IEEE Control Systems Letters paper presented at
2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

Observer-driven Switching Design for Stabilizing Continuous-time Switched Autonomous Linear Systems

Miaomiao Wang† and Zhendong Sun ‡
† Key Laboratory of Systems & Control, Academy of Mathematics & Systems Science
Chinese Academy of Sciences, Beijing 100190, China
‡ College of Electrical Engineering & Automation
Shandong University of Science & Technology, Qingdao 266590, China

Abstract— **For switched autonomous linear systems, feedback stabilization is to seek a proper switching strategy to steer the system exponentially stable. For continuous-time switched autonomous linear systems, under the mild assumption that the system is switched observable, we propose a novel hybrid observer that could exactly estimate the system state in any given time interval. For a stabilizable switched system, by incorporating the observer into the pathwise switching mechanism, the resultant observer-driven switching law could achieve exponential stability.**

I. Introduction

A hybrid system consists of continuous dynamics, discrete/logic dynamics, and their interactions. While there is no formal formulation yet, it is widely recognized that switching and impulsive behaviors are two basic characterizations of hybrid dynamical systems, and the interaction between them poses interesting and challenging issues that are theoretically appealing [11], [12]. In particular, switched systems constitute a specific yet fundamental class of hybrid systems with wide representability and powerful control ability. The reader is referred to the monographs [16], [8] for background and progress of switched and hybrid systems.

For a switched linear autonomous system with freely designed switching signal, the stabilization problem is to design, when possible, a proper switching law that steers the system exponentially stable. The stabilization problem has long been a core & classical problem that attracts numerous attention in the literature, and huge progress has been made over the past decades. As the dominate approach for addressing the stability/stabilizability of switched systems, various Lyapunov methods including the multiple Lyapunov function technique [5], and the composite Lyapunov function technique [9], have been successfully applied to stability analysis and stabilizing design of switched systems. However, it was found that, even for planar switched linear systems, a stabilizable system might not admit any convex (control-) Lyapunov function [4]. This means that convex-set-based searching is not enough for finding Lyapunov function candidates. Another major approach is the optimization approach. Indeed, for discrete-time switched linear systems, the seminal work [20] established a general framework for solving the switched LQ problem, which also provides constructive design for the stabilization problem. Unfortunately, the iterating design idea is not applicable to continuous-time switched systems. There are some other approaches developed in the literature, for instance, the automata-driven switching scheme [7], the dissipativity approach [21], the small gain approach [19], [13], the structural decomposition approach [14], [23], the dwell-time switching scheme [1], [6], and the phase portrait method [18], yet they are only applicable to specific classes of switched systems.

When the system state is not totally available, state observers/estimators were designed to asymptotically estimate the state. In Ref. [17], an observer design method was proposed for a class of hybrid systems with both switching and impulse. The observer itself is also a hybrid system, which could asymptotically converge to the actual state under persistent switching. Other observer design schemes could be found in [3], [22], [2]. For switched linear systems, it has been established that observability is dual with reachability, and observer design is dual with state feedback stabilizing design [15]. As revealed in [15, Example 5.23], generally there is no one-to-one feedback control law for the feedback stabilizability. By the duality principle, when the observer is also a switched system with Luenburger-like subsystems, it is impossible to assign one gain matrix for a subsystem, which implies that more-than-one gain matrices have to be designed for one subsystem. Since the stabilization problem is still largely unsolved for continuous-time switched linear systems, the observer design is also a challenging and difficult issue yet to be addressed.

In this work, we investigate the dynamic-output-feedback stabilization problem for continuous-time switched linear autonomous systems with measured outputs. To estimate the state, a hybrid observer with state impulse is proposed. By incorporating the observer into the switched system, we design a common switching law that stabilizes the extended system. The contributions of the work include: i) Under the mild assumption of switched observability, a hybrid observer with state impulse is proposed; the observer itself is a switched impulsive system that shares a common switching law with the original system, and we prove that the observer could achieve exact state reconstruction in any given time interval via a properly designed switching signal and an impulsive time sequence; and ii) We design an observer-driven switching law that stabilizes both the original switched

Copyright ©2023 IEEE

system and the observer; the switching law is a proper concatenation of a set of piecewise norm-contractive switching paths via observer-driven mechanism at the concatenating time instants, and the switching law is always well posed.

