for $i \in I[1, 2^m]$,

$$\begin{aligned} E_i &= \begin{bmatrix} \Lambda_i F & 0 \\ 0 & \Lambda_i \end{bmatrix} \in \mathbf{R}^{2m\times(n+m)}, \\ \overline{F}_i &= [E_i^{\mathrm{T}}\ \ I_{n+m}]^{\mathrm{T}} \in \mathbf{R}^{(3m+n)\times(n+m)}, \end{aligned} \tag{5}$$

where Λ_i is as defined in (3). Let $\xi = [x^{\mathrm{T}}\ \ \mathrm{dz}_{\mu_i}(Fx)^{\mathrm{T}}]^{\mathrm{T}}$. Then, within Γ_i, we define a piece-wise quadratic function as follows,

$$\begin{aligned} V_{Gi}(x) &= \begin{bmatrix} x \\ \mathrm{dz}_{\mu_i}(Fx) \end{bmatrix}^{\mathrm{T}} \overline{F}_i^{\mathrm{T}} T \overline{F}_i \begin{bmatrix} x \\ \mathrm{dz}_{\mu_i}(Fx) \end{bmatrix} \\ &= \xi^{\mathrm{T}} P_{Gi} \xi, \end{aligned}$$

where $T \in \mathbf{R}^{(3m+n)\times(3m+n)}$ and $P_{Gi} \in \mathbf{R}^{(n+m)\times(n+m)}$ is a symmetric matrix to be determined later. We then define the following Lyapunov function candidate,

$$V_G(x) = V_{Gi}(x), \ \text{if}\ x \in \Gamma_i, \ i \in I[1, 2^m], \tag{6}$$

whose level set is given as

$$\mathcal{E}_{\mathrm{Union}} = \cup_{i=1}^{2^m} (\mathcal{E}(P_{Gi}) \cap \Gamma_i),$$

with

$$\mathcal{E}(P_{Gi}) = \{x \in \mathbf{R}^n : \xi^{\mathrm{T}} P_{Gi} \xi \leq 1\}.$$

Partition T as $T = \begin{bmatrix} T_1 & T_2 \\ T_2^{\mathrm{T}} & T_3 \end{bmatrix}$, where $T_1 \in \mathbf{R}^{2m\times 2m}$, $T_2 \in \mathbf{R}^{2m\times(m+n)}$ and $T_3 \in \mathbf{R}^{(n+m)\times(n+m)}$. We have

$$\begin{aligned} \xi^{\mathrm{T}} P_{Gi} \xi &= \xi^{\mathrm{T}} \overline{F}_i^{\mathrm{T}} T \overline{F}_i \xi \\ &= \xi^{\mathrm{T}} [E_i^{\mathrm{T}}\ \ I_{n+m}] \begin{bmatrix} T_1 & T_2 \\ T_2^{\mathrm{T}} & T_3 \end{bmatrix} [E_i^{\mathrm{T}}\ \ I_{n+m}]^{\mathrm{T}} \xi \\ &= \xi^{\mathrm{T}} \big(E_i^{\mathrm{T}} T_1 E_i + \mathrm{He}(E_i^{\mathrm{T}} T_2) + T_3\big) \xi. \end{aligned} \tag{7}$$

Consider two intersected regions Γ_i and Γ_j. It is clear that there is only one different element between Λ_i and Λ_j, that is, there is only one $k \in I[1,m]$ such that $\lambda_{ik} \neq \lambda_{jk}$.

Denote the boundary between Γ_i and Γ_j as Ω_{ij}, then for each $x \in \Omega_{ij}$, the following facts hold:

- There must be one input $u_k = F_k x$ such that $F_k x = \mathrm{dz}(F_k x) = 0$. Then one could have $\lambda_{ik} F_k x = \lambda_{jk} F_k x = 0$ and $\lambda_{ik}\mathrm{dz}(F_k x) = \lambda_{jk}\mathrm{dz}(F_k x) = 0$.
- Since $\lambda_{ik} \neq \lambda_{jk}$, for each $l \in I[1,m]\backslash\{k\}$, we have $\Lambda_{il} F_l x = \Lambda_{jl} F_l x$ and $\lambda_{il}\mathrm{dz}(F_l x) = \lambda_{jl}\mathrm{dz}(F_l x)$.

