2024 IEEE 63rd Conference on Decision and Control (CDC)
December 16-19, 2024. MiCo, Milan, Italy

On implicit discretization of prescribed-time stabilizers

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***Abstract*—An implicit Euler discretization scheme is given for a linear system driven by the prescribed-time stabilizing control algorithm from [1] in the presence of matched disturbances and measurement noise. The discretized version preserves all main properties of the continuous-time counterpart, and can be recursively applied on infinite horizon rather than confined to the prescribed-time interval. In addition, the discretized estimation error is robustly stable with respect to the measurement noise with a linear gain. The efficiency of the suggested discretization is illustrated through numeric experiments.**

I. Introduction

Robust stabilization of dynamical systems using noisy state measurements and in the presence of external bounded disturbances is a well-studied problem, which has many popular and well established solutions based on linearization, passivation, backstepping, sliding mode and other approaches [2]–[9]. There are different performance criteria to be imposed on the closed-loop system by the feedback, with varying importance dependent on applications, which is a reason for existence of many design methods. The main performance characteristics include (but not limited to): the time of convergence of the state to the origin, asymptotic precision in the noise-free case (the steady-state error), measurement noise sensitivity and the implementation complexity.

A popular control solution is based on the concept of prescribed-time convergence [1], [10]–[14]. In such a case, a linear feedback with time-varying gains is designed in a way to guarantee that for any initial conditions, in the noise-free scenario, the estimation error in the closed-loop system vanishes by the specified time instant, and this zero-settling property of the error is independent in the matched perturbations (*i.e.*, exact uniform stabilization in a prescribed time). However, there is a price to pay for these advantageous performances: 1) noise dependence of stabilization error is complicated and can be properly handled under strong restrictions on the perturbations only [13], 2) the definition of control law after the settling time requires a special attention, and the error behavior after the convergence is usually not considered (since the control gains take infinite values at this instant of time, then a commutation to another control law is needed [15], [16]). All these drawbacks, which are unusual for other control frameworks, complicate the applicability of the prescribed-time converging stabilizers (frequently in applications, the control is commuted to another solution before the settling instant).

Inspired by [17], [18], where implicit discretization of a hyperexponentially (asymptotically) converging differentiator and controller was studied, in this note we are going to analyze an implicit discretization scheme for the prescribed-time controller. Our contribution is in establishing that such a discretization *preserves the uniform finite-time convergence* of the continuous-time counterpart, while being *robust with respect to the measurement noise* and avoiding the infinite gain implementation problem. Moreover, the resulting control law proves to be well-posed on the infinite horizon. Its efficacy is illustrated by simulations, which demonstrate that the obtained discrete-time linear time-varying feedback has very advantageous features. Note that for linear systems, the explicit and implicit discretization schemes have often similar computational complexity, since the latter ones can be easily reduced to en explicit form.

The paper is organized as follows. Brief preliminaries are given in Section II. The problem statement is introduced in Section III. The properties of the considered control in continuous time are recalled in Section IV. The properties of its implicit Euler discretization are investigated in Section V. The results of numeric simulation of the control algorithm are shown in Section VI. The proofs are omitted due to space limitations.

Notation

- $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, where $\mathbb{R}$ is the set of real numbers; $\mathbb{Z}$ is the set of integer numbers, $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$.
- $|\cdot|$ denotes the absolute value in $\mathbb{R}$, $\|\cdot\|$ is used for the Euclidean norm on $\mathbb{R}^n$.
- For a (Lebesgue) measurable function $d : \mathbb{R}_+ \to \mathbb{R}^m$ define the norm $\|d\|_\infty = \text{ess}\sup_{t\in\mathbb{R}_+} \|d(t)\|$, and the set of d with the property $\|d\|_\infty < +\infty$ we further denote as $\mathcal{L}_\infty^m$ (the set of essentially bounded measurable functions).
- For a sequence $d_k \in \mathbb{R}^m$ with $k \in \mathbb{Z}_+$ define its norm by $|d|_\infty = \sup_{k\in\mathbb{Z}_+} \|d_k\|$ and the set of d with $|d|_\infty < +\infty$ we denote by l_∞^m.
- A continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}$ if $\alpha(0) = 0$ and it is strictly increasing. The

