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Bipartite Containment Control of Nonuniform Delayed Fractional-Order Multi-agent Systems Over Signed Networks

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Abstract— **In this study, the bipartite containment control problem of fractional-order multi-agent systems with nonuniform time delays is addressed. An in-depth analysis of the system stability and bipartite containment control performance from a delay margin perspective is provided. Theoretically, the corresponding delay margin (maximum allowable time delay) over undirected and directed signed networks is obtained in the presence of nonuniform time delays, respectively. In addition, numerical relationships between the delay margin and the control coefficients, fractional order, and topology parameters are established, thus enabling easy and direct calculation of the maximum allowable time delay and facilitating distributed controller design and controller parameter tuning. Finally, some simulation examples are given to verify the effectiveness of the proposed bipartite containment controller and the obtained delay margin.**

I. Introduction

Over the past two decades, distributed cooperative control of multi-agent systems (MASs) has received much attention due to its wide applications, such as multi-robots [1], multi-UAVs [2], smart grid systems [3] and others. Consensus as the fundamental research topic of distributed cooperative control has drawn an increasing interest. In particular, the consensus problem can be divided into average consensus, leader-following consensus tracking, and containment control depending on the number of leaders in the network. Among these cooperative behaviors, containment control of MASs under multiple leaders has attracted extensive interest from researchers. A common application of containment control is to prevent a group of autonomous vehicles from exploring potentially dangerous regions. Some classic references on containment control can be found in [4]–[7].

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Remarkably, the above results of containment control concentrate on the cooperative control of networked agents. Nevertheless, the coexistence of competition and cooperative interactions between agents is realistic in some practical applications and scenarios, like opinion dynamics of social networks and multi-robot systems [8], [9]. This also means that traditional interaction networks are unable to describe complex relationships between agents. Fortunately, signed networks are introduced to represent positive weights as cooperative relationships between agents, and negative weights as competitive relationships between agents [10], [11]. Particularly, Altafini [10] designed the classic bipartite consensus protocol. Inspired by this work, the bipartite containment control problem was well defined in [12] and it was demonstrated therein that followers can converge together at the convex hull that contains the trajectory of each leader. More important results on the bipartite containment control of MASs can be found in [13]–[15].

In practical applications, due to the limited speed of signal transmission over long distances, time delays in information exchanges between agents and the presence of state measurements of agents themselves are often unavoidable. The existing methods for analyzing the stability conditions of MASs in the presence of time delays are mainly the Lyapunov method and the frequency domain method. The work in [16] investigated the bipartite containment control problem of MASs with input delay over switching signed directed typologies. According to the fractional Razumikhin and Lyapunov method, a practical strategy was suggested to address those issues caused by time delays. Moreover, a follower-based observer was designed to achieve bipartite containment control of MASs with mixed time delays in [17]. However, usually the Lyapunov-based method can only prove whether a closed-loop error system is stable or not, and cannot give an explicit maximum allowable time delay. In contrast, the frequency domain method in the classic control theory can provide an explicit upper bound on the allowable time delay by analyzing the delay margin of the system, despite that it also has a significant drawback in its application, i.e., it is only applicable to linear systems.

In addition, most studies on bipartite containment control focus on integer-order systems. Notably, several physical processes that occur in the actual world can naturally be described by fractional-order MASs [18], for example, the multi-vehicle system in the form of a desert or multiple individuals moving on nonuniform snow. Until now, there have been few reported works on MASs with nonuniform delays and fractional-order dynamics, which encourages us to initiate the present study.

According to the above discussion, this paper addresses the bipartite containment control problem of fractional-order MASs with nonuniform time delays over signed networks. The distributed bipartite containment controller is designed for each follower agent to achieve the convergence to the (symmetric) convex hull of multiple leaders. The main contributions of this paper are summarized as follows.

1) With the help of the frequency domain method, the delay margin (maximum allowable time delay) of fractional-order MASs with nonuniform time delays is obtained, which is related to the controller parameters, the fractional order, and the eigenvalues of the Laplacian matrix. Using the direct delay margin easily facilitates the distributed controller design and controller parameter tuning.

2) The bipartite containment control convergence conditions of the delayed fractional-order agents in undirected or directed signed networks are respectively generated. These conditions are presented simply in the form of inequalities which are easy to calculate.

The rest parts of this paper are organized as follows. In Section II, the preliminaries of graph theory and fractional-order derivatives are introduced. Section III describes the bipartite containment control problem of fractional-order MASs over signed networks. Section IV provides the main results for the distributed bipartite containment control controller and the delay margin of MASs with undirected and directed topologies, respectively. In Section V, some simulation results are given for validation of the delay margin. Section VI concludes the whole paper and provides some future research topics.

