2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

Optimality Conditions in Infinite-Horizon Optimal Control Problem with Vanishing Discounting

Ilya Shvartsman

Abstract— **We consider an optimal control problem on infinite horizon with a vanishing discounting factor, state sufficient conditions of optimality and illustrate them with examples.**

I. INTRODUCTION

In this paper, we consider the controlled system

$$\begin{aligned} &\dot{x}(t) = f(x(t), u(t)),\ t \geq 0, \\ &x(0) = x_0, \\ &x(t) \in X, \\ &u(t) \in U, \end{aligned} \tag{1}$$

where $f(\cdot, \cdot) : I\!\!R^n \times U \to I\!\!R^n$, $U \subset I\!\!R^m$, and $X \subset I\!\!R^n$ plays a role of a state constraint.

A control $u(\cdot)$ and the pair $(x(\cdot), u(\cdot))$ are called admissible control and an admissible process, respectively, if $u(\cdot)$ is measurable, $x(\cdot)$ is absolutely continuous, and the relationships (1) are satisfied. The set of admissible controls is denoted by $\mathcal{U}(x_0)$, which we assume to be not empty.

The optimal control problem often considered on the trajectories of (1) is that of finding the infimum over admissible processes of

$$\int_0^\infty e^{-\lambda t} g(x(t), u(t))\, dt, \tag{2}$$

where $\lambda > 0$ is a parameter and $g(x, u) : I\!\!R^n \times U \to I\!\!R$ is a given function. Here $e^{-\lambda t}$ is the discounting factor, which is introduced to ensure convergence of the integral when appropriate assumptions on g are in place. λ in (2) is often chosen arbitrarily. If it is desired to take λ close to zero, then the following optimization problem with a *vanishing discounting factor* can be considered:

$$\inf_{u(\cdot)\in\mathcal{U}(x_0)} \limsup_{\lambda\to 0^+} \lambda \int_0^\infty e^{-\lambda t} g(x(t), u(t))\, dt. \tag{3}$$

The multiplicative factor λ is introduced here to ensure that the limit as $\lambda \to 0^+$ is bounded. We could as well have taken $\liminf\limits_{\lambda\to 0^+}$ rather than $\limsup\limits_{\lambda\to 0^+}$ in (3), but we chose the latter because taking $\inf\limits_{u(\cdot)\in\mathcal{U}(x_0)} \limsup\limits_{\lambda\to 0^+}$ can be interpreted as "minimization in the worst case scenario". In this paper we establish sufficient optimality conditions in problem (3).

It should also be mentioned that there are optimality criteria on infinite horizon that do not involve a discounting factor; see, e.g., [2], [12] and references therein.

We assume that $f(\cdot, \cdot)$ and $g(\cdot, \cdot)$ are Borel measurable and that g is bounded below. We do not assume that f and g are continuous. Neither do we assume any structure of U and X such as closedness or compactness.

If we interchange the infimum and the limit in (3), if the latter exists, we obtain the so-called *Abel limit*

$$\lim_{\lambda\to 0^+} \inf_{u(\cdot)\in\mathcal{U}(x_0)} \lambda \int_0^\infty e^{-\lambda t} g(x(t), u(t))\, dt. \tag{4}$$

The so-called *Cesàro limit* of the long-run averages

$$\lim_{T\to\infty} \inf_{u(\cdot)\in\mathcal{U}_T(x_0)} \frac{1}{T} \int_0^T g(x(t), u(t))\, dt \tag{5}$$

is closely related to it. (Here $\mathcal{U}_T(x_0)$ is the set of admissible controls on the interval $[0, T]$.)

A lot of literature is devoted to estabishing conditions of existence and equality of Cesàro and Abel limits in problems of dynamic programming and optimal control in discrete and continuous time, see, e.g., [6], [9], [11]. Relatively weak conditions ensuring equality of the limits (4) and (5) are established in the recent paper [6].

Optimality conditions in the problem

$$\inf_{u(\cdot)\in\mathcal{U}_T(x_0)} \liminf_{T\to\infty} \frac{1}{T} \int_0^T g(x(t), u(t))\, dt,$$

as well as in its discrete counterpart, were studied in [1], [6], [7], [8]. Optimality conditions for the discrete-time version of problem (3) recently appeared in [10]. Although the ideas in treatment of discrete-time and continuous-time problems are similar, there are also significant differences; e.g., nonsmooth analysis is used in the present paper, but not in discrete time.

The rest of the paper is organized as follows. In Section 2 we formulate sufficient optimality conditions in problem (3) and illustrate them by examples in Section 3.