II. Preliminaries

Notation. Let n, m, p be positive integers. Let $\|\cdot\|$ be any given vector norm or induced matrix norm. For a matrix A, let A^T be its transpose. Let I_n be the identity matrix with dimension n.

The switched linear autonomous system in this work is described by

$$\begin{aligned} \dot{x}(t) &= A_{\sigma(t)}x(t) \\ y(t) &= C_{\sigma(t)}x(t), \end{aligned} \tag{1}$$

where $x(t) \in \mathbf{R}^n$ is the system continuous state, $\sigma(t) \in M \triangleq \{1, 2, \cdots, m\}$ is the system discrete state, also known as switching signal, $y(t) \in \mathbf{R}^p$ is the system output, and $A_1, \cdots, A_m$ and $C_1, \cdots, C_m$ are real constant matrices with compatible dimensions. Let $\phi(\cdot; t_0, x_0, \sigma)$ be the state trajectory starting from $x(t_0) = x_0$ along switching signal σ. We assume without loss of generality that $t_0 = 0$.

Definition 2.1: Switched system (1) is said to be (exponentially) stabilizable, if there are positive real numbers α and β, such that for any $x_0 \in \mathbf{R}^n$, there exists a switching signal $\sigma : [0, +\infty) \to M$ satisfying

$$\|\phi(t; 0, x_0, \sigma)\| \le \beta e^{-\alpha t}\|x_0\|, \quad \forall\ t \ge 0.$$

A switching path is a switching signal defined over a finite time interval. For a switching path θ defined over $[0, s)$, the length of θ, s, is denoted by $|\theta|$. The state transition matrix along θ is denoted Φ_θ. It is clear that

$$\phi(|\theta|; 0, x_0, \theta) = \Phi_\theta x_0, \quad \forall\ x_0 \in \mathbf{R}^n.$$

Let θ_1 and θ_2 be two switching paths. The concatenation of θ_1 and θ_2, denoted by $\theta_1 \sqcup \theta_2$, is defined to be

$$\theta_1 \sqcup \theta_2(t) = \begin{cases} \theta_1(t) & \text{if } t \in [0, |\theta_1|) \\ \theta_2(t) & \text{if } t \in [|\theta_1|, |\theta_1| + |\theta_2|). \end{cases}$$

Concatenation of more than two switching paths could be defined in the same manner.

Lemma 2.1: [16, Thm. 4.30] Switched system (1) is stabilizable iff for any positive real number λ, there exist a positive integer κ, and switching paths $\theta_1, \ldots, \theta_\kappa$, such that

$$\min_{i=1}^{\kappa} \|\phi(|\theta_i|; 0, x_0, \theta_i)\| \le \lambda\|x_0\|, \quad \forall\ x_0 \in \mathbf{R}^n. \tag{2}$$

The set of switching paths satisfying (2) is said to be a λ-norm-contractive switching path set. A constructive design procedure was developed in [16, §4.4.2] to compute a set of pathwise contractive switching paths.

Definition 2.2: A state observer for switched system (1) is a dynamical system

$$\Delta z(t) = f(t, z(t), \sigma(t), y(t)), \quad z(0) = z_0, \tag{3}$$

where f is a proper vector function, and $\Delta z(t)$ is either the differential operator $\dot{z}(t)$ or the impulsive operator $z(t+)$. For a switching signal σ, the observer is said to be σ-asymptotic if $\lim_{t\to+\infty} \|x(t) - z(t)\| = 0$ for any x_0 and z_0. When there exists a $T \ge 0$ such that $x(t) = z(t)$ for all $t \ge T$, the observer is said to be σ-deadbeat.

Definition 2.3: Switched system (1) is said to be dynamic output stabilizable, if there exist a state observer and an observer-driven switching signal that steer the switched system and the observer exponentially convergent.

Definition 2.4: For switched system (1), state x_0 is said to be unobservable if it is indistinguishable from the origin, that is, for any switching signal σ we have

$$C_{\sigma(t)}\phi(t; 0, x_0, \sigma) = 0, \quad \forall\ t \ge 0.$$

The switched system is said to be completely (switched) observable if the unobservable set is $\{0\}$.

Lemma 2.2: Suppose that switched system (1) is completely observable. Then, there exist a positive integer $l \le \frac{n(n+1)}{2} - 1$, and a sequence of indices $j_0, j_1, \cdots, j_l$, such that matrix

$$\begin{bmatrix} C_{j_