Based on the above observations, it can be shown that $E_i\xi = E_j\xi$ and $\overline{F}_i\xi = \overline{F}_j\xi$ hold if x is on the boundary Ω_{ij}. This guarantees the continuity of $V_G(x)$ in the whole state space.

Remark 1: Note that in the case that $T_1 = 0_{2m\times 2m}$ and $T_2 = 0_{2m\times(m+n)}$, $V_G(x)$ reduces to the asymmetric Lyapunov function proposed in [7] with a common positive definite matrix shared in each region Γ_i. Moreover, if $T_1 = 0_{2m\times 2m}$ and $T_2 \neq 0_{2m\times(m+n)}$, V_G becomes the asymmetric Lyapunov function with different positive definite matrices for each Γ_i. This type of asymmetric Lyapunov functions were also proposed in [7]. Apparently, the presence of T_1 allows further enlargement of the differences between each P_{Gi}. This will result in a larger estimate of the domain of attraction. And the relaxed conditions that will be presented later could transform the proposed Lyapunov function into a sign-indefinite one (the matrices associated with Lyapunov functions are not required to be positive definite), which in turn reduces the conservatism. Therefore, the proposed Lyapunov function in this paper is more general than those in [7]. Also, compared with [7], where each P_{Gi} is constructed by several different variables, the newly proposed Lyapunov function shows a more compact form, that is, matrix T is the only parameter that needs to be determined. In fact, such a form is inspired by [16]. However, the method provided in [16] is different from ours. In particular, although the specific partition of the state space and the stability analysis for each partition in [16] contribute to an LMI-based method that is easy to solve, such a method is not directly applicable to the estimation of the domain of attraction. This is because the stability analysis in [16] is to verify the stability of each predetermined partition in the state space while the key idea of estimation is to find a set in the state space where trajectories converge to the origin. Furthermore, instead of piece-wise Lyapunov functions [15], quadratic Lyapunov functions, which can be very conservative, are designated to each partition in [16], leading to less favorable results.

The following lemmas present some useful properties of saturation functions, which will help to develop the results in the next section.

Lemma 1: [5] For any given diagonal matrix $S > 0 \in \mathbf{R}^{m\times m}$, for any $v = [v_1\ v_2\ \cdots\ v_m]^{\mathrm{T}} \in \mathbf{R}^m$ where $|v_j| \leq \mu$, $\forall\, j \in I[1,m]$, the following inequality holds,

$$\mathrm{dz}_\mu^{\mathrm{T}}(Fx)S(Fx - \mathrm{dz}_\mu(Fx) + v) \geq 0.$$

Let $\phi_\mu(Fx) \in \mathbf{R}^m$ be the directional derivative of $\mathrm{dz}_\mu(Fx)$ at x along $\dot{x}$, which can be defined as

$$\phi_\mu(Fx) = \lim_{t\to 0^+} \frac{\mathrm{dz}_\mu(Fx + t\dot{x}) - \mathrm{dz}_\mu(Fx)}{t}.$$

Lemma 2: [15] For any given diagonal matrices $S_1,\ S_2 \in \mathbf{R}^{m\times m}$, the following sector-like conditions hold for any $x \in \mathbf{R}^n$,

$$\phi_\mu^{\mathrm{T}}(Fx)S_1(F\dot{x} - \phi_\mu(Fx)) \equiv 0,$$
$$\mathrm{dz}_\mu^{\mathrm{T}}(Fx)S_2(F\dot{x} - \phi_\mu(Fx)) \equiv 0.$$

III. Main Results

In this section, we establish sufficient conditions under which a level set of the proposed Lyapunov function is contractively invariant for system (4).