II. SUFFICIENT OPTIMALITY CONDITIONS

Denote

$$d(x_0) := \sup_{(\psi,\eta)} \inf_{(x,u,p)} \{g(x, u) + (\psi(x_0) - \psi(x)) + pf(x, u)\}, \tag{6}$$

where supremum is taken over functions $\psi : I\!\!R^n \to I\!\!R$ non-decreasing along admissible trajectories, that is, such that

$$t \mapsto \psi(x(t)) \text{ is non-decreasing for any admissible } x(\cdot), \tag{7}$$

and bounded locally Lipschitz $\eta : I\!\!R^n \to I\!\!R$ (we write $\eta \in BL$). Infimum in (6) is taken over $(x, u) \in X \times U$ and $p \in \partial\eta(x)$, where $\partial\eta$ stands for the Clarke's generalized gradient ([3]), which for Lipschitz η is equal to the convex hull of the limits of its gradients, that is, has representation

$$\partial\eta(x) = \operatorname{conv} \{p |\, p = \lim_{i\to\infty} \nabla\eta(x_i) \text{ for some } x_i \to x\}.$$

I. Shvartsman is with the Dept. of Mathematics and Computer Science, Penn State Harrisburg, Middletown, PA 17057, USA, ius13@psu.edu

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Since g is assumed to be bounded below, by taking constant ψ and η in (6), we see that $d(x_0) > -\infty$.

For a fixed $\lambda > 0$ denote

$$h_\lambda(x_0) := \lambda \inf_{u(\cdot)\in\mathcal{U}(x_0)} \int_0^\infty e^{-\lambda t} g(x(t), u(t))\, dt. \quad (8)$$

Since g is bounded below, so is $h_\lambda(x_0)$. Assume in addition that there exists an admissible process $(x(\cdot), u(\cdot))$ such that $g(x(t), u(t)) \le M$ for all t. Then $h_\lambda(x_0) \le M$ for all λ.

The following theorem, which is the main result of the paper, provides conditions ensuring the existence of the limit $\lim_{\lambda\to 0^+} h_\lambda(x_0)$ and of optimality of a given process in problem (3). We state it without proof.

Theorem 2.1: Assume that a pair $(\bar\psi, \bar\eta)$ of maximizers in problem (6) exists and for some admissible process $(x^*(\cdot), u^*(\cdot))$ and all $t \ge 0$,

$$(x^*(t), u^*(t)) = \operatorname{argmin}_{(x,u)}\{g(x,u) - \bar\psi(x) + pf(x,u)\} \text{ for all } p \in \partial\bar\eta(x) \quad (9)$$

and

$$\psi(x^*(t)) = \text{const}. \quad (10)$$

Then

(a) there exists the limit $h(x_0) := \lim_{\lambda\to 0^+} h_\lambda(x_0)$;

(b) there is equality

$$V(x_0) = h(x_0) = d(x_0), \quad (11)$$

where

$$V(x_0) := \inf_{u(\cdot)\in\mathcal{U}(x_0)} \limsup_{\lambda\to 0^+} \lambda \int_0^\infty e^{-\lambda t} g(x(t), u(t))\, dt$$

is the value function in (3);

(c) the process $(x^*(\cdot), u^*(\cdot))$ is optimal in (3).

Remark 1. If function $\bar\eta$ is smooth, then (9) becomes

$$(x^*(t),u^*(t)) = \operatorname{argmin}_{(x,u)}\{g(x,u) - \bar\psi(x) + \nabla\bar\eta(x) f(x,u)\}. \quad (12)$$

However, as we will see in examples below, $\bar\eta$ may be nonsmooth even in simple situations.

Remark 2. The limiting function h may be discontinuous and functions $\bar\psi, \bar\eta$ may depend on x_0, as shown in Example 2 below.

Remark 3. For (9) to hold it is necessary that for all t

$$u^*(t) = \operatorname{argmin}_{u\in U}\{g(x^*(t), u) + pf(x^*(t), u)\} \text{ for all } p \in \partial\bar\eta(x)\big|_{x=x^*(t)}, \quad (13)$$

which implies the optimal feedback control law

$$u^f[x] = \operatorname{argmin}_{u\in U}\{g(x,u) + pf(x,u)\} \text{ for all } p \in \partial\bar\eta(x).$$

In the case when $\bar\eta$ is smooth, the latter becomes

$$u^f[x] = \operatorname{argmin}_{u\in U}\{g(x,u) + \nabla\bar\eta(x) f(x,u)\}. \quad (14)$$

If $\bar\eta$ is known, this law can be used to construct an optimal control in (3). If the maximizing η is not known, it may be possible to approximate it and to construct a control close to the optimal. This approach is demonstrated in [5] in a problem with a fixed discounting factor. In the case of the vanishing dicounting factor, developing a method for constructing an approximately optimal control may be a subject of further research.