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function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and it is increasing to infinity.
- A finite series of integers $1, 2, ..., n$ is denoted by $\overline{1,n}$, and $\{\overline{1,n}\} = \{1, 2, ..., n\}$.
- Denote the identity matrix of dimension $n \times n$ by I_n and the matrix of zeros of dimension $m \times n$ by $0_{m\times n}$.
- diag$\{g\}$ represents a diagonal matrix of dimension $n\times n$ with a vector $g \in \mathbb{R}^n$ on the main diagonal.
- The relation $P \prec 0$ ($P \succeq 0$) means that a symmetric matrix $P \in \mathbb{R}^{n\times n}$ is negative (positive semi-) definite, $\lambda_{\min}(P)$ denotes the minimal eigenvalue of such a matrix P.
- Denote $\mathbf{e} = \exp(1)$.

II. Preliminaries

The standard stability notions are used throughout and their definitions can be found in [3].

A. Uniform prescribed-time stability

Consider a non-autonomous differential equation:

$$\frac{dx(t)}{dt} = f(t, x(t), d(t)),\ t \geq t_0,\ t_0 \in \mathbb{R}_+, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $d(t) \in \mathbb{R}^m$ is the vector of external disturbances and $d \in \mathcal{L}^m_\infty$; $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuous function with respect to x, d and piecewise continuous with respect to t, $f(t, 0, 0) = 0$ for all $t \in \mathbb{R}_+$. A solution of the system (1) for an initial condition $x_0 \in \mathbb{R}^n$ at time instant $t_0 \in \mathbb{R}_+$ and some $d \in \mathcal{L}^m_\infty$ is denoted by $X(t, t_0, x_0, d)$, and we assume that f ensures existence and uniqueness of solutions $X(t, t_0, x_0, d)$ at least locally in forward time.

The following definition is inspired by [11], [13], and it is specified for a control-free system (1) while also presuming that the settling-time instant has been assigned at the designer will.

Definition 1. Given $T > 0$ and a set $\mathbb{D} \subset \mathcal{L}^m_\infty$, the system (1) is called *uniformly prescribed-time stable* (T-uPTS) if there exist $\sigma_1, \sigma_2 \in \mathcal{K}$ such that for all $x_0 \in \mathbb{R}^n \setminus \{0\}$, $t_0 \in \mathbb{R}_+$ and $d \in \mathbb{D}$:

$$\|X(t, t_0, x_0, d)\| \leq \max\{\sigma_1(\|x_0\|), \sigma_2(\|d\|_\infty)\},$$
$$0 < \|X(t, t_0, x_0, 0)\|$$

for all $t \in [t_0, t_0 + T)$, and

$$\lim_{t\to t_0+T} \|X(t, t_0, x_0, d)\| = 0.$$

It is important to highlight that the boundedness of solutions is claimed on a finite interval $[t_0, t_0 + T)$ only, and the solutions of (1) may be undefined for $t > T$. Hence, a uPTS system may not demonstrate a Lyapunov stable behavior, and it is a variant of short-time stability (frequently also called finite-time one) as in [19]. In comparison with the concept of fixed-time stability [20], an important feature of a prescribed-time stable system is that its settling time is *the same for all initial conditions* out of the origin in the disturbance-free setting. In this definition the uniformity of convergence is understood in double meaning: as independence in both the initial time t_0 and in the input $d \in \mathbb{D}$. Despite it is assumed that $\mathbb{D} \subset \mathcal{L}^m_\infty$, any other admissible class of inputs can be considered.