Theorem 1: Consider system (4). If there exist a symmetric matrix $T \in \mathbf{R}^{(3m+n)\times(3m+n)}$, diagonal matrices S_{1i}, S_{2i}, $S_{3i} \in \mathbf{R}^{m\times m}$ with $S_{1i} > 0$, and matrix $H_i \in \mathbf{R}^{m\times(n+m)}$, $i \in I[1, 2^m]$, such that the following matrix inequalities hold,

$$\begin{aligned}\Theta_i = \mathrm{He}\Big(&\mathcal{O}P_{Gi}\mathcal{A} + \mathcal{I}_1^{\mathrm{T}}S_{1i}(\mathcal{G}_1 + H_i\mathcal{G}_2)\\ &+ \mathcal{I}_2^{\mathrm{T}}S_{2i}\mathcal{G}_3 + \mathcal{I}_1^{\mathrm{T}}S_{3i}\mathcal{G}_3\Big) < 0,\end{aligned} \tag{8}$$

and

$$\begin{bmatrix} \mu_i^2 & h_{ij} \\ \star & P_{Gi} \end{bmatrix} > 0, \tag{9}$$

where h_{ij} is the j^{th} row of H_i and

$$\begin{aligned}
P_{Gi} &= \overline{F}_i^{\mathrm{T}} T \overline{F},\\
\mathcal{O} &= \begin{bmatrix} I_{n+m} \\ 0_{m\times(n+m)} \end{bmatrix},\\
\mathcal{A} &= \begin{bmatrix} A+BF & -B & 0_{n\times m} \\ 0_{m\times n} & 0_{m\times m} & I_m \end{bmatrix},\\
\mathcal{I}_1 &= [0_{m\times n}\ \ I_m\ \ 0_{m\times m}],\\
\mathcal{G}_1 &= [F\ \ -I_m\ \ 0_{n\times m}],\\
\mathcal{G}_2 &= [I_{n+m}\ \ 0_{(n+m)\times m}],\\
\mathcal{I}_2 &= [0_{m\times(n+m)}\ \ I_m],\\
\mathcal{G}_3 &= [F(A+BF)\ \ -FB\ \ -I_m],
\end{aligned} \tag{10}$$

then the level set $\mathcal{E}_{\mathrm{Union}}$ is a contractively invariant set of system (4).

Proof: To prove the theorem, we need to show that the value of the Lyapunov function decreases towards zero along the system trajectory at each nonzero $x \in \mathcal{E}_{\mathrm{Union}}$. Since the proposed Lyapunov function V_G is continuous across the boundaries among the partitions, it is sufficient to verify that inequality $\dot{V}_G(x) < 0$ holds at each nonzero $x \in \mathcal{E}(P_{Gi}) \cap \Gamma_i$, $i \in I[1, 2^m]$.

Recall that $\xi = [x^{\mathrm{T}}\ \ \mathrm{dz}_{\mu_i}(Fx)^{\mathrm{T}}]^{\mathrm{T}}$ and $\phi_{\mu_i}(Fx)$ is the directional derivative of $\mathrm{dz}_{\mu_i}(Fx)$ at x along $\dot{x}$, $i \in I[1, 2^m]$. By the Schur complement, inequalities (9) is equivalent to $P_{Gi} \geq \frac{1}{\mu_i^2} h_{ij}^{\mathrm{T}} h_{ij}$. Thus, for each $x \in \mathcal{E}(P_{Gi})$, we have $|h_{ij}\xi| \leq \mu_i$. Let $v_i = H_i\xi$. Then, by Lemma 1, the regional sector condition is given as

$$\mathrm{dz}_{\mu_i}^{\mathrm{T}}(Fx)S_{1i}(Fx - \mathrm{dz}_{\mu_i}(Fx) + H_i\xi) \geq 0,$$

which is equivalent to

$$\Phi_{i1} = \eta^{\mathrm{T}}\mathcal{I}_1^{\mathrm{T}}S_{1i}(\mathcal{G}_1 + H_i\mathcal{G}_2)\eta \geq 0, \tag{11}$$

where $\eta = [x^{\mathrm{T}}\ \mathrm{dz}_{\mu_i}(Fx)^{\mathrm{T}}\ \phi_{\mu_i}(Fx)]^{\mathrm{T}}$.

Besides, the sector-like conditions introduced in Lemma 2 can be written as

$$\Phi_{i2} = \eta^{\mathrm{T}} \mathcal{I}_2^{\mathrm{T}} S_{2i} \mathcal{G}_3 \eta \equiv 0, \tag{12}$$

and

$$\Phi_{i3} = \eta^{\mathrm{T}} \mathcal{I}_1^{\mathrm{T}} S_{3i} \mathcal{G}_3 \eta \equiv 0, \tag{13}$$

respectively.

