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Superposition theorems for input-to-output stability of infinite dimensional systems

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Abstract— **We characterize input-to-output stability of a general class of both continuous-time and discrete-time infinite dimensional systems in terms of weaker stability properties. Our results generalize the corresponding criteria for ordinary differential equations achieved by Ingalls et al. [1] and those for infinite dimensional systems for which the output equals the state [2]. This way, we investigate the relation between several stability and attractivity properties for infinite dimensional systems with outputs by providing the according implications and giving counterexamples, respectively.**

Index Terms— **Distributed parameter systems; Stability of nonlinear systems; Nonlinear systems; Input-to-state stability; Input-to-output stability**

I. Introduction

Input-to-state stability (ISS) was first introduced for systems of ordinary differential equations (ODEs) [3], and then developed for other classes of finite-dimensional control systems such as switched [4], hybrid [5], and impulsive systems [6]. More recently, the ISS theory was extended to infinite dimensional systems, including time-delay systems [7], [8], partial differential equations (PDEs) [9] and general evolution equations in Banach spaces [10], [11]. For more details, we refer to the survey [11].

Yet, these developments are confined to systems with full-state output. The notion of input-to-output stability (IOS) introduced for ODE systems in [12] extends ISS to output systems. IOS combines the uniform global asymptotic stability of the output dynamics with its robustness w.r.t. external inputs. If the output equals to the state, IOS and ISS coincide.

For finite-dimensional systems, the IOS theory is quite rich. Lyapunov characterizations of IOS have been shown in [13] based on some earlier developments in [12]. A so-called IOS superposition theorem was obtained in [1]. It states that a forward-complete ODE system satisfying both output Lagrange stability (OL) and output-limit property (OLIM), is necessarily IOS. For the special case of ISS, a corresponding result has been shown in [14] and extended to infinite dimensional systems in [2].

Trajectory-based small-gain theorems for interconnections of two IOS systems have been obtained in [15] and generalized to interconnections of n IOS systems in [16]. Lyapunov-based small-gain theorems for couplings of $n \in \mathbb{N}$ interconnected IOS systems have been reported in [17, Sec. 3.3.4].

IOS is paramount in numerous applications including multi-agent systems [18], coverage controllers [19] and neural networks [20].

In time-delay context, IOS for infinite dimensional systems serves for controller design in networked systems, which is applied to teleoperating systems, though in this case only weaker than IOS properties for the control system are obtained [21]. The work [22] develops finite-dimensional observer-based controllers for a linear reaction-diffusion system. In [23], [24], small-gain theorems for the so-called maximum formulation of the IOS property are presented. For time-delay systems, Lyapunov characterizations of IOS were developed (cf. [8]).

Nevertheless, despite its practical relevance, infinite dimensional IOS theory remains largely unexplored [11].

In the following, we characterize the IOS property for infinite dimensional systems in terms of weaker properties, such as the output-uniform asymptotic gain property (OUAG), output-uniform local stability (OULS), output continuity at the equilibrium point (OCEP) and other notions. Furthermore, we consider the influence of output-Lagrange stability (OL) on the IOS property by establishing a superposition theorem for systems which are OL and IOS. We provide a superposition theorem for OL. We point out differences between the ISS case and general IOS case by several (counter)examples.

The characterizations from [1] cannot be extended straight forwardly to infinite dimensional systems. In [2, Ex. 1], it is shown directly that the output-limit property (OLIM) and OL are not sufficient to imply IOS for the case of full-state output-linear infinite dimensional systems. Similarly, the IOS characterization for finite-dimensional systems in terms of OL and the output-asymptotic gain property (OAG) cannot be extended to the infinite dimensional setting in the same formulation, even in the ISS case (i.e., if $h(x, u) = x$), because trajectory-wise asymptotic stability does not imply uniform asymptotic stability as argued in [2, Lem. 9].

A different problem arises due to the fact that nonlinear forward complete infinite dimensional systems do not necessarily have bounded reachability sets, in contrast to nonlinear ODE systems [2]. As we discuss in [25, Sec. VI], one of the consequences of this problem is the breakdown of the equivalence between several types of uniform asymptotic gain properties, in contrary to the finite dimensional case. In view of this, the investigation of the IOS of *infinite dimensional nonlinear systems* becomes challenging.

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IOS superposition theorems are a meta-tool that helps to prove other important theoretical results including Lyapunov theory and small-gain theorems. Recently, in [26], ISS characterizations have been used to prove Lyapunov-Krasovskii theorems with pointwise dissipation for ISS of nonlinear time-delay systems. Our IOS characterizations can be a basis that will help to extend those results to IOS Lyapunov-Krasovskii theorems.