II. Preliminaries

In this section, some notations of matrix and vector are defined. Then the preliminaries of graph theory and fractional-order derivatives are introduced.

A. Notations

A diagonal matrix is identified by the notation $\mathbf{diag}_i^n[\alpha_i] \triangleq \mathbf{diag}\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, and $\mathbf{col}_i^n[\alpha_i] \triangleq [\alpha_1^T, \alpha_2^T, \ldots, \alpha_n^T]^T$ or $\mathbf{col}^n[\alpha] \triangleq [\alpha^T, \alpha^T, \ldots, \alpha^T]_n^T$ defines a column vector. All one vector $\mathbf{1}_n$ is expressed as $\mathbf{1}_n \triangleq \mathbf{col}^n[1]$. In addition, the identity matrix is defined as $I_n = \mathbf{diag}_i^n[1] \triangleq \mathbf{diag}\{1, 1, \ldots, 1\}$.

B. Graph theory

Consider the connected signed network topology $\mathcal{G}$ including n followers indexed as $\mathcal{V}_F = \{1, 2, ..., n\}$ and m leaders indexed as $\mathcal{V}_L = \{n+1, n+2, ..., n+m\}$ in this paper. The information exchange relationship among agents can be represented as the corresponding topology $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$. Here $\mathcal{V}$ indicates the node set and satisfies $\mathcal{V} = \mathcal{V}_F \cup \mathcal{V}_L$, $\mathcal{E}$ denotes the edge set with the condition $(j, i) \in \mathcal{E}$ indicating that there exists information interaction between agent j and agent i, and $\mathcal{A} = [a_{ij}]^{(n+m)\times(n+m)}$ is a weighted adjacency matrix with $a_{ij} \neq 0$ if $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. In addition, if the graph $\mathcal{G}$ is an undirected topology, then $\mathcal{A}$ is a symmetric matrix, that is $a_{ij} = a_{ji}$. In general, assume that the leaders in the connected networks have no neighbour. Hence, the corresponding Laplacian matrix $\mathcal{L} \in \mathbb{R}^{(n+m)\times(n+m)}$ is expressed as $\mathcal{L} = \mathcal{D} - \mathcal{A}$, where the degree matrix $\mathcal{D} = \mathbf{diag}_i^{n+m}[d_i]$ with the elements $d_i = \sum_{k=1}^{n+m} a_{ik}$.

Assumption 1: Consider the undirected or directed subgraph of the topology $\mathcal{G}$ being composed of n followers. For each of the n followers, there exists at least one leader with a directed path to this follower.

Based on the relationship between the leaders and the followers, one has $\mathcal{L} = \begin{bmatrix} \mathcal{L}_F & \mathcal{L}_L \\ \mathbf{0}_{m\times n} & \mathbf{0}_{m\times m} \end{bmatrix}$, where $\mathcal{L}_F \in \mathbb{R}^{n\times n}$ and $\mathcal{L}_L \in \mathbb{R}^{n\times m}$.

Assumption 2: The signed graph $\mathcal{G}$ is structurally balanced. Moreover, the leaders are cooperative with a group of followers and the corresponding node set is $\mathcal{V}_1$; otherwise, they are competitive with the other group of followers and the corresponding node set is $\mathcal{V}_2$.

If Assumption 2 holds, then a diagonal matrix of the structurally balanced signed graph $\mathcal{W} = \mathbf{diag}_i^{n+m} = [w_i]$ can be formed, where $w_i = 1$ if $i \in \mathcal{V}_1$, and $w_i = -1$ if $i \in \mathcal{V}_2$. Thus, the corresponding Laplacian matrix is $\overline{\mathcal{L}} = \begin{bmatrix} \overline{\mathcal{L}}_F & \overline{\mathcal{L}}_L \\ \mathbf{0}_{m\times n} & \mathbf{0}_{m\times m} \end{bmatrix}$.

Lemma 1 *([19]):* Under Assumptions 1 and 2, the matrix $\overline{\mathcal{L}}_F$ is a nonsingular M-matrix. In addition, each row sum of the matrix $-\overline{\mathcal{L}}_F^{-1}\overline{\mathcal{L}}_L$ equals one, that is, $-\overline{\mathcal{L}}_F^{-1}\overline{\mathcal{L}}_L\mathbf{1}_m = \mathbf{1}_n$, and all entries of this matrix are nonnegative.