Note that the time derivative of $V_G(x)$ can be computed as

$$\begin{aligned} \dot{V}_G(x) &= 2\begin{bmatrix} x \\ \mathrm{dz}_{\mu_i}(Fx) \end{bmatrix}^{\mathrm{T}} P_{Gi} \begin{bmatrix} \dot{x} \\ \mathrm{dz}_{\mu_i}(Fx) \\ \phi_{\mu_i}(Fx) \end{bmatrix} \\ &= 2\xi^{\mathrm{T}} P_{Gi} \begin{bmatrix} A+BF & -B & 0_{n\times m} \\ 0_{m\times n} & 0_{m\times m} & I_m \end{bmatrix} \eta \\ &= \eta^{\mathrm{T}} \mathrm{He}(\mathcal{O} P_{Gi} \mathcal{A}) \eta. \end{aligned} \tag{14}$$

Then incorporating the sector and sector-like conditions (11), (12) and (13) into (14), we have

$$\dot{V}_G(x) \le \dot{V}_G(x) + \eta^{\mathrm{T}} \mathrm{He}\Big(\sum_{j=1}^{3} \Phi_{ij}\Big)\eta = \eta^{\mathrm{T}} \Theta_i \eta. \tag{15}$$

In view of inequalities (8), it can be verified that $\dot{V}_G(x) < 0$ for all nonzero $x \in \mathcal{E}(P_{Gi}) \cap \Gamma_i$, $i \in I[1, 2^m]$. This completes the proof. □

From inequalities (9) in Theorem 1, we can see that matrices P_{Gi}, $i \in I[1, 2^m]$, are required to be positive definite such that the positiveness of $V_G(x)$ can be guaranteed for all $x \in \mathbf{R}^n$. However, this seems to be unnecessarily conservative since it is sufficient to show that $V_{Gi}(x) > 0$ for each $x \in \Gamma_i$. Let $K_i \in \mathbf{R}^{(n+m)\times(n+m)}$ be a symmetric matrix whose elements are all non-negative. Recall that

$$E_i \xi = \begin{bmatrix} \Lambda_i F x \\ \Lambda_i \mathrm{dz}_{\mu_i}(Fx) \end{bmatrix}.$$

For each $x \in \Gamma_i$, $i \in I[1, 2^m]$, all elements of $E_i \xi$ are non-negative. Thus, we have

$$\xi^{\mathrm{T}} E_i^{\mathrm{T}} K_i E_i \xi \ge 0, i \in I[1, 2^m], \tag{16}$$

Then, the positiveness of $V_{Gi}(x) = \xi^{\mathrm{T}} P_{Gi} \xi$ can be guaranteed if

$$\xi^{\mathrm{T}} P_{Gi} \xi - \xi^{\mathrm{T}} E_i^{\mathrm{T}} K_i E_i \xi \ge 0,$$

which can be further ensured by

$$P_{Gi} - E_i^{\mathrm{T}} K_i E_i \ge 0. \tag{17}$$

It is clear from (17) that P_{Gi} is not required to be positive definite to guarantee $V_G(x) > 0$, $x \in \mathbf{R}^n \setminus \{0\}$. Based on the analysis above, we establish the following theorem.

Theorem 2: Consider system (4). If there exist symmetric matrices K_i, $M_i \in \mathbf{R}^{(n+m)\times(n+m)}$ with non-negative entries, symmetric matrix $T \in \mathbf{R}^{(3m+n)\times(3m+n)}$, diagonal matrices S_{1i}, S_{2i}, $S_{3i} \in \mathbf{R}^{m\times m}$ with $S_{1i} > 0$, and matrix $H_i \in \mathbf{R}^{m\times(n+m)}$, $i \in I[1, 2^m]$, such that the following inequalities hold,

$$\tilde{\Theta}_i = \Theta_i + \mathcal{O} E_i^{\mathrm{T}} M_i E_i \mathcal{O}^{\mathrm{T}} < 0, \tag{18}$$

and

$$\begin{bmatrix} \mu_i^2 & h_{ij} \\ \star & P_{Gi} - E_i^{\mathrm{T}} K_i E_i \end{bmatrix} > 0, \tag{19}$$

where h_{ij} is the j^{th} row of H_i and matrices Θ_i and $\mathcal{O}$ are as defined in (8) and (10), respectively, then the level set $\mathcal{E}_{\mathrm{Union}}$ is a contractively invariant set of system (4).