These IOS characterizations can be applied to extend a small-gain theorem to infinite networks of infinite dimensional IOS subsystems. For ISS, [27] provided such a general small-gain theorem based on the ISS superposition theorems [2]. Our work will serve as a basis for IOS small-gain theorems for infinite networks. Furthermore, one could formulate stronger IOS small-gain theorems for time-delay systems, which will go far beyond existing results even in the ISS case.

Due to the page limit, most of the proofs are omitted here. A preprint of the version of this work with further results and insights, detailed proofs, and counterexamples can be found in [25], to be submitted to a journal.

We denote the nonnegative integers by $\mathbb{N}_0$, the natural numbers by $\mathbb{N}$, the real numbers by $\mathbb{R}$, the nonnegative real numbers by $\mathbb{R}_0^+$ and the balls of radius r around zero in Banach spaces X, U and $\mathcal{U}$, respectively, by B_r, $B_{r,U}$ and $B_{r,\mathcal{U}}$. We define the standard classes of comparison functions (cf. [28, p. xvi]) by

$$\begin{aligned}
\mathcal{K} &:= \{\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \,|\, \gamma(0) = 0,\ \gamma \text{ is continuous} \\
&\qquad \text{and strictly increasing}\},\\
\mathcal{K}_\infty &:= \{\gamma \in \mathcal{K} \,|\, \gamma \text{ is unbounded}\},\\
\mathcal{L} &:= \{\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \,|\, \gamma \text{ is continuous and decreasing} \\
&\qquad \text{with } \lim_{t\to\infty} \gamma(t) = 0\},\\
\mathcal{KL} &:= \{\beta \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+ \,|\, \beta(\,\cdot\,, t) \in \mathcal{K},\ \forall t \geq 0,\\
&\qquad \beta(r, \,\cdot\,) \in \mathcal{L},\ \forall r > 0\}.
\end{aligned}$$

II. Preliminaries

Definition 1: Consider a quadruple $\Sigma = (I, X, \mathcal{U}, \phi)$ consisting of:

1) A *time set* $I \in \{\mathbb{N}_0, \mathbb{R}_0^+\}$.
2) A normed vector space $(X, \|\cdot\|_X)$, called the *state space*.
3) A vector space U of input values and a normed vector space of inputs $(\mathcal{U}, \|\cdot\|_{\mathcal{U}})$, where $\mathcal{U}$ is a linear subspace of $\{u \,|\, u : I \to U\}$. We assume that the following invariance axioms hold:
   - *Axiom of shift invariance*: for all $u \in \mathcal{U}$ and all $\tau \in I$, the time-shifted function $u(\,\cdot\, + \tau)$ belongs to $\mathcal{U}$ with $\|u\|_{\mathcal{U}} \geq \|u(\,\cdot\, + \tau)\|_{\mathcal{U}}$.
   - *Axiom of restriction invariance*: for each $u \in \mathcal{U}$ and for all $t_2 \geq t_1 \geq 0$ the restriction of u to time interval $[t_1, t_2]$ given by $u|_{[t_1,t_2]}$ belongs to $\mathcal{U}$ and $\|u|_{[t_1,t_2]}\|_{\mathcal{U}} \leq \|u\|_{\mathcal{U}}$.
4) A map $\phi\colon D_\phi \to X$, $D_\phi \subset I \times X \times \mathcal{U}$, called *transition map*, so that for all $(x, u) \in X \times U$ it holds that $D_\phi \cap (I \times \{(x, u)\}) = [0, t_m) \times \{(x, u)\}$, for a certain $t_m = t_m(x, u) \in (0, +\infty]$. The corresponding interval $[0, t_m)$ is called the *maximal domain of definition* of the mapping $t \mapsto \phi(t, x, u)$, which we call a *trajectory* of the system.

The quadruple Σ is called a *(control) system* if it satisfies the following axioms:

(Σ1) *Identity property*: for all $(x, u) \in X \times U$, it holds that $\phi(0, x, u) = x$.

(Σ2) *Causality*: for all $(t, x, u) \in D_\phi$ and $\widetilde{u} \in \mathcal{U}$ such that $u(s) = \widetilde{u}(s)$ for all $s \in [0, t]$, it holds that $[0, t] \times \{(x, \widetilde{u})\} \subset D_\phi$ and $\phi(t, x, u) = \phi(t, x, \widetilde{u})$.