C. Fractional-Order Derivatives

In this subsection, we introduce the fractional calculus theory and present two commonly used symbols. The notable Euler's Gamma function is initially described as $\Gamma(q) = \int_0^\infty e^{-t}t^{q-1}dt$, where the parameter q is an arithmetic value. Besides, the binomial coefficient of the generalized Newton formula is $\mathcal{C}_k^q = \frac{\Gamma(q+1)}{\Gamma(k+1)\Gamma(q-k+1)}$.

Based on the two definitions above, the widely used Caputo fractional-order derivative is given below.

Definition 1 *([20]):* For $q \in \mathbb{R}$, the common q-order Caputo fractional-order derivative is defined as

$$ {}_a^C\mathbf{D}_t^q f(t) = \frac{1}{\Gamma(m-q)} \int_a^t (t-\tau)^{m-q-1} f^{(m)}(\tau) d\tau, $$

where a is the Caputo derivatives' base point and m is the integer with $m - 1 < q \leq m$.

For the fractional-order derivative ${}_0^C\mathbf{D}_t^q f(t)$ presented with respect to time t previously, we use $\mathbf{D}_t^q f(t)$ in the following parts for notational simplicity. Then the Laplace transform of the fractional-order derivative $\mathbf{D}_t^q f(t)$ is:

1) when the fractional order $0 < q \leq 1$,

$$ \mathbb{L}\{\mathbf{D}_t^q f(t)\} = s^q \mathbb{L}\{f(t)\} - s^{q-1} f(0); $$

2) when the fractional order $1 < q < 2$,

$$ \mathbb{L}\{\mathbf{D}_t^q f(t)\} = s^q \mathbb{L}\{f(t)\} - s^{q-2} \dot{f}(0) - s^{q-1} f(0). $$

Definition 2 *([21]):* A subset $K \subset \mathbb{R}^n$ is said to be convex if $(1-\beta)x + \beta y \in K$ for $x \in K, y \in K$ and

$0 < \beta < 1$. The convex hull of a specified subset $S \subset \mathbb{R}^n$ is the intersection of all convex sets that contain S and is denoted by $Co(S)$. The convex hull $Co(X)$ of a finite subset $X = \{x_1, x_2, ..., x_n\} \in \mathbb{R}^n$ is made up of all vectors of the form $(\beta_1 x_1 + \beta_2 x_2 + ... + \beta_n x_n)$ with the coefficients $\beta_1 \geq 0, \beta_2 \geq 0, ..., \beta_n \geq 0$ and $\beta_1 + \beta_2 + ... + \beta_n = 1$.

III. Problem Description

Consider the single-integrator fractional-order networked system consisting of n followers and m leaders. The dynamics of each follower agent can be expressed as follows:

$$\mathbf{D}_t^q x_i(t) = u_i(t),\ i \in \mathcal{V}_F, \tag{1}$$

where q is the fractional order, $x_i(t)$ is the position of the i-th follower agent, and $u_i(t)$ is the i-th agent's control input.

In addition, the dynamics of the leader agents can be expressed as $\mathbf{D}_t^q x_i(t) = 0,\ i \in \mathcal{V}_L$, where $x_i(t)$ denotes the position of the i-th leader. The above dynamics means that the leader agents' position is stationary.

Remark 1: The leaders in MASs are uncontrolled agents ($u_i(t), i \in \mathcal{V}_L$) who just send messages to the followers. The primary task is to provide a fully distributed control algorithm for fractional-order followers, which enables bipartite containment tracking for the leaders.

***Definition** 3:* For single-integrator MASs with fractional-order dynamics, the bipartite containment control is called being implemented if the follower agents' positions asymptotically converge to the convex hull created by the leader agents. In other words, for any initial condition, there holds that

$$\lim_{t\to\infty}\left| x_i(t) - w_i \sum_{j=n+1}^{m+n} \varepsilon_{ij} x_j(t) \right| = 0$$

with $w_i = \pm 1$, where $\varepsilon_{ij} \in \mathbb{R}, \varepsilon_{ij} \geq 0$ and $\sum_{j=n+1}^{m+n} \varepsilon_{ij} = 1, i \in \mathcal{V}_F, j \in \mathcal{V}_L$.

IV. Main Results

A. Containment Control of Fractional-Order MASs Without Time Delay

In this part, we provide a traditional bipartite containment control protocol without time delay, and obtain the stability condition of the closed-loop error system with undirected topology as well as directed topology, respectively.