Proof: Recall the definition of E_i, ξ and η. It is clear that, for any $x \in \Gamma_i$, all the elements of $E_i \xi$ and $E_i \mathcal{O}^{\mathrm{T}} \eta$ are non-negative. Since each element in the symmetric matrices K_i and M_i is non-negative, for all $x \in \Gamma_i$, we have (16) and

$$\eta^{\mathrm{T}} \mathcal{O} E_i^{\mathrm{T}} M_i E_i \mathcal{O}^{\mathrm{T}} \eta \ge 0. \tag{20}$$

Noting that inequalities (19) imply that $P_{Gi} - E_i^{\mathrm{T}} K_i E_i > 0$, we have $\xi^{\mathrm{T}} P_{Gi} \xi - \xi^{\mathrm{T}} E_i^{\mathrm{T}} K_i E_i \xi > 0$. By conditions (16), the positive definiteness of $V_G(x)$ can be guaranteed. Furthermore, from (15) and (20), we have

$$\begin{aligned} \dot{V}_G(x) &\le \dot{V}_G(x) + \eta^{\mathrm{T}} \mathrm{He}\Big(\sum_{j=1}^{3} \Phi_{ij}\Big)\eta + \eta^{\mathrm{T}} \mathcal{O} E_i^{\mathrm{T}} M_i E_i \mathcal{O}^{\mathrm{T}} \eta \\ &= \eta^{\mathrm{T}} \tilde{\Theta}_i \eta. \end{aligned}$$

By inequalities (18), it can be verified that $\dot{V}_G(x) < 0$ for all nonzero $x \in \mathcal{E}(P_{Gi}) \cap \Gamma_i$, $i \in I[1, 2^m]$. Hence, the level set $\mathcal{E}_{\mathrm{Union}}$ is contractively invariant for system (4). □

Remark 2: If K_i and M_i are set to be zeros, Theorem 2 will reduce to Theorem 1. Moreover, conditions (16) and (20) play different roles in reducing the conservativeness of Theorem 1. conditions (16) are used to relax the positive definiteness constraint on P_{Gi}, while conditions (20) weaken the negative definiteness constraint on Θ_i, which is defined in (8).

Remark 3: Obviously, we can reduce our proposed Lyapunov function into a quadratic one by fixing some parts of T to be zeros. This implies that the selection of $(P_{Gi}, S_{1,i}, H_i)$ can be inherited from the results obtained by this quadratic Lyapunov function. Under such a selection of $(P_{Gi}, S_{1,i}, H_i)$, let $S_{3i} = 0_m$ and $K_i = M_i = 0_{n+m}$, and then we can always find a diagonal matrix $S_{2,i}$ with sufficiently large diagonal elements such that conditions in Theorems 1 and 2 are feasible [20].

IV. Optimization Problems

Theorems 1 and 2 present sufficient conditions under which the level set $\mathcal{E}_{\mathrm{Union}}$ is an estimate of the domain of attraction of system (4). Based on these conditions, we will formulate optimization problems to obtain the estimates as large as possible. Considering that P_{Gi} in Theorem 2 is not required to be positive definite, we need to find a subset of $\mathcal{E}_{\mathrm{Union}}$, which is characterized by some positive definite matrices, to measure the size of $\mathcal{E}_{\mathrm{Union}}$. The following proposition, which is modified from the one in [17], presents a condition under which a reference set is a subset of $\mathcal{E}(P_{Gi})$.