A simple scalar example of a uPTS system (1) for $t_0 = 0$ is [1]:

$$\dot{x}(t) = -\frac{T}{T-t}x(t) + d(t),\ t \in [0, T),\ T > 0,$$

with $x(t), d(t) \in \mathbb{R}$, whose solutions admit an estimate:

$$|x(t)| \leq \left(\frac{T-t}{T}\right)^T |x(0)| + \iota(t)\|d\|_\infty,\ t \in [0, T)$$

for any $x(0) \in \mathbb{R}$ and $d \in \mathcal{L}^1_\infty$, where

$$\iota(t) = \begin{cases} \frac{T(\frac{T-t}{T} - (\frac{T-t}{T})^T)}{T-1} & \text{if } T \neq 1, \\ (1-t)\ln\frac{1}{1-t} & \text{if } T = 1. \end{cases}$$

Then by Definition 1, it follows that $\sigma_1(s) = s$ and $\sigma_2(s) = \iota_{\max} s$ with

$$\iota_{\max} = \begin{cases} \frac{T^{(1-T)^{-1}} - T^{(1-T)^{-T}}}{1-T^{-1}} & \text{if } T \neq 1, \\ \mathbf{e}^{-1} & \text{if } T = 1. \end{cases}$$

B. Auxiliary property

The following block matrix inversion formulas are used in the sequel [21]:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1}BS_1CA^{-1} & -A^{-1}BS_1 \\ -S_1CA^{-1} & S_1 \end{bmatrix}$$
$$= \begin{bmatrix} S_2 & -S_2BD^{-1} \\ -D^{-1}CS_2 & D^{-1} + D^{-1}CS_2BD^{-1} \end{bmatrix},$$
$$S_1 = (D - CA^{-1}B)^{-1},\ S_2 = (A - BD^{-1}C)^{-1},$$

where A, B, C and D are matrices of appropriate dimensions (A, D, S_1 and S_2 should be nonsingular).

C. Useful linear matrix inequalities

Define

$$A = \begin{bmatrix} 0_{n-1\times 1} & I_{n-1} \\ 0 & 0_{1\times n-1} \end{bmatrix},\ b = \begin{bmatrix} 0_{n-1\times 1} \\ 1 \end{bmatrix},$$
$$c = \begin{bmatrix} 1 \\ 0_{n-1\times 1} \end{bmatrix}^\top,\ H = \text{diag}\{[0\ 1 \ldots n-1]^\top\}.$$

Lemma 1. *[22] For any $a_1 > 0$ there exist constants $d_0 > 0$, $d_1 \geq 0$, symmetric matrices $P_o, P_c \in \mathbb{R}^{n\times n}$ and vectors $K_o, K_c \in \mathbb{R}^n$ such that*

$$(A + bK_c^\top)^\top P_c + P_c(A + bK_c^\top) \leq -d_0P_c,$$
$$(A + K_oc)^\top P_o + P_o(A + K_oc) \leq -d_0P_o,$$
$$P_c \succ 0,\ -a_1P_c \leq P_cH + HP_c \leq d_1P_c,$$
$$P_o \succ 0,\ -a_1P_o \leq P_oH + HP_o \leq d_1P_o.$$

III. Problem statement

Consider a single-input single-output linear system in the Brunovsky canonical form with a matched disturbance and measurement noise:

$$\dot{x}(t)=Ax(t)+b(u(t)+d(t)),\ y(t)=x(t)+v(t), \quad (2)$$

where $x(t)\in\mathbb{R}^n$ is the state, $u(t)\in\mathbb{R}$ is the control to be designed; $d(t)\in\mathbb{R}$ is the disturbance, $d\in\mathcal{L}_\infty^1$; $y(t),v(t)\in\mathbb{R}^n$ are the measured output and noise, $v\in\mathcal{L}_\infty^n$; $A\in\mathbb{R}^{n\times n}$ and $b\in\mathbb{R}^n$ are in the canonical form as above.

Problem. Given $T>0$, it is required to design a feedback $u(t)=u(t,y(t))$, which for the closed-loop system provides T-uPTS property for $d\in\mathbb{D}=\mathcal{L}_\infty^1$ while $\|v\|_\infty=0$, and additionally guarantees a bounded state behavior for $v\in\mathcal{L}_\infty^1$ after a proper discretization.

In continuous time, this problem has been studied using different approaches in [1], [10], [13], [23], whereas the subsequent discrete-time extension is the main contribution of the present work.