(Σ3) *Cocycle property*: for all $x \in X$, $u \in \mathcal{U}$ and $t, s \geq 0$ so that $[0, t + s] \times \{(x, u)\} \subset D_\phi$, we have $\phi(t + s, x, u) = \phi(s, \phi(t, x, u), u(t + \,\cdot\,))$.

Definition 2: A (time-invariant) *control system with outputs* $\Sigma := (I, X, \mathcal{U}, \phi, Y, h)$ is given by an abstract control system $(I, X, \mathcal{U}, \phi)$ together with

1) a normed vector space $(Y, \|\cdot\|_Y)$ called the *output-value space* or *measurement-value space*; and
2) a map $h\colon X \times U \to Y$, called the *output* (or: *measurement*) *map*.

We also denote $y(\,\cdot\,, x, u) := h(\phi(\,\cdot\,, x, u), u(\,\cdot\,))$ for all $(x, u) \in X \times \mathcal{U}$.

The following definition is taken from [2].

Definition 3: We call a control system $(I, X, \mathcal{U}, \phi)$ *forward complete (FC)*, if for each $x \in X$, $u \in \mathcal{U}$ and $t \in I$ the value $\phi(t, x, u) \in X$ is well-defined.

In the following, we always consider a forward complete control system with outputs $\Sigma = (I, X, \mathcal{U}, \phi, Y, h)$.

Definition 4: We call Σ *output continuous at the equilibrium point (OCEP)* if for every $\tau \in I$ and every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, \tau) > 0$ such that

$$t \in I\colon t \leq \tau,\ \|x\|_X \leq \delta,\ \|u\|_{\mathcal{U}} \leq \delta \implies \|y(t, x, u)\|_Y \leq \varepsilon.$$

Definition 5: Σ is said to have *bounded output reachability sets (BORS)* if for all $C > 0$ and $\tau \in I$ it holds that

$$\sup_{\|x\|_X,\ \|u\|_{\mathcal{U}} < C,\ t < \tau} \|y(t, x, u)\|_Y < \infty.$$

Definition 6: We call the output map h *$\mathcal{K}$-bounded* if there are $\sigma_1, \gamma_1 \in \mathcal{K}$ so that for all $x \in X$ and all $u \in \mathcal{U}$ we have

$$\|h(x, u)\|_Y \leq \sigma_1(\|x\|_X) + \gamma_1(\|u\|_{\mathcal{U}}). \tag{1}$$

Let us define the main concept of this paper.

Definition 7: Σ is called *input-to-output stable (IOS)*, if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that $\forall x \in X$, $\forall u \in \mathcal{U}$ the following holds:

$$\|y(t, x, u)\|_Y \leq \beta(\|x\|_X, t) + \gamma(\|u\|_{\mathcal{U}}), \qquad t \in I. \tag{2}$$

The concept of IOS was introduced in [15], generalizing input-to-state stability as given in [3].

Definition 8: Σ is called *input-to-state stable (ISS)*, if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that $\forall x \in X$, $\forall u \in \mathcal{U}$ the following holds:

$$\|\phi(t, x, u)\|_X \leq \beta(\|x\|_X, t) + \gamma(\|u\|_{\mathcal{U}}), \qquad t \in I. \tag{3}$$

Remark 9: A special case of output systems is given for $Y = X$, $h(x, u) \equiv x$ and $y(t, x, u) = \phi(t, x, u)$ for all $t \in I$,

$x \in X$ and $u \in \mathcal{U}$. We call such systems as *systems with full-state output*. For such systems, IOS reduces to ISS.

A. Stability properties

In this section, we introduce several stability properties needed for the characterization of IOS.

Definition 10 ([12]): We call Σ *output Lagrange stable (OL)* if there exist $\sigma, \gamma \in \mathcal{K}_\infty$ such that for all $x \in X$ and $u \in \mathcal{U}$, it holds that

$$\|y(t,x,u)\|_Y \leq \sigma(\|y(0,x,u)\|_Y) + \gamma(\|u\|_{\mathcal{U}}), \qquad t \in I.$$

We call Σ *locally output Lagrange stable (locally OL)* if there exist $\sigma, \gamma \in \mathcal{K}_\infty$ and $r > 0$ such that for all $x \in X$ and $u \in B_{r,\mathcal{U}}$ such that $\|y(0,x,u)\| < r$, it holds that

$$\|y(t,x,u)\|_Y \leq \sigma(\|y(0,x,u)\|_Y) + \gamma(\|u\|_{\mathcal{U}}), \qquad t \in I.$$

The following notions generalize the classical concepts of uniform local/global stability (cf. [2]) to systems with outputs.