Proposition 1: Given a symmetric matrix $P_{Gi} \in \mathbf{R}^{(n+m)\times(n+m)}$. If there exists a symmetric matrix $L_i \in \mathbf{R}^{(n+m)\times(n+m)}$ with non-negative entries, a positive definite matrix $\hat{P}_i \in \mathbf{R}^{n\times n}$ and a diagonal positive definite matrix $W_i \in R^{m\times m}$ such that

$$\mathcal{I}_4^{\mathrm{T}} \hat{P}_i \mathcal{I}_4 - P_{Gi} - E_i^{\mathrm{T}} L_i E_i - \mathrm{He}\Big(\mathcal{I}_5^{\mathrm{T}} W_i \mathcal{G}_1\Big) > 0, \qquad (21)$$

where $\mathcal{I}_4 = [I_n \;\; 0_{n\times m}]$, $\mathcal{I}_5 = [0_{m\times n} \;\; I_m]$, $\mathcal{G}_1$ and E_i are as defined in (10) and (5), respectively, then $\mathcal{E}(\hat{P}_i) \cap \Gamma_i$ with $\mathcal{E}(\hat{P}_i) := \{x \in \mathbf{R}^n : x^{\mathrm{T}} \hat{P}_i x \le 1\}$ is a subset of $\mathcal{E}(P_{Gi})$.

Let $\alpha := \frac{1}{2^m} \sum_{i=1}^{2^m} \mathrm{tr}(\hat{P}_i)$. The minimization of α represents the maximization of $\cup_{i=1}^{2^m} \mathcal{E}(\hat{P}_i) \cap \Gamma_i$, which in turn implies the maximization of the level set $\mathcal{E}_{\mathrm{Union}}$. Then the optimization problem associated with Theorem 2 is formulated as follows,

$$\min_{P_{Gi}, \hat{P}_i > 0, S_{1i} > 0, S_{2i}, S_{3i}, H_i, K_i, M_i, L_i, i \in I[1,2^m]} \alpha, \qquad (22)$$
$$\text{s.t. Inequalities (18), (19) and (21).}$$

Because of the presence of $S_{1i} H_i$ in (18), the optimization problem (22) is a BMI problem. Let $Z_i = S_{1i} H_i$, and noting that $\mu_i^2 s_{1ij}^2 + \frac{1}{\mu_i^2} \ge 2 s_{1ij}$, we replace (19) with

$$\begin{bmatrix} 2 s_{1ij} - \frac{1}{\mu_i^2} & z_{ij} \\ \star & P_{Gi} - E_i^{\mathrm{T}} K_i E_i \end{bmatrix} > 0, \qquad (23)$$

where s_{1ij} is the j^{th} diagonal element of S_{1i} and z_{ij} is the j^{th} row of Z_i, $i \in I[1, 2^m]$, $j \in I[1, m]$. As a result, (22) reduces to the following LMI problem,

$$\min_{P_{Gi}, \hat{P}_i > 0, S_{1i} > 0, S_{2i}, S_{3i}, Z_i, K_i, M_i, L_i, i \in I[1,2^m]} \alpha, \qquad (24)$$
$$\text{s.t. Inequalities (18), (21) and (23).}$$

The optimal solution to (24) is a suboptimal solution of the BMI problem (22). Furthermore, by applying this optimal solution as the initial values, we develop the following direct-iterative algorithm for solving the optimization problem (22), and the result obtained by this iterative algorithm is at least as good as the result obtained by solving (24).

Algorithm 1 : Iterative Algorithm for Enlarging $\mathcal{E}_{\mathrm{Union}}$.

Step 1: Solve the LMI problem (24) and denote the optimal solution as $(\hat{\alpha}, P_{Gi}, \hat{P}_i, S_{1i}, S_{2i}, S_{3i}, Z_i, K_i, M_i, L_i)$. Let $t = 0$, $H_i = S_{1i}^{-1} Z_i$, $i \in I[1, 2^m]$ and $\alpha_0 = \hat{\alpha}$.
Step 2: Let $k \leftarrow k + 1$. Fix H_i and solve optimization problem (22). Denote the optimal solution as $(\hat{\alpha}, P_{Gi}, \hat{P}_i, S_{1i}, S_{2i}, S_{3i}, K_i, M_i, L_i)$.
Step 3: Fix S_{1i} and solve optimization problem (22). Denote the optimal solution as $(\hat{\alpha}, P_{Gi}, \hat{P}_i, S_{2i}, S_{3i}, H_i, K_i, M_i, L_i)$. Let $\alpha_k = \hat{\alpha}$.
Step 4: If $\mid \alpha_k - \alpha_{k-1} \mid \le \theta$, a pre-determined tolerance, stop, then P_{Gi}, $i \in I[1, 2^m]$, are feasible solutions. Else go to Step 2.