On the basis of the work in [12], the containment control protocol for the fractional-order MASs is designed as

$$u_i(t) = \sum_{j \in \mathcal{V}_F \cup \mathcal{V}_L} |a_{ij}|(\mathrm{sgn}(a_{ij}) x_j(t) - x_i(t)),\ i \in \mathcal{V}_F, \tag{2}$$

where sgn(.) represents the sign function. According to the equations (2) and (1), there holds that

$$\begin{aligned} \mathbf{D}_t^q x_i &= \sum_{j \in \mathcal{V}_F \cup \mathcal{V}_L} |a_{ij}|(\mathrm{sgn}(a_{ij}) x_j(t) - x_i(t)),\ i \in \mathcal{V}_F, \\ \mathbf{D}_t^q x_i &= 0,\ i \in \mathcal{V}_L. \end{aligned} \tag{3}$$

Define the vectors: $X_F = \{x_1, x_2, ..., x_n\}$ and $X_L = \{x_{n+1}, x_{n+2}, ..., x_{n+m}\}$. Hence, the compact form of the closed-loop system is expressed as

$$\mathbf{D}_t^q X_F = -\overline{\mathcal{L}}_F \mathcal{W}_1 X_F - \overline{\mathcal{L}}_L \mathcal{W}_2 X_L, \quad \mathbf{D}_t^q X_L = 0, \tag{4}$$

where $\mathcal{W}_1 = \mathbf{col}_{i=1}^n[w_i]$ and $\mathcal{W}_2 = \mathbf{col}_{i=n+1}^n[w_i]$. Then the coordinate transformation $Z_1 = \mathcal{W}_1 X_F + \overline{\mathcal{L}}_F^{-1} \overline{\mathcal{L}}_L \mathcal{W}_2 X_L$ is introduced for the subsequent analysis, where $Z_1 = \mathbf{col}_i^n[z_{1i}]$ and its corresponding fractional-order derivative is given by

$$\mathbf{D}_t^q Z_1 = -\overline{\mathcal{L}}_F Z_1. \tag{5}$$

Remark 2: According to Definition 3 and Lemma 1, the containment control for MASs can be implemented if the condition $\lim_{t\to\infty} |z_{1i}| = 0$ holds.

***Lemma** 2 ([22]):* For the above closed-loop system (5), if the fractional order q satisfies $0 < q < \frac{2\theta}{\pi}$ and the real part of the eigenvalues of the matrix $\overline{\mathcal{L}}_F$ is greater than zero, then $\lim_{t\to\infty} |Z_1(t)| = 0$, and meanwhile $\theta = \min\{\pi - \arg\{\lambda_i(\overline{\mathcal{L}}_F)\}\}$, where $\arg\{.\}$ indicates the phase of the variable, and x's phase satisfies $\arg\{x\} \in (-\pi, \pi]$ for $x \in \mathbb{C}$.

In addition, based on Lemma 2, the following two properties are derived about undirected and directed topologies.

1) If the topology subgraph composed of n followers is undirected, then for the Laplacian matrix $\overline{\mathcal{L}}_F$, all of its eigenvalues are positive real number and $\theta_i = \pi$. At the same time, under the condition $0 < q < 1$, the bipartite containment control of MASs is achieved.
2) If the topology subgraph composed of n followers is directed, then for the Laplacian matrix $\overline{\mathcal{L}}_F$, all of its eigenvalues are positive real number and $\arg\{\lambda_i(\overline{\mathcal{L}}_F)\} \in [-\pi/2, \pi/2]$. At the same time, under the condition $0 < q < \frac{2\theta}{\pi}$, the multi-agent containment control is achieved.

B. Containment Control of Fractional-Order MASs With Nonuniform Time Delays

In the previous part, a normal containment controller is presented for fractional-order MASs without time delay. Based on the above control protocol (2), this subsection designs a containment controller for fractional-order MASs under undirected topology with nonuniform delay. Specifically, the controller for the i-th agent is designed as

$$u_i(t) = \sum_{j \in \mathcal{V}_F \cup \mathcal{V}_L} |a_{ij}|(\mathrm{sgn}(a_{ij}) x_j(t - \tau_{ij}) - x_i(t - \tau_{ij})), \tag{6}$$

where $i \in \mathcal{V}_F$ and τ_{ij} represents the time delay between agent j's acceptance of the state information and agent i's acquisition and processing of the state information. Therefore, it is noteworthy that the time delay will appear in two situations: one situation is in the information interaction among agents and the other, called as "input delay", is associated with the agent's own state variable in the controller of the dynamic equation.