IV. Stabilization in continuous time

For later use, let us recall a possible control law solving the stated uPTS problem in the closed loop for $v\equiv 0$ [1], [12], [23]:

$$u(t)=-KD(t)y(t), \quad (3)$$
$$D(t)=\mathrm{diag}\{[\varrho^n(t)\ \varrho^{n-1}(t)\dots\varrho(t)]^\top\},$$

where $K=[K_1\dots K_n]\in\mathbb{R}^{1\times n}$ is the control gain that will be selected later, $\varrho(t)=\frac{T}{T-t}$ is a strictly growing and unbounded function of time for $t\in[0,T)$.

Remark 1. Introducing mild modifications in the forthcoming analysis, any continuous strictly growing and unbounded (for $t\to T$) function of time $\varrho:[0,T)\to\mathbb{R}_+$ with $\varrho(0)=1$, having also unbounded integral, can be used in (3) (see also [24, Lemma 4.2]).

Remark 2. As it is common for the prescribed-time converging systems [14], in the presence of escaping to infinity at $t=T$ gain $\varrho(t)$, the right-hand side of (3) is defined on a finite interval of time $[0,T)$. Due to this, for $t\geq T$, frequently, another stabilization or estimation algorithm is applied, since $D(t)$ is not yet defined behind T. Such ax extension is natural due to well-posedness of the system on $[0,T)$, and next for small regulation errors other solutions can be used [15]. However, such a switching does not take into account the advantageous uniformity of convergence in the disturbance magnitude feasible with prescribed-time controllers, which may be difficult to ensure through other methods. In this work, we will later consider another approach in the discrete-time setting by replacing $t\in\mathbb{R}_+$ with $\mathrm{mod}\,(t,T)\in[0,T)$. In such a case $\varrho\,(\mathrm{mod}(t,T))$ periodically ranges from 1 till $+\infty$ while t passes from iT to $(i+1)T$, correspondingly, for any $i\in\mathbb{Z}_+$.

Define the dynamics of the closed-loop system (2), (3):

$$\dot{x}(t)=(A-bKD(t))\,x(t)+bd(t)-bKD(t)v(t)$$

and introduce for future reference auxiliary variables:

$$\Gamma(t)=\varrho^{-n}(t)D(t)$$
$$=\mathrm{diag}\{[1\quad \varrho^{-1}(t)\dots\varrho^{1-n}(t)]^\top\}$$

and

$$\tilde{\iota}_i(t)=T\begin{cases}\frac{\varrho^{-2(i-1)}(t)-\varrho^{-T}(t)}{T-2(i-1)} & \text{if } T\neq 2(i-1),\\ \varrho^{-T}(t)\ln\varrho(t) & \text{if } T=2(i-1)\end{cases}$$

for $i=\overline{1,n+1}$. Then, the result of [1], [23] can be repeated for the considered uPTS scenario:

Lemma 2. *Let $P=P^\top\in\mathbb{R}^{n\times n}$, $U\in\mathbb{R}^n$, $\gamma_1>0$ and $\gamma_2>0$ be such that the linear matrix inequalities are verified:*

$$P^{-1}\succ 0,\ \begin{bmatrix} Q_{11} & b & -b\\ b^\top & -\gamma_1 & 0\\ -b^\top & 0 & -\gamma_2\end{bmatrix}\preceq 0,$$
$$Q_{11}=P^{-1}\left(A-\frac{1}{T}H\right)^\top+\left(A-\frac{1}{T}H\right)P^{-1}$$
$$-bU-U^\top b^\top+P^{-1}.$$

Then for $K=UP$ and any $x(0)\in\mathbb{R}^n$, $d\in\mathcal{L}_\infty^1, v\in\mathcal{L}_\infty^n$ in (2), (3)*:*

$$\sqrt{\lambda_{\min}(P)}\begin{bmatrix}|x_1(t)|\\|x_2(t)|\\ \vdots\\ |x_n(t)|\end{bmatrix}\leq\begin{bmatrix}1\\ \varrho(t)\\ \vdots\\ \varrho^{n-1}(t)\end{bmatrix}$$
$$\times(\varrho^{-T/2}(t)\sqrt{x(0)^\top Px(0)}+\sqrt{\gamma_1\tilde{\iota}_{n+1}(t)}\|d\|_\infty$$
$$+\sqrt{n\gamma_2}\|v\|_\infty\sum_{i=1}^n|K_i|\sqrt{\tilde{\iota}_i(t)})$$

for all $t\in[0,T)$.