Remark 4: Note that Theorem 2 reduces to Theorem 1 when $K_i = M_i = 0$. Thus, setting K_i and M_i to be zeros in the optimization problems (22) and (24), we obtain the corresponding BMI-based and LMI-based optimization problems, respectively, with their constraints obtained from Theorem 1. Correspondingly, by setting $K_i = M_i = 0$, Algorithm 1 can also be used to solve the resulting BMI-based optimization problem from (22).

V. Numerical Examples

Consider the following system taken from [7],

$$A = \begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix}, \; B = \begin{bmatrix} 0.8030 & 0.9455 \\ 0.0839 & 0.9159 \end{bmatrix},$$
$$F = \begin{bmatrix} -1.2031 & 1.0926 \\ -0.4441 & -1.5447 \end{bmatrix},$$
$$\overline{\mu} = [2 \;\; 1]^{\mathrm{T}}, \;\; \underline{\mu} = [1 \;\; 2]^{\mathrm{T}}.$$

We first estimate the domain of attraction by solving the LMI-based optimization problems, corresponding to both Theorems 1 and 2. To do this, we solve the LMI-based optimization problem (24) with $K_i = M_i = 0$, which is formulated according to Theorem 1, and obtain the estimate $\mathcal{E}_{\mathrm{Union}}^{\text{Th1-LMI}}$ with $\alpha_{\mathrm{opt}}^{\text{Th1-LMI}} = 0.8910$.

Then solving the optimization problem (24) whose constraints are derived from Theorem 2, we obtain the estimate $\mathcal{E}_{\mathrm{Union}}^{\text{Th2-LMI}}$ with $\alpha_{\mathrm{opt}}^{\text{Th2-LMI}} = 0.6659$ and matrix P_{G1} with its eigenvalues being $\{0.7600, 0.1470, 0.0027, -0.0001\}$. It is clear that P_{G1} is not positive definite.

For comparison, by using the asymmetric Lyapunov function presented in [7], we get the estimate $\mathcal{E}_{\mathrm{Union1}}^{[7]}$ with $\alpha_{\mathrm{opt1}}^{[7]} = 0.9027$. Also, if we use the asymmetric Lyapunov function in [7] as the basic construction and then incorporate the sector conditions into the construction such that the associated matrices are not required to be positive definite ([6], [17]), the resulting estimate $\mathcal{E}_{\mathrm{Union1}}^{[6,17]}$ is with $\alpha_{\mathrm{opt1}}^{[6,17]} = 1.1449$.

We plot these estimates and the actual domain of attraction in Fig. 1. As apparent in Fig. 1, the estimates based on the methods proposed in this paper, especially Theorem 2, are significantly larger than the existing approaches.

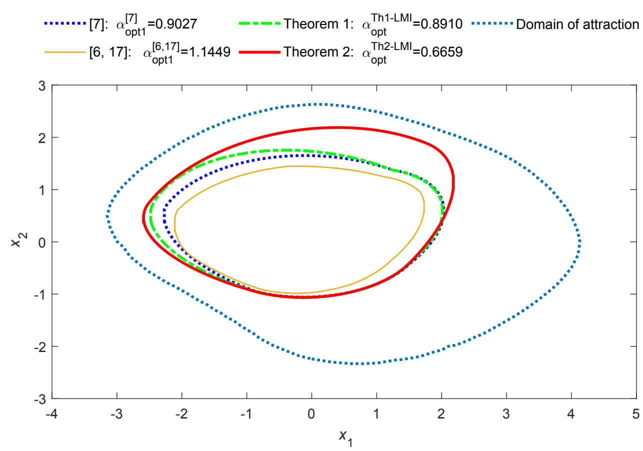

Fig. 1. The estimates of the domain of attraction obtained by LMI-based optimization problems.