Definition 11: We call system Σ

1) *output-uniformly locally stable (OULS)* if there exist $r > 0$ and $\sigma, \gamma \in \mathcal{K}_\infty$ such that for all $x \in B_r$ and $u \in B_{r,\mathcal{U}}$, it holds that

$$\|y(t,x,u)\|_Y \leq \sigma(\|x\|_X) + \gamma(\|u\|_{\mathcal{U}}), \qquad t \in I.$$

2) *output-uniformly globally stable (OUGS)* if there exist $\sigma, \gamma \in \mathcal{K}_\infty$ such that for all $x \in X$ and all $u \in \mathcal{U}$, it holds that

$$\|y(t,x,u)\|_Y \leq \sigma(\|x\|_X) + \gamma(\|u\|_{\mathcal{U}}), \qquad t \in I.$$

An equivalent characterization of local OL and OULS in ε-δ-notation is given by the following

Lemma 12: Consider a control system with outputs $\Sigma = (I, X, \mathcal{U}, \phi, Y, h)$.

1) Σ is locally OL if and only if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|y(0,x,u)\|_Y \leq \delta, \quad \|u\|_{\mathcal{U}} \leq \delta, \quad t \in I \implies \|y(t,x,u)\|_Y \leq \varepsilon.$$

2) System Σ is OULS if and only if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|x\|_X \leq \delta,\ \|u\|_{\mathcal{U}} \leq \delta,\ t \in I \implies \|y(t,x,u)\|_Y \leq \varepsilon.$$

Proof: The proof is analogous to the proof of [2, Lem. 2]. ■

The notions of OULS and local OL (OUGS and OL) coincide for systems with full-state output. For systems with full-state output, OULS and local OL become uniform local stability (ULS), OUGS and OL are the same as uniform global stability (UGS). Similarly, many of the other notions are derived from a concept for systems with full-state output which has the same name except the word *output* in the beginning.

B. Attractivity properties

Following [1], we define several attractivity-like properties for systems with inputs and outputs, and use them to characterize IOS.

Definition 13: Σ has the

1) *output-global uniform asymptotic gain property (OGUAG)* if there exists $\gamma \in \mathcal{K}_\infty$ such that for every $\varepsilon > 0$, and every $r > 0$ there exists $\tau = \tau(\varepsilon, r) \in I$ such that

$$\|y(t,x,u)\|_Y \leq \varepsilon + \gamma(\|u\|_{\mathcal{U}}), \quad x \in B_r,\ u \in \mathcal{U},\ t \in I\colon t \geq \tau.$$

2) *output-uniform asymptotic gain property (OUAG)* if there exists $\gamma \in \mathcal{K}_\infty$ such that for every $\varepsilon, r, s > 0$ there exists $\tau = \tau(\varepsilon, r, s) \in I$ such that

$$\|y(t,x,u)\|_Y \leq \varepsilon + \gamma(\|u\|_{\mathcal{U}}), \quad x \in B_r,\ u \in B_{s,\mathcal{U}},\ t \in I\colon t \geq \tau.$$

A system is OGUAG and OUAG, respectively, if all outputs converge to the ball with radius $\gamma(\|u\|_\infty)$. The difference between the two is that for OUAG, the convergence rate is dependent on the norm of the input and the norm of the state of the system, and for OGUAG it depends on the norm of the state, but not on the applied input.

C. Weak attractivity properties

Weak attractivity for dynamical systems was introduced in [29]. Its extension of the limit property (LIM) to control systems with full-state output [14] is essential for ISS superposition theorems. To characterize ISS for infinite dimensional systems, several variations of the LIM property have been introduced in [2]. We extend these notions to systems with general outputs.

Definition 14: Σ is said to possess the *output-limit property (OLIM)* if there exists $\gamma \in \mathcal{K}_\infty$ such that for all $\varepsilon > 0$, all $x \in X$ and all $u \in \mathcal{U}$ there exists a $t = t(\varepsilon, x, u) \in I$, such that

$$\|y(t,x,u)\|_Y \leq \varepsilon + \gamma(\|u\|_{\mathcal{U}}).$$

In other words, system Σ is OLIM, if for any input u and any initial state, its output function can approach the ball of radius $\gamma(\|u\|_{\mathcal{U}})$ arbitrarily close.