The linear matrix inequalities formulated in Lemma 2 are feasible due to the result of Lemma 1.

Theorem 1. *Under conditions of Lemma 2, if $T>2n-2$ and $\|v\|_\infty=0$, then* (2), (3) *is T-uPTS for $d\in\mathcal{L}_\infty^1$.*

The result of this theorem means that (3) is a prescribed-time stabilizer, and the class of disturbances $d\in\mathcal{L}_\infty^1$, for which the uniformity of the convergence is kept, can be enlarged by ones satisfying the constraint

$$\lim_{t\to T}d(t)\varrho^{n-1}(t)\sqrt{\tilde{\iota}_{n+1}(t)}=0.$$

Remark 3. Note that any arbitrary time of convergence T can be assigned by substituting $\varkappa t\to t$ in ϱ for any $\varkappa>1$. Moreover, this technical constraint disappears after the discretization proposed in the next section.

Remark 4. The upper bound on the transients of (2), (3) calculated in Lemma 2 implies that the gain of $|x_i(t)|$ in

$v(t)$ is proportional to $\varrho^{i-1}(t)$ for $i \in \overline{1,n}$, hence, it becomes infinite at $t = T$ for $i > 1$. It is a serious drawback for a non-vanishing noise v often met in practical implementation. Frequently, to avoid this issue, the precision is sacrificed by stopping the growth of controller gains slightly before the prescribed time instant T (it actually implies that the system looses the prescribed-time convergence quality).

Let us investigate what happens after discretization of (2), (3).

V. Discretization of (2), (3)

Note that (2), (3) is modeled by a linear time-varying system with external inputs $d(t)$ and $v(t)$. Since the time-varying gain $D(t)$ is strictly growing to infinity, the explicit Euler discretization cannot be used for all $t \in [0,T]$ (the resulted discrete-time dynamics will be unstable for any sampling rate with the growth of $\varrho(t)$), however, the implicit one can be effectively applied [25]. Let $h > 0$ be constant discretization step, denote by $t_k = hk$ for $k \in \mathbb{Z}_+$ the discretization time instants, and slightly loosing generality assume that there exists $N \in \mathbb{Z}_+$ such that $T = Nh$, then application of the implicit Euler discretization method to (2), (3) gives for $k \in \{\overline{0, N-1}\}$:

$$\xi_{k+1} = F^{-1}(t_{k+1})\left(\xi_k + hbd_{k+1} + L(t_{k+1})v_{k+1}\right), \quad (4)$$
$$F(t) = I_n - h\left(A - bKD(t)\right), \; L(t) = -hbKD(t),$$

where $\xi_k \in \mathbb{R}^n$ is an approximation of $x_k = x(t_k)$ (*i.e.*, $\xi_k \to x_k$ as $h \to 0$), $v_k = v(t_k)$ and $d_k = d(t_k)$. For the discrete-time part we assume that v_k and d_k take finite values with bounded norms as before, *i.e.*, in this section these sequences $d \in L^1_\infty$ and $v \in L^n_\infty$.

In order to calculate the expression of $F^{-1}(t)$ and to further analyze its properties, let us use the first block matrix inversion formula given in the preliminaries. To this end, represent this matrix as follows

$$F(t) = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C}(t) & \mathcal{D}(t) \end{bmatrix},$$

where

$$\mathcal{A} = I_{n-1} - h\begin{bmatrix} 0_{n-2\times 1} & I_{n-2} \\ 0 & 0_{1\times n-2} \end{bmatrix}, \; \mathcal{B} = \begin{bmatrix} 0_{n-2\times 1} \\ -h \end{bmatrix},$$
$$\mathcal{C}(t) = h\left[K_1\varrho^n(t)\dots K_{n-1}\varrho^2(t)\right], \; \mathcal{D}(t) = 1 + hK_n\varrho(t),$$

then

$$F^{-1}(t) = \frac{W(t)}{O(t)},$$
$$O(t) = \mathcal{D}(t) - \mathcal{C}(t)\mathcal{A}^{-1}\mathcal{B} = 1 + \sum_{i=1}^{n} K_i h^{n-i+1}\varrho^{n-i+1}(t),$$
$$W(t) = \begin{bmatrix} \mathcal{A}^{-1}(O(t)I_{n-1} + \mathcal{B}\mathcal{C}(t)\mathcal{A}^{-1}) & -\mathcal{A}^{-1}\mathcal{B} \\ -\mathcal{C}(t)\mathcal{A}^{-1} & 1 \end{bmatrix}.$$