We next estimate the domain of attraction by solving the BMI-based optimization problems, corresponding to both Theorem 1 and Theorem 2. To obtain estimation based on

Theorem 1, we let K_i and M_i be zeros and choose the optimal solution of the resulting LMI-based problem (24) as the initial values for the resulting BMI problem (22). Carrying out the resulting Algorithm 1, we obtain $\mathcal{E}_{\text{Union}}^{\text{Th1-BMI}}$ with $\alpha_{\text{opt}}^{\text{Th1-BMI}} = 0.7463$.

We also carry out Algorithm 1 associated with Theorem 2, and obtain $\mathcal{E}_{\text{Union}}^{\text{Th2-BMI}}$ with $\alpha_{\text{opt}}^{\text{Th2-BMI}} = 0.5377$. Specially, the associated matrix P_{G1} has eigenvalues $\{0.7405, 0.1448, 0.0025, -0.0001\}$, that is, P_{G1} is not positive definite. Moreover, numerical computation shows that each matrix Θ_i, $i \in I[1,4]$, in (18) has positive eigenvalues. This reflects the discussion in Remark 2 that Theorem 2 leads to less conservative results than Theorem 1.

For further comparison, we apply the same iterative strategy to solve the BMI problems formulated by using the Lyapunov functions in [7] and [6], [17]. We obtain $\mathcal{E}_{\text{Union2}}^{[7]}$ with $\alpha_{\text{opt2}}^{[7]} = 0.7566$ and $\mathcal{E}_{\text{Union2}}^{[6,17]}$ with $\alpha_{\text{opt2}}^{[6,17]} = 0.6880$, respectively. Besides, in order to show the superiority of piece-wise Lyapunov functions used in this paper, we reformulate Theorems 1 and 2 using the quadratic Lyapunov functions suggested in [16], based on which we then establish and solve the BMI problems to obtain quadratic-Lyapunov-function-based estimates $\mathcal{E}_{\text{Union}}^{\text{Th1-Q}}$ with $\alpha_{\text{opt}}^{\text{Th1-Q}} = 0.7419$ and $\mathcal{E}_{\text{Union}}^{\text{Th2-Q}}$ with $\alpha_{\text{opt}}^{\text{Th2-Q}} = 0.6493$, respectively. All of these results and the actual domain of attraction are depicted in Fig. 2. As observed in Fig. 2, the proposed approach contributes to improving the estimates of the domain of attraction for linear systems with asymmetric actuator saturation. Also, it is apparent that the piece-wise Lyapunov function used in this paper for each subspace yields less conservatism than the quadratic one in [16].

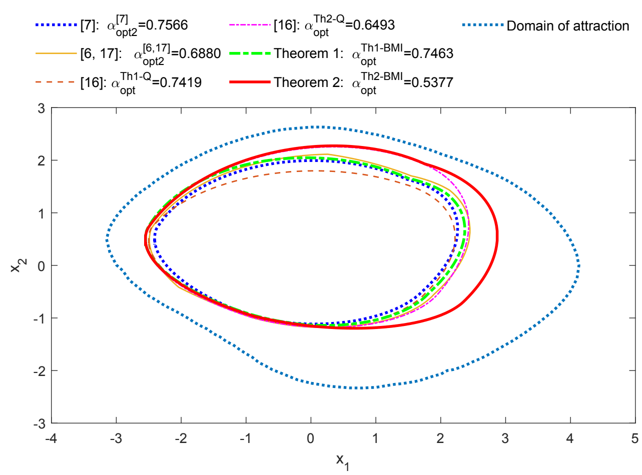

Fig. 2. The estimates of the domain of attraction obtained by BMI-based optimization problems.

VI. Conclusions

This paper proposes a generalized asymmetric Lyapunov function for the estimation of the domain of attraction for linear systems subject to asymmetric actuator saturation. Such a Lyapunov function is less conservative as the associated matrices are not required to be positive definite. Based on the proposed Lyapunov function, sufficient conditions for stability analysis are established and then relaxed conditions are provided to further reduce the conservativeness. The estimation of the domain of attraction is formulated as an optimization problem. An iterative algorithm is developed to solve the optimization problem. Simulation results show that our approach has the ability to significantly enlarge the estimate of the domain of attraction.

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