Theorem 2.1 establishes sufficient conditions of optimality in terms of the maximizing functions ψ, η in (6). Theorem 2.2 below demonstrates a possible way for finding one such pair of functions. (It may be not unique.) We also state this theorem without proof.

Theorem 2.2: Let the pointwise limit $h(x_0) = \lim_{\lambda\to 0^+} h_\lambda(x_0)$ exist for all $x_0 \in X$ and $\bar\eta(\cdot) \in BL$ be such that

$$\inf_{(x,u,p)}\{g(x,u) - h(x) + pf(x,u)\} = 0, \quad (15)$$

where infimum is taken over $(x,u) \in X \times U$ and $p \in \partial\bar\eta(x)$. Then the supremum in (6) is reached at the functions $\psi = h$ and $\eta = \bar\eta$.

Remark. If $\bar\eta$ is smooth, (15) becomes

$$\inf_{(x,u)}\{g(x,u) - h(x) + \nabla\bar\eta(x) f(x,u)\} = 0. \quad (16)$$

III. EXAMPLES

In this section, applications of Theorems 2.1 and 2.2 are demonstrated.

Example 1.

Consider the problem

$$\inf_{u(\cdot)\in\mathcal{U}(x_0)} \limsup_{\lambda\to 0^+} \lambda \int_0^\infty e^{-\lambda t}(1 - x(t))^2\, dt \quad (17)$$

on the trajectories of the system

$$\begin{aligned} &\dot x(t) = (1 - x(t))^2 u(t),\ t > 0,\\ &x(0) = x_0,\\ &x(t) \in (0,2),\\ &u \in [-1,1]. \end{aligned} \quad (18)$$

In this example, $g(x) = (1-x)^2$ and $X = (0,2)$.

It is obvious that the control that makes the system approach $x = 1$ as quickly as possible, is optimal. That is, the optimal feedback control law is

$$u^f[x] = \begin{cases} 1, & x \in (0,1),\\ -1, & x \in (1,2),\\ \text{any}, & x = 1, \end{cases} \quad (19)$$

and the corresponding optimal trajectory is

$$x^*(t) = \begin{cases} 1 - \frac{1}{t + 1/(1-x_0)}, & x_0 \in (0,1)\\ 1 + \frac{1}{t+1/(x_0-1)}, & x_0 \in (1,2),\\ 1, & x_0 = 1. \end{cases} \quad (20)$$

Let us show that $h(x) = 0$ for all $x \in X$. We see from (20) that the integral

$$\int_0^T (1 - x^*(t))^2 \, dt$$

is uniformly bounded with respect to T, hence

$$h(x_0) = \lim_{\lambda \to 0^+} \lambda \int_0^\infty e^{-\lambda t}(1 - x^*(t))^2 = 0.$$

Next we will show that (15) holds with $\bar{\eta}(x) = |1 - x|$. We have

$$\partial\bar{\eta}(x) = \begin{cases} -1, & x < 1, \\ 1, & x > 1, \\ [-1, 1], & x = 1 \end{cases}$$

and for $p \in \partial\bar{\eta}(x)$ we have

$$\begin{aligned} & g(x) - h(x) + pf(x, u) = (1 - x)^2 + p(1 - x)^2 u \\ & = (1 - x)^2(1 + pu) \\ & = \begin{cases} (1 - x)^2(1 - u), & x \in (0, 1), \\ (1 - x)^2(1 + u), & x \in (1, 2), \\ 0, & x = 1. \end{cases} \end{aligned} \tag{21}$$

Therefore,

$$\min_{(x,u,p)} \{g(x) - h(x) + pf(x, u)\} = 0, \tag{22}$$

that is, (15) holds. Due to Theorem 2.2, maximizing functions in (6) are $\bar{\psi} = 0$ and $\bar{\eta} = |1 - x|$.

From (19) and (21) one can see that, for $u^*(t) := u^f[x^*(t)]$ we have for all t and $p(t) \in \partial\bar{\eta}(x)\big|_{x=x^*(t)}$

$$g(x^*(t)) - h(x^*(t)) + p(t)f(x^*(t), u^*(t)) = 0.$$

This implies via (22) that

$$(x^*(t), u^*(t)) = \mathrm{argmin}_{(x,u,p)}\{g(x) - \bar{\psi}(x) + pf(x, u)\}$$
for all $p \in \partial\bar{\eta}(x)$,

and, since $h(x) = 0$, we have

$$h(x^*(t)) = \text{const}.$$

Thus, (9) and (10) hold, hence, the process $(x^*(\cdot), u^*(\cdot))$ is optimal due to Theorem 2.1, which agrees with our earlier observation.