Moreover, the direct computations show that

$$\mathcal{A}^{-1} = \begin{bmatrix} 1 & h & \cdots & h^{n-2} \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & h \\ 0 & \cdots & 0 & 1 \end{bmatrix}, \; \mathcal{A}^{-1}\mathcal{B} = -\begin{bmatrix} h^{n-1} \\ h^{n-2} \\ \vdots \\ h \end{bmatrix},$$
$$\mathcal{C}(t)\mathcal{A}^{-1} = \left[\left(\mathcal{C}(t)\mathcal{A}^{-1}\right)_1 \dots \left(\mathcal{C}(t)\mathcal{A}^{-1}\right)_{n-1}\right],$$

where

$$\left(\mathcal{C}(t)\mathcal{A}^{-1}\right)_j = \sum_{i=1}^{j} K_i h^{j-i+1}\varrho^{n-i+1}(t), \; j = 1, \dots, n-1.$$

As we can conclude performing these computations, the discrete state transition matrix $F^{-1}(t)$ is nonsingular (since $O(t) > 0$ for all $t \geq 0$ due to $K_i > 0$, $i = \overline{1,n}$ for a stabilizing control gain K) and elementwise bounded for all $t \geq 0$ (since $O(t)$ and $\mathcal{C}(t)\mathcal{A}^{-1}$ are polynomial functions of $\varrho(t)$ of order n, and $O(t)$ appears as the common denominator of $F^{-1}(t)$). Moreover,

$$\lim_{t\to T} F^{-1}(t) = -\begin{bmatrix} 0 & 0 & \cdots & 0 \\ h^{-1} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ h^{1-n} & \cdots & h^{-1} & 0 \end{bmatrix},$$

which implies that, after a finite time, the own dynamics of (4) becomes a static linear system with a nilpotent matrix of index n. The measurement noise v gain matrix

$$F^{-1}(t)L(t) = \begin{bmatrix} h^n \\ \vdots \\ h^2 \\ h \end{bmatrix} K\frac{D(t)}{O(t)}$$

is also elementwise bounded with

$$\lim_{t\to T} F^{-1}(t)L(t) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ h^{-1} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ h^{1-n} & 0 & \cdots & 0 \end{bmatrix},$$

which evaluates the terminal noise sensitivity (only the first component of v affects the system dynamics at the end of the time interval). And, obviously, the effect of the disturbance d is still annihilated by the control with

$$hF^{-1}(t)b = \frac{1}{O(t)}\begin{bmatrix} h^n \\ \vdots \\ h \end{bmatrix}.$$

To investigate stability and the rate of convergence in (4), we will consider first a finite interval of time with $k \in \{\overline{0, N-2}\}$, and next possible extensions for $k \in \mathbb{Z}_+$.

A. Analysis for $k \in \{\overline{0, N-2}\}$

Let us define a time-varying Lyapunov function candidate:

$$V_k = \xi_k^\top \Pi_k \xi_k,\ \Pi_k = \Gamma(t_{k+1}) P \Gamma(t_{k+1}),\ \forall k \in \{\overline{0, N-2}\},$$

where $P = P^\top \succ 0$ is as in Lemma 2 (more precise requirements will be defined below). Note that for $k = N-1$, by definition $t_{k+1} = T$, hence, $\Gamma(T) = \text{diag}\{[1 \quad 0 \dots 0]^\top\}$ and $\Gamma(T) P \Gamma(T)$ is a singular matrix, then such a choice of Π_k is admissible for $k \in \{\overline{0, N-2}\}$ only (it also explains our division of the stability analysis on two cases: for $k \leq N-2$ and $k = N-1$).