As shown in [2, Ex. 1] for the special case of ISS, OLIM and OL are in general not sufficient to imply IOS for infinite dimensional systems. Therefore, we introduce the following new notions, which are stronger as compared to OLIM.

Definition 15: We say Σ possesses the *output-global uniform limit property (OGULIM)* if there exists $\gamma \in \mathcal{K}_\infty$ such that for all $\varepsilon, r > 0$ there exists $\tau = \tau(\varepsilon, r) \in I$ such that for all $x \in B_r$ and all $u \in \mathcal{U}$ there exists $t \in I$, $t \leq \tau$ such that

$$\|y(t,x,u)\|_Y \leq \varepsilon + \gamma(\|u\|_{\mathcal{U}}).$$

Definition 16: We say Σ possesses the *output-uniform limit property (OULIM)* if there exists $\gamma \in \mathcal{K}_\infty$ such that for all $\varepsilon, r, s > 0$ there exists $\tau = \tau(\varepsilon, r, s) \in I$ such that

TABLE I

LIST OF SYSTEM PROPERTIES AND ABBREVIATIONS

Abbr.	Property	Def.
BORS	bounded output reachability sets	5
FC	forward completeness	3
IOS	input-to-output stability	7
ISS	input-to-state stability	8
local OL	local output Lagrange stability	10
OBORS	output-bounded output reachability sets	22
OCEP	output continuity at the equilibrium point	4
OGUAG	output-global uniform asymptotic gain property	13
OGULIM	output-global uniform limit property	15
OL	output Lagrange stability	10
OLIM	output-limit property	14
OOULIM	output-to-output uniform limit property	21
OUAG	output-uniform asymptotic gain property	13
OUGS	output-uniform global stability	11
OULIM	output-uniform limit property	16
OULS	output-uniform local stability	11

for all $x \in B_r$ and all $u \in B_{s,\mathcal{U}}$ there exists $t \in I$, $t \leq \tau$ such that

$$\|y(t,x,u)\|_Y \leq \varepsilon + \gamma(\|u\|_{\mathcal{U}}) .$$

In the case of OLIM, the approaching speed towards the ball of radius $\gamma(\|u\|_{\mathcal{U}})$ depends on input and initial state. For OULIM, this speed only depends on the norm of the input and the initial state. And in the case of OGULIM, the speed of approach is also uniform in the input and does only depend on the norm of the initial state.

III. MAIN RESULTS

The main result of this paper is summarized in Figure 1. First, we establish several equivalent characterizations of IOS in Theorem 17. Then, we will show equivalences for IOS $\wedge$ OL in Proposition 19. By Example 25, it becomes clear that the notions of IOS and OL are independent of each other. Furthermore, from Lemma 18, it follows that IOS implies OULIM $\wedge$ OUGS, but the converse implication does not hold true in general as explained in Example 26.

A. IOS superposition theorem

We start by stating the following characterization of IOS.

Theorem 17 (IOS superposition theorem): Let $\Sigma = (I, X, \mathcal{U}, \phi, Y, h)$ be a forward complete control system with outputs. Then, the following statements are equivalent:

1) Σ is IOS.
2) Σ is OUAG, OCEP and BORS.
3) Σ is OUAG, OULS and BORS.
4) Σ is OUAG and OUGS.

Next, we present the technical lemmas, which we use in the proof of Theorem 17.

Lemma 18: Let $\Sigma = (I, X, \mathcal{U}, \phi, Y, h)$ be a forward complete control system with outputs. Then, the implications depicted in Figure 2 hold true.

Proposition 19 (IOS $\wedge$ OL superposition theorem): Let $\Sigma = (I, X, \mathcal{U}, \phi, Y, h)$ be a forward complete control system with outputs. Then the following statements are equivalent:

1) Σ is IOS and OL.
2) Σ is OUAG, OL, and h is $\mathcal{K}$-bounded.
3) Σ is OULIM, OL, and h is $\mathcal{K}$-bounded.

The proofs of the previous results are omitted due to the restriction of the page limit. Detailed proofs can be found in the preprint of the journal version [25].

B. Full-state output case

As a corollary of Theorem 17, we obtain the ISS superposition theorem proved in [2, Thm. 5]. We refer to [2] for the definition of the corresponding notions.

Corollary 20: Consider a system Σ with full-state output. Then the following statements are equivalent:

1) Σ is ISS.
2) Σ is UAG $\wedge$ CEP $\wedge$ BRS.
3) Σ is ULIM $\wedge$ UGS.
4) Σ is ULIM $\wedge$ ULS $\wedge$ BRS.