In this paper, it is assumed that there are M_1 varying forms of communication delay among the followers and M_2

varying kinds of communication delay between the leaders and followers, which satisfy $M_1 \leq n \times (n-1)$ and $M_2 \leq m \times n$, respectively. Therefore, all kinds of communication time delay M satisfy the inequality $M \leq M_1 + M_2$. Consider the existence of the communication delay τ_m and redefine the topology subgraph $\mathcal{G}_m$. Its corresponding Laplacian matrix $\overline{\mathcal{L}}_m$ is expressed as

$$\overline{\mathcal{L}}_m = \begin{bmatrix} \overline{\mathcal{L}}_{Fm} & \overline{\mathcal{L}}_{Lm} \\ \mathbf{0}_{m\times n} & \mathbf{0}_{m\times m} \end{bmatrix}. \tag{7}$$

where $\overline{\mathcal{L}}_{Fm} \in \mathbb{R}^{n\times n}$ and $\overline{\mathcal{L}}_{Lm} \in \mathbb{R}^{n\times m}$. It is obvious that $\sum_{m=1}^{M} \overline{\mathcal{L}}_{Fm} = \overline{\mathcal{L}}_F$ and $\sum_{m=1}^{M} \overline{\mathcal{L}}_{Lm} = \overline{\mathcal{L}}_L$. Accordingly, the compact form of the closed-loop system is given by

$$\mathbf{D}_t^q Z_1(t) = -\sum_{m=1}^{M} \overline{\mathcal{L}}_{Fm} Z_1(t-\tau_m). \tag{8}$$

We now give the following main result.

Theorem 1: Consider the fractional-order MASs with communication time delays and undirected topology. By adopting the protocol (6), the bipartite containment control is achieved if all time delays satisfy $\tau_m < \bar{\tau}$ and related parameters are given by

$$\bar{\tau} = \frac{\pi(2-q)}{2\bar{\omega}}, \quad \bar{\omega} = (\lambda_n)^{\frac{1}{q}}. \tag{9}$$

Proof: We use the frequency domain method to analyze the stability of the fractional-order closed-loop system (8). Assume that the Laplace transform of $Z_1(t)$ is $Z_1(s)$ and $Z_1(s) = G_\tau^T Z_1(0)$, where $G_\tau^{-1}(s) = s^q I_n + \sum_{m=1}^{M} \mathcal{L}_m e^{-\tau_m s}$.

Let $\det[G_\tau(s)]$ denote the characteristic polynomial of $G_\tau(s)$, and the root of the polynomial is called the characteristic root of the closed-loop system. Based on the above analysis, if there is no time delay in the fractional-order system, then the corresponding nonzero characteristic root has negative real parts. If there exists a time delay in the system, then the characteristic roots of the system will also change with the increase of τ_m. Once the characteristic roots change from the negative half-plane to the positive half-plane, the fractional-order system will become unstable. Therefore, the magnitude of the nonzero characteristic roots determines the delay margin corresponding to the imaginary axis.

The imaginary characteristic roots of the closed-loop system with time delay is expressed as $s = -j\omega \neq 0$, and the corresponding eigenvector is $u = u_1 \otimes [1,0]^T + u_2 \otimes [0,1]^T$. And the vector norm satisfies $\|u\| = 1, u_1, u_2 \in \mathbb{C}^n$. Then

$$\left[(-j\omega)^q I_n + \sum_{m=1}^{M} \overline{\mathcal{L}}_{Fm} e^{-j\omega\tau_m}\right] u = 0. \tag{10}$$

It is universally known that nonzero characteristic roots usually appear in a complex conjugate form, while we just focus on the situation that ω is greater than zero. Then multiplying u^H (also called the conjugate transpose of u) onto the left-hand side of (10), we get $u^H\big[(-j\omega)^q I_n + \sum_{m=1}^{M} \overline{\mathcal{L}}_{Fm} e^{-j\omega\tau_m}\big] u = 0$, which leads to

$$\begin{aligned} \sum_{m=1}^{M} \frac{u^H \overline{\mathcal{L}}_{Fm} u}{u^H u} e^{-j\omega\tau_m} &= -u(-j\omega)^q = -\omega^q (-j)^q \\ &= -\omega^q \left\{ \cos\left(\frac{-\pi}{2}\right) + j \sin\left(\frac{-\pi}{2}\right) \right\} \\ &= -\omega^q e^{j(-\frac{\pi q}{2})} = \omega^q e^{j\frac{\pi(2-q)}{2}}. \end{aligned} \tag{11}$$