The following result can be formulated:

Theorem 2. *Let for some $\phi > 0$, $\psi > 0$ and $\sigma > 0$ there exist $P = P^\top \in \mathbb{R}^{n \times n}$, $K \in \mathbb{R}^{1 \times n}$ such that the matrix inequalities are verified:*

$$P \succ 0,\ (A - bK) P^{-1} + P^{-1} (A - bK)^\top \preceq 0,$$
$$H P^{-1} + P^{-1} H \succeq 0,$$
$$(A - bK) P^{-1} (A - bK)^\top - \psi^{-1} P^{-1}$$
$$-\phi^{-1} b K K^\top b^\top - \sigma^{-1} b b^\top \succeq 0.$$

Then for any $\xi_0 \in \mathbb{R}^n$, $v \in L_\infty^n$ and $d \in L_\infty^1$ in (4):

$$\|\xi_k\|^2 \leq \varrho^{2(n-1)}(t_k) \left(\frac{\psi}{h^2}\right)^k \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \|\xi_0\|^2 \prod_{i=0}^{k-1} \varrho^{-2}(t_{i+1})$$
$$+ \frac{\varrho^{2(n-1)}(t_k)}{\lambda_{\min}(P)} \sum_{i=0}^{k-1} \left(\frac{\psi}{h^2}\right)^{k-1-i} [\phi |v|_\infty^2 \sum_{l=1}^{n} \varrho^{2(1-l)}(t_{i+1})$$
$$+ \sigma \varrho^{-2n}(t_{i+1}) |d|_\infty^2] \prod_{j=i+1}^{k-1} \varrho^{-2}(t_{j+1})$$

for all $k \in \{\overline{0, N-2}\}$.

The feasibility of the imposed linear matrix inequalities again follows Lemma 1.

As we can see from the obtained results, the convergence in the initial error is similar to prescribed-time one for $t \in [0, T)$, and the dependence in d becomes infinitesimal at $k = N-2$, while the noise gain admits a static linear upper bound.

B. Analysis for $k \geq N-1$

For $k = N-1$, $t_{k+1} = T$ then all eigenvalues of the matrix $F^{-1}(T)$ are zero, hence, there is $\bar{P} = \bar{P}^\top \succ 0$ such that the Lyapunov equation reads the inequality $F^{-\top}(T) \bar{P} F^{-1}(T) \prec \beta \bar{P}$ for $\beta \in [0, 1)$. If we would like to extend the analysis to $k > N-1$ we need to introduce a definition of $F^{-1}(t)$ and $L(t)$ in (4) for $t > T$. A possible approach is just to take

$$F^{-1}(t) = F^{-1}(T),\ L(t) = L(T),\ \forall t > T.$$

In such a case let us modify our time-varying Lyapunov function $V_k = e_k^\top \Pi_k e_k$ as follows:

$$\Pi_k = \begin{cases} \Gamma(t_{k+1}) P \Gamma(t_{k+1}) & \text{if } k \in \{\overline{0, N-2}\} \\ \bar{P} & \text{if } k \geq N-1 \end{cases},$$

then it has been already proven an accelerated convergence of the state for $k \in \{\overline{0, N-2}\}$ in Theorem 2, while for $k \geq N-1$ we get the error dynamics (note that $F^{-1}(T) b = 0$)

$$\xi_{k+1} = F^{-1}(T) \xi_k + F^{-1}(T) L(T) v_{k+1}$$
$$= F^{-1}(T) \xi_k + \begin{bmatrix} 1 \\ h^{-1} \\ \vdots \\ h^{1-n} \end{bmatrix} v_{1,k+1},$$

which is independent in d_{k+1} and all $v_{i,k+1}$ for $i = \overline{2, n}$. Moreover, in the noise-free case $\xi_{N+n-1} = 0$, and we recover the prescribed-time convergence in n steps after T. This behavior can be interpreted as continuation for $t \geq T$ of the system trajectories in the noise-free setting by considering zero control input.