Example 2.

Consider the problem

$$\inf_{u(\cdot)\in\mathcal{U}(x_0)} \limsup_{\lambda \to 0^+} \lambda \int_0^\infty e^{-\lambda t}(-x(t)) \, dt \tag{23}$$

on the trajectories of the system

$$\begin{aligned} & \dot{x}(t) = x(t)u(t), \; t \geq 0, \\ & x(0) = x_0, \\ & x(t) \in [0, 1], \\ & u \in [-1, 1]. \end{aligned}$$

In this example, $g(x) = -x$ and $X = [0, 1]$.

It is clear that the feedback control law below is optimal in problem (23):

$$u^f[x] = \begin{cases} \text{any } u, & x = 0, \\ 1, & x \in (0, 1), \\ 0, & x = 1. \end{cases} \tag{24}$$

If $x_0 = 0$, then $x(t) \equiv 0$ and $h_\lambda(0) = 0$ for all $\lambda > 0$. Otherwise, if $x_0 \in (0, 1]$, the optimal trajectory reaches $x = 1$ at some time τ independent of λ and stays there for $t \geq \tau$. Therefore,

$$\begin{aligned} h_\lambda(x_0) &= \lambda \int_0^\tau e^{-\lambda t}(-x^*(t)) \, dt + \lambda \int_\tau^\infty e^{-\lambda t}(-1) \, dt \\ &= \lambda \int_0^\tau e^{-\lambda t}(-x^*(t)) \, dt - e^{-\lambda\tau}, \end{aligned}$$

from which we conclude that $h(x_0) = \lim_{\lambda \to 0^+} h_\lambda(x_0) = -1$. Thus,

$$h(x) = \lim_{\lambda \to 0^+} h_\lambda(x) = \begin{cases} 0, & x = 0, \\ -1, & x \in (0, 1]. \end{cases} \tag{25}$$

Notice that h is discontinuous.

Let us construct $\bar{\eta}_{x_0}$ such that (15) holds. (In this example $\bar{\eta}_{x_0}$ depends on x_0, for this reason we keep it in the subscript.)

For $x_0 = 0$ set $\bar{\eta}_{x_0}(x) \equiv 0$. In this case

$$g(x) - h(x) + \nabla\bar{\eta}_{x_0}(x)f(x, u) = \begin{cases} 0, & x = 0, \\ -x + 1, & x \in (0, 1], \end{cases}$$

and (16) holds.

If $x_0 \in (0, 1]$ set

$$\bar{\eta}_{x_0}(x) = \begin{cases} x_0 - \ln x_0, & x \in [0, x_0) \\ x - \ln x, & x \in [x_0, 1]. \end{cases}$$

In this case, at the points $x \neq x_0$ where $\bar{\eta}_{x_0}$ is differentiable, we have

$$\begin{aligned} & g(x) - h(x) + \nabla\bar{\eta}_{x_0}(x)f(x, u) = \\ & \begin{cases} 0, & x = 0, \\ -x + 1, & x \in (0, x_0), \\ -x + 1 + (1 - 1/x)xu = (1 - x)(1 - u), & x \in (x_0, 1]. \end{cases} \end{aligned} \tag{26}$$

At $x = x_0$ function $\bar{\eta}_{x_0}$ is not differentiable and from the properties of the generalized gradient it follows that $g(x_0) - h(x_0) + \partial\bar{\eta}_{x_0}(x_0)f(x_0, u)$ is equal to the interval between the points $-x_0 + 1$ and $(1 - x_0)(1 - u)$. From these formulas we see that (15)-(16) also hold.

Due to Theorem 2.2, maximizing functions in (6) are $\bar{\psi} = h$ given by (25) and $\bar{\eta}_{x_0}$. As seen from (24) and the bottom line of (26), for $u^*(t) := u^f[x^*(t)]$ we have for all $t > 0$

$$g(x^*(t)) - h(x^*(t)) + \nabla\bar{\eta}_{x_0}(x^*(t))f(x^*(t), u^*(t)) = 0.$$

This implies via (16) and (26) that

$$(x^*(t), u^*(t)) = \mathrm{argmin}_{(x,u)}\{g(x) - \bar{\psi}(x) + \nabla\bar{\eta}_{x_0}(x)f(x, u)\}.$$

From (25) we see that

$$h(x^*(t)) = \text{const}$$

(if $x_0 = 0$ then $h(x^*(t)) \equiv 0$, if $x_0 \in (0, 1]$ then $h(x^*(t)) \equiv -1$.)

Thus, (9) and (10) hold, hence, the process $(x^*(\cdot), u^*(\cdot))$ is optimal due to Theorem 2.1, which agrees with the observation made above.

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