Accordingly, we can obtain $\sum_{m=1}^{M} \alpha_m e^{j\omega\tau_m} = \omega^q e^{j\frac{\pi(2-q)}{2}}$, where $\alpha_m = \frac{u^H \overline{\mathcal{L}}_{Fm} u}{u^H u}$. Thus, we have $\sum_{m=1}^{M} \alpha_m e^{j\omega\tau_m} = \omega^q e^{j\frac{\pi(2-q)}{2}} = \mathcal{F}(\omega)$. Taking the modulus operation on both sides of the above equation produces

$$\begin{aligned} \mathcal{M}(\omega) = \|\mathcal{F}(\omega)\| &= \left\| \sum_{m=1}^{M} \alpha_m e^{j\omega\tau_m} \right\| \leq \left\| \sum_{m=1}^{M} \alpha_m \right\| \\ &= \frac{u^H \overline{\mathcal{L}}_F u}{u^H u} \leq \lambda_n. \end{aligned} \tag{12}$$

If the condition (9) is satisfied, then the inequality $\mathcal{M}(\omega) \leq \lambda_n$ holds if and only if $\omega \leq \bar{\omega}$. In other words, the above result (12) holds. After that, we can obtain the angle of the complex $\mathcal{F}(\omega)$ as $\theta(\omega) = \arg[\mathcal{F}(\omega)] = \frac{\pi(2-q)}{2}$, where $\theta(\omega) \in (0, \pi)$.

Define $\tau(\omega) = \frac{\theta(\omega)}{\omega} = \frac{\pi(2-q)}{2\omega}$. The first-order derivative of $\tau(\omega)$ with respect to ω is $\mathcal{D}_1(\omega) = \frac{d\tau(\omega)}{d\omega} = -\frac{\pi(2-q)}{2\omega^2} \leq 0$. From these results, we can attain that the function $\tau(\omega)$ is descending about ω, where $\omega \leq \bar{\omega}$ and

$$\bar{\tau} = \tau(\bar{\omega}) \leq \tau(\omega). \tag{13}$$

It should be noted that the aforementioned conclusion is predicated on the assumption that the system has a characteristic root on the imaginary axis. Clearly, if all the time-delays τ_m satisfy $\tau_m < \bar{\tau}$, then one has

$$\begin{aligned} \tau(\omega) = \frac{\theta(\omega)}{\omega} &= \frac{\arg(\sum_{m=1}^{M} \alpha_m e^{j\omega\tau_m})}{\omega} \\ &\leq \frac{\max\{\omega\tau_m\}}{\omega} < \frac{\omega\bar{\tau}}{\omega} = \bar{\tau}. \end{aligned} \tag{14}$$

However, the equation (14) is contrary to the result of the equation (13), that is, as long as all time-delays τ_m satisfy $\tau_m < \bar{\tau}$, the characteristic roots of the fractional-order system cannot cross the imaginary axis to the right half-plane. Accordingly, the fractional-order closed-loop system (8) with time delays can keep the stability and achieve containment control. Besides, while $\tau_m > \bar{\tau}$, the fractional-order closed-loop system will have characteristic roots that are in the right half-plane. So in virtue of the stability principle, the system will be unstable and cannot achieve containment control. On the other hand, if $\tau_m = \bar{\tau}$, then there exists an imaginary characteristic root $j\bar{\omega}$, and its corresponding time-delay is called the critical time-delay. To sum up, the proof of Theorem 1 is complete. ■

Note that Theorem 1 accomplishes the design of the bipartite containment control protocol for fractional-order MASs with nonuniform delays over undirected signed networks. If the topology is directed, then there exists $\tau_{ij} \neq$

τ_{ji}. Therefore, in the following we consider the bipartite containment control over directed signed networks.

***Theorem** 2:* Consider the fractional-order MASs with symmetric nonuniform multiple time delays and directed signed graph. By adopting the control protocol (6), the bipartite containment control is achieved if all the time delays satisfy $\tau_m < \bar{\tau}$ and the related parameters are given by

$$\bar{\tau} = \min_{\|\lambda_i\|}\left\{\frac{\frac{\pi(2-q)}{2} - \arg(\lambda_i)}{\bar{\omega}_i}\right\}, \quad \bar{\omega}_i = \|\lambda_i\|^{\frac{1}{q}}, \tag{15}$$

where $q \in (0, \frac{2\theta}{\pi})$, and $\theta = \min\{\pi - \arg(\lambda_i)\}$.

Proof: The subsequent analysis is similar to that of Theorem 1. We define the imaginary characteristic roots of the closed-loop system with time delays as $s = -j\omega \neq 0$, and the corresponding eigenvector is $u = u_1 \otimes [1,0]^T + u_2 \otimes [0,1]^T$. Then we can get

$$\begin{aligned}\mathcal{B}_\gamma &= \sum_{m=1}^{M} \alpha_m e^{j\omega\tau_m} = -(-j\omega)^q = (-1)(-j)^q(\omega)^q \\ &= e^{j\pi}e^{-\frac{\pi q}{2}}(\omega)^q = \omega^q e^{j\frac{2\pi - \pi q}{2}} = \omega^q e^{j\frac{\pi(2-q)}{2}}.\end{aligned} \tag{16}$$

Take ω as a function of $\|\mathcal{B}_\gamma\|$. Through the modulus operation on both sides, we can obtain $\omega(\|\mathcal{B}_\gamma\|) = \|\mathcal{B}_\gamma\|^{\frac{1}{q}}$. Clearly, the function $\omega(\|\mathcal{B}_\gamma\|)$ is incremental.

Considering the argument amplitude on both sides of (16), respectively, we can obtain $\arg(\mathcal{B}_\gamma) = \frac{\pi(2-q)}{2}$. Based on the definition of $\mathcal{B}_\gamma$, one gets that $\arg(\mathcal{B}_\gamma) \leq \arg\left(\sum_{m=1}^{M}\alpha_m\right) + \max(\omega\bar{\tau}_m)$ and

$$\max(\omega\bar{\tau}_m) \geq \frac{\pi(2-q)}{2} - \arg\left(\sum_{m=1}^{M}\gamma_m\right). \tag{17}$$

Notice that

$$\sum_{m=1}^{M}\alpha_m = \sum_{m=1}^{M}\left(\frac{u^H\overline{\mathcal{L}}_{Fm}u}{u^H u}\right) = \frac{u^H\overline{\mathcal{L}}_F u}{u^H u},$$

and each possible value of $\sum_{m=1}^{M}\alpha_m$ ought to be a nonzero eigenvalue of the matrix $\overline{\mathcal{L}}_F$, that is, $\sum_{m=1}^{M}\alpha_m = \lambda_i, i \in \mathcal{V}_F$. Hence, we can get that $\|\mathcal{B}_\gamma\| \leq \|\lambda_i\|$ and $\omega(\|\mathcal{B}_\gamma\|) \leq \omega(\|\lambda_i\|) = \bar{\omega}_i = \|\lambda_i\|^{\frac{1}{q}}$. If all time delays satisfy $\tau_m < \bar{\tau}$, then the following condition holds:

$$\begin{aligned}\max(\omega\tau_m) < \bar{\omega}_i\bar{\tau} &= \min_{\|\lambda_i\|}\left\{\frac{\frac{\pi(2-q)}{2} - \arg(\lambda_i)}{\bar{\omega}_i}\right\}\bar{\omega}_i \\ &\leq \frac{\pi(2-q)}{2} - \arg\left(\sum_{m=1}^{M}\alpha_m\right).\end{aligned} \tag{18}$$

But the equation (18) is contrary to the result of the equation (17). Namely, as long as all time delays satisfy $\tau_m < \bar{\tau}$, the characteristic roots of the fractional-order system under directed graph cannot cross the imaginary axis to the right half-plane. Accordingly, the fractional-order closed-loop system (8) with time delays can keep the stability and achieve containment control. Besides, while $\tau_m > \bar{\tau}$, the fractional-order closed-loop system will have characteristic roots which are in the right half-plane. In virtue of the stability principle, the system will be unstable and the followers cannot converge into the convex hull formed by the leaders. On the other hand, if $\tau_m = \bar{\tau}$, then there exists an imaginary characteristic root $j\bar{\omega}$, and $\bar{\tau}$ is also the delay margin. Therefore, the proof of Theorem 2 is complete. ■

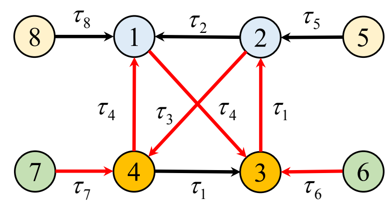

Fig. 1. Directed topology among agents.

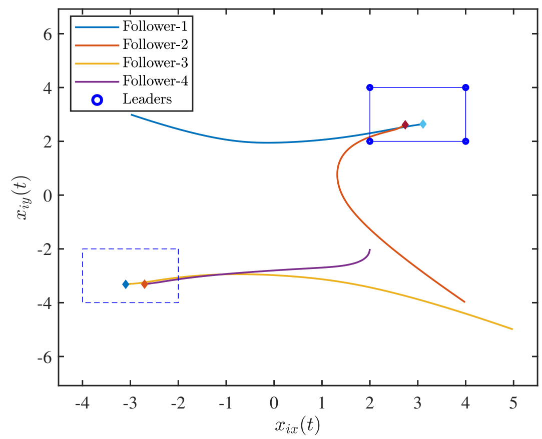

Fig. 2. Trajectories of agents ($\tau_m = 0$).

V. Numerical Simulations

In this section, some simulation results are given to verify the effectiveness of the proposed delay margin for bipartite containment control of fractional-order MASs.

The directed topology among the agents is shown in Fig. 1. The node set of follower agents is $\mathcal{V}_F = \{1,2,3,4\}$ and the node set of leader agents is $\mathcal{V}_L = \{5,6,7,8\}$. In Fig. 1, the black connecting lines represent cooperative relationships and are assigned a weight of 1, while the red connecting lines represent competitive relationships and are assigned a weight of -1. According to Fig. 1, we can get that followers 1 and 2 belong to the same group, while followers 3 and 4 are located in another group. In addition, there exist cooperative relationships among the leaders and followers 1 and 2. The fractional order of the agents is chosen as $q = 0.8$.

In the simulation, we consider the bipartite containment control problem in two dimensions (xy-coordinates) and define the initial values of the states of the leader are $\mathbf{col}_i[x_{ix}] = [2,4,2,4]^T$, $\mathbf{col}_i[x_{iy}] = [2,4,4,2]^T, i \in \mathcal{V}_L$ and the initial values of the states of the followers are $\mathbf{col}_i[x_{ix}] = [-3,4,5,2]^T$, $\mathbf{col}_i[x_{iy}] = [3,-4,-5,-2]^T, i \in \mathcal{V}_F$. According to the equation (15), the delay margin of the directed topology is $\bar{\tau} = 0.3707s$. Three cases are set up to compare the bipartite containment control performance of MASs with different time delays, namely, $\tau_m = 0$, $\tau_m < \bar{\tau}$ and $\tau_m > \bar{\tau}$. The simulation results are exhibited in Fig. 2–4. It can be seen that when all time delays are smaller than the delay margin, the MAS can achieve bipartite containment

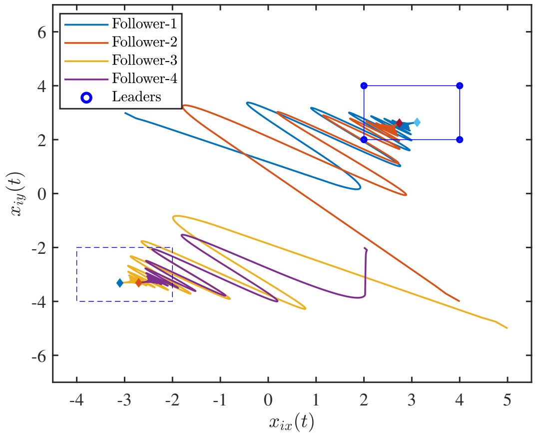

Fig. 3. Trajectories of agents ($\tau_m < \bar{\tau}$).

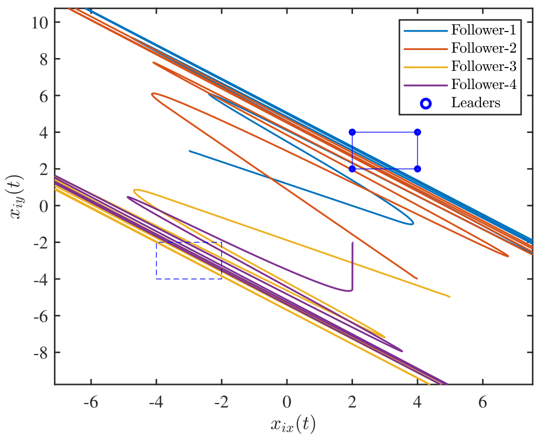

Fig. 4. Trajectories of agents ($\tau_m > \bar{\tau}$).

control. That is, a group of the followers converge into the convex hull formed by the leaders, while the other group of the followers move into in the symmetric convex hull formed by the leaders. However, when all time delays are larger than the delay margin, the state of the agents will diverge and no collaborative behavior can be achieved.

VI. Conclusion

This paper has investigated the bipartite containment control problem of fractional-order MASs with nonuniform time delays. The cooperation and competition interactions among agents are described by signed networks. We have first designed the bipartite containment controller for each follower agent and then obtained the delay margin of the closed-loop system over undirected/directed topology based on the frequency domain method, respectively. The delay margin is related to the controller parameters, the fractional order, and the eigenvalue of the Laplacian matrix of the topology. In turn, it is easy to obtain the maximum allowable time delay for the MAS and facilitate tuning the distributed bipartite containment controller parameters. In the future work, we will extend the control structure and analysis method to solve the bipartite containment control problem of double-integrator and multiple-integrator fractional-order MASs, respectively. We will also try to obtain the corresponding delay margin.

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