Proof: Theorem 17 states the equivalence ISS $\iff$ UAG $\wedge$ CEP $\wedge$ BRS as all of these notions for systems with outputs reduce accordingly.

Next, OL defines stability on the output-value space which is equivalent to UGS for systems with full-state output. As ISS already implies UGS, Proposition 19 is a strict generalization of the equivalence ISS $\iff$ ULIM $\wedge$ UGS to systems with outputs.

By Lemma 23, we have

$$\text{OOULIM} \wedge \text{local OL} \wedge \text{OBORS} \implies \text{OL},$$

which for systems with full-state output reads precisely as ULIM $\wedge$ ULS $\wedge$ BRS $\implies$ UGS. The converse implication UGS $\implies$ ULS $\wedge$ BRS follows from Lemma 18. ■

IV. SUFFICIENT CONDITION FOR OL

In the following, we derive sufficient conditions for the OL property. To this aim, we introduce a modified version of OULIM and OGULIM. The difference between the newly defined OOULIM as compared to OULIM and OGULIM lies in the choice of the uniformliness with respect to the initial condition. For OOULIM, the initial condition x is chosen such that the output $y(0,x,u)$ is in a bounded ball whereas for OULIM and OGULIM the initial state x itself is bounded. Similarly, we modify BORS.

Definition 21: We say Σ possesses the *output-to-output uniform limit property (OOULIM)* if there exists $\gamma \in \mathcal{K}_\infty$ such that for all $\varepsilon, r, s > 0$ there exists $\tau = \tau(\varepsilon, r, s) \in I$ such that for all $x \in X$ and all $u \in B_{s,\mathcal{U}}$ such that $y(0,x,u) \in B_{r,Y}$, there exists $t \in I$, $t \leq \tau$ satisfying

$$\|y(t,x,u)\|_Y \leq \varepsilon + \gamma(\|u\|_{\mathcal{U}}) .$$

In the following, we derive sufficient conditions for the OL property.

Definition 22: System Σ is said to have *output-bounded output reachability sets (OBORS)* if for all $C > 0$ and $\tau \in I$ it holds that

$$\sup_{x \in X, \|u\|_{\mathcal{U}}, \|y(0,x,u)\|_Y < C,\ t < \tau} \|y(t,x,u)\|_Y < \infty.$$

Lemma 23: Let $\Sigma = (I, X, \mathcal{U}, \phi, Y, h)$ be a forward complete control system with outputs. Let Σ be OOULIM, locally OL and OBORS. Then, Σ is OL.

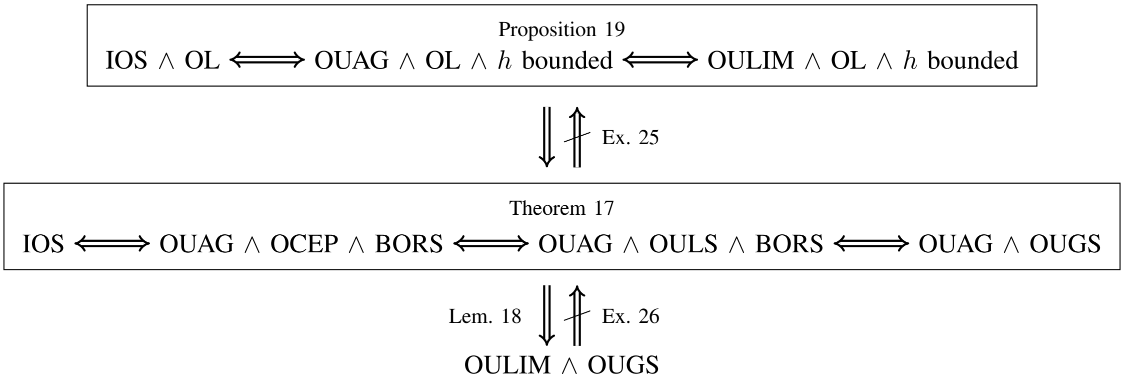

Fig. 1. Diagram of implications.

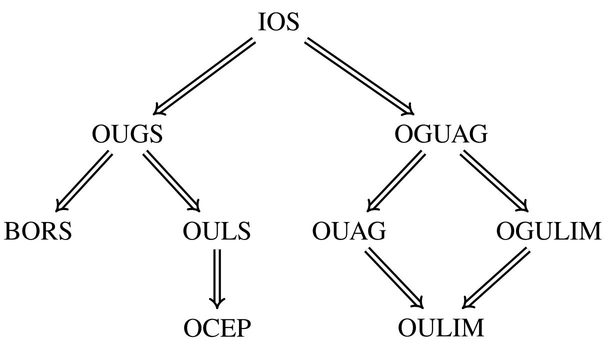

Fig. 2. Diagram of implications of Lemma 18.

V. Counterexamples

In this section, we provide several counterexamples to show that certain implications do not hold.

Remark 24: In [2, Ex. 1], it is shown that in general OL ∧ OLIM $\not\Longrightarrow$ IOS even in the case of full-state output.

Example 25 (IOS $\not\Longrightarrow$ OL): Let us consider the following system with a scalar state and output

$$\Sigma\colon \quad \dot{x} = -x, \quad y(t, x_0) = \sin(\phi(t, x_0)).$$

where $\phi = \phi(t, x_0)$ is the transition map (independent of u) of system Σ as given in Definition 1.

This system is IOS since

$$|y(t, x_0)| = |\sin(\phi(t, x_0))| \le |\phi(t, x_0)| = e^{-t} |x_0| .$$

However, for $x_0 = \pi$, $u \equiv 0$ it follows that $y(0, x_0) = 0$ but $y(1, x_0) = \sin(\pi e^{-1}) \neq 0$. Therefore, the system is not OL.

For systems with full-state output, the notions of OL and OUGS both reduce to UGS and both OGULIM and OOULIM become ULIM. However, these notions differ for general output systems and the implication ISS $\Longleftrightarrow$ ULIM ∧ UGS (Cor. 20) cannot be extended to output systems in a naive way as stated in the following example.

Example 26: We show the following:
OGULIM ∧ OOULIM ∧ OUGS $\not\Longrightarrow$ IOS ∨ OL. We consider a two-dimensional uncontrolled system with state $x = (x_1, x_2)^T \in \mathbb{R}^2$ given in polar coordinates $r = \sqrt{x_1^2 + x_2^2} = \|x\|_2$ and $\theta = \arg(x_1 + ix_2)$ by

$$\Sigma\colon \quad \dot{\theta} = 1, \quad \dot{r} = 0, \quad y(t, x) = \phi_1(t, x)$$

with transition map (in Cartesian coordinates) $\phi(\,\cdot\,, x_0) = (\phi_1(\,\cdot\,, x_0), \phi_2(\,\cdot\,, x_0))^T$ of Σ corresponding to the initial condition x_0 represented by (θ_0, r_0) in polar coordinates. The system Σ is OGULIM and OOULIM as it holds that

$$\phi(t, x_0) = \begin{pmatrix} r_0 \cos(t + \theta_0) \\ r_0 \sin(t + \theta_0) \end{pmatrix},$$

i.e., $y(t, x_0) = 0$ for $t \in \pi\left(\mathbb{N}_0 + \frac{1}{2}\right) - \theta_0$.

Hence, we can choose the uniform bound $\tau = \pi$ for which for any initial condition x_0 there exists $t \le \tau$ such that $y(t, x_0) = 0$, which implies OGULIM and OOULIM.

Moreover, Σ is OUGS as $|y(t, x_0)| \le \|\phi(t, x_0)\|_2 \le r_0$ $\forall t \in I$ but it is not IOS as $y(t, x_0) = r_0$ for $t = 2\pi\mathbb{N} - \theta_0$.

Furthermore, the system is not OL as for $x_0 = (0, 1)^T$ it holds that $y(0, x_0) = 0$, but $y(\frac{3}{2}\pi, x_0) = 1$.

Also, the implication ULIM ∧ ULS ∧ BRS $\Longrightarrow$ ISS or even ULIM ∧ ULS ∧ BRS $\Longrightarrow$ UGS cannot be generalized to output systems as stated in the following example.

Example 27 (OGULIM ∧ local OL ∧ OBORS $\not\Longrightarrow$ OL): We consider the uncontrolled system $x = (x_1, x_2)^T \in \mathbb{R}^2$ with polar coordinates $r = \sqrt{x_1^2 + x_2^2} = \|x\|_2$, $\theta = \arg(x_1 + ix_2)$ given by

$$\dot{\theta} = \operatorname{sat}\left(\tfrac{1}{r}\right), \qquad \dot{r} = -\operatorname{sat}(r),$$
$$y(t, x_0) = \sqrt{\phi_1(t, x_0)^2 + \operatorname{sat}(\phi_2^2(t, x_0))},$$

with transition map $\phi(\,\cdot\,, x_0) = (\phi_1(\,\cdot\,, x_0), \phi_2(\,\cdot\,, x_0))^T$, and $\operatorname{sat}\colon \mathbb{R}_0^+ \to [0, 1]$, $\operatorname{sat}(r) = \min\{r, 1\}$. First consider the following: Due to

$$\dot{r} = -\min\{r, 1\} < 0, \qquad r > 0, \tag{4}$$

$\|\phi(t, x_0)\|_2$ is strictly decreasing to zero in time and for all $\varepsilon > 0$ and all $r_0 \in [0, 1]$, it holds that

$$|y(t, x_0)| = \|\phi(t, x_0)\|_2 = e^{-t} \|x_0\|_2 \le \varepsilon \tag{5}$$

for all $t \ge \tau_1(\varepsilon, r_0) = \max\left\{\ln\left(\frac{r_0}{\varepsilon}\right), 0\right\}$ and all $x_0 : \|x_0\|_2 \le r_0$. Here, we used (4) and that for $r_0 \le \varepsilon$, the bound is already satisfied at $t = 0$. For $\|x_0\|_2 > 1$, $t = \|x_0\|_2 - 1$, it holds that $|y(t, x_0)| \leq \|\phi(t, x_0)\|_2 \leq \|x_0\|_2 - t = 1$. Hence, OGULIM follows by $\tau := \tau_1 + \max\{\|x_0\|_2 - 1, 0\}$.

For $\|x_0\|_2 < 1$, the system is OULS by (5). Therefore, as $\|x_0\|_2 = y(0, x_0)$ for $\|x_0\|_2 < 1$ and $\|x_0\|_2 \geq 1$ implies $y(0, x_0) \geq 1$, it follows $y(t, x_0) \leq \|x_0\|_2^2 = y(0, x_0)$ for all $x \in B_1$, $t \in I$, i.e. the system is locally OL.

Furthermore, the system is OBORS as due to

$$\dot{y}(t, x_0) = \begin{cases} -\operatorname{sat}(r), & \text{if } |\phi_2(t, x_0)| \leq 1, \\ -\frac{r\cos^2(\theta) + r^2\cos(\theta)\sin(\theta)\frac{1}{r}}{\sqrt{r^2\cos^2(\theta)+1}}, & \text{if } |\phi_2(t, x_0)| > 1, \end{cases}$$
$$\leq 0 + \frac{r|\cos(\theta)|}{\sqrt{r^2\cos^2(\theta)+1}} \cdot |\sin(\theta)| \leq 1 \cdot |\sin(\theta)| \leq 1,$$

it holds that $y(t, x_0) \leq y(0, x_0) + t$.

The system is not OL as for $\|x_0\|_2 > 1$, $t < \|x_0\|_2 - 1$, it holds that $\|\phi(t, x_0)\|_2 = \|x_0\|_2 - t$, $\theta(t) = \theta_0 + \ln(\|x_0\|_2) - \ln(\|x_0\|_2 - t)$, and thus for $x_0 = (0, c)$, $c > e^{\frac{\pi}{2}}$ and $t^* := \|x_0\|_2 \left(1 - e^{-\frac{\pi}{2}}\right)$, it holds that $y(0, x_0) = 1$, $\theta(t^*) = \frac{\pi}{2}$, and $y(t^*, x_0) = |\phi_1(t^*, x_0)| = ce^{-\frac{\pi}{2}} \to \infty$ for $c \to \infty$.

VI. Conclusion

We introduced several notions of stability and attractivity for infinite dimensional systems with outputs. We gave superposition theorems for IOS and related the stability and attractivity notions by implications and counterexamples. By this, we gave generalizations of the results of [1] for infinite dimensional systems. It turns out that some of the characterizations for ODE systems do not hold anymore in the infinite dimensional setting (e.g., OGUAG $\not\Rightarrow$ IOS). Moreover, we generalized the characterizations in [2] in terms of considering systems in which the output is not necessarily equal to the state. It turns out that the output equivalent of OULIM in combination with OUGS is not sufficient to conclude IOS as opposed to the case where the output equals the state [2, Thm. 5].

We proved a sufficient condition for OL in terms of OOULIM, local OL and OBORS.

The omitted proofs as well as further results, insights and counterexamples can be found in the preprint of the journal version [25].

As the next step, we want to develop a Lyapunov theory for infinite dimensional systems with outputs and investigate interconnections of IOS systems with the aim of small-gain theorems.

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