Another approach is to recursively apply the control (3) always staying onto the interval $k \in \{\overline{0, N-2}\}$ for the values of matrices F^{-1} and L:

$$\hat{x}_{k+1} = F^{-1}(\tilde{t}_{k+1}) \left(\xi_k + h b d_{k+1} + L(\tilde{t}_{k+1}) v_{k+1} \right), \quad (5)$$
$$\tilde{t}_{k+1} = \text{mod}\,(t_{k+1}, T)$$

for all $k \in \mathbb{Z}_+$. Then, the dependence on initial conditions is eliminated for $t_k \leq T$, and for $t_k > T$ such a regulator compensates the influence of any bounded disturbances d and filters the noise v following the advantageous performance evaluated in Theorem 2.

Remark 5. Recalling Remark 2, note that in the continuous-time noise-free case, a similar modification can be applied to (3) extending its application to all $t \geq 0$, but in the presence of an arbitrary bounded noise v the state x may become infinite while $\text{mod}\,(t, T) \to T$. Moreover, definition of solutions and analysis of well-posedness for $t \geq 0$ in a noise-free scenario is a subject of a separate research. In the discrete-time setting, (5) is naturally well-posed since F^{-1} and L are continuous functions on the entire segment $[0, T)$.

Let us illustrate the efficiency of the proposed iterative scheme in simulations.

VI. Simulations

For simulations, let us take $n = 2$ and

$$v(t) = \begin{bmatrix} \sin(2t) + \text{rnd}(1) \\ \cos(\pi t) + \text{rnd}(1) \end{bmatrix},\ d(t) = -\text{rnd}(1),$$

where rnd(1) generates a uniformly distributed in the interval $[0, 1]$ random number. For $T = 3 > 2n - 2$,

$$K = [1\ 1]$$

and the initial conditions $x(0) = [50\ 50]^\top$, the results of simulations are presented in figures 1 and 2 for $h = 0.1$ and $h = 0.01$, respectively. On the plots, the norm of the state vector is plotted for three scenarios: 1) green dash line for $v \equiv 0$ and $d \equiv 0$; 2) blue dot line for $v \equiv 0$ and d as above; 3) red solid line corresponds to the results with the noise

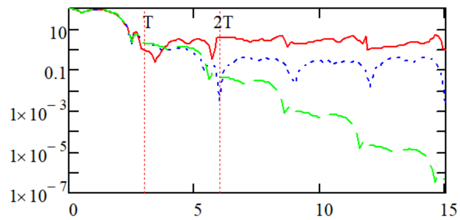

Figure 1. The state norm versus time t, $h = 0.1$

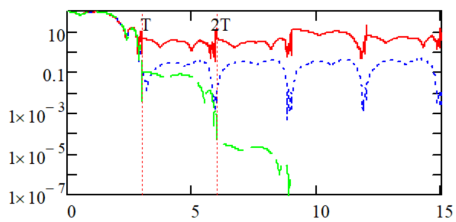

Figure 2. The state norm versus time t, $h = 0.01$

and disturbance. As we can conclude from these results, in the absence of noise and disturbance the state norm decays till a vicinity of zero at the end of the interval $[0, T)$ (the size of the vicinity depends on h), and further continues to converge to the origin (the behavior for $t \geq 9$ is not shown since it is out the scale). Moreover, the convergence rate is faster than exponential. In the noise-free case the influence of the disturbance is compensated at the end of each interval $[(i-1)T, iT)$ with $i = \overline{1,5}$, but later it is relaxed at the beginning of the next interval when the gains are resetted. In the presence of noises the behavior is similar, but more excited (note that noise has rather big amplitude with random and harmonic components). Recall that application of the explicit Euler discretization method to this example results in unbounded trajectories.

VII. Conclusion

A new simple discretization scheme for the prescribed time stabilizing control from [1], [23] is proposed, which guarantees an accelerated rate of convergence of the estimation error in discrete time, while remaining prescribed-time exact in the noise-free case. It has also good robustness with respect to the measurement noises. The tuning rules are formulated using feasible linear matrix inequalities. Development to the sampled-and-hold control implementation or extension to the MIMO case can be considered as directions for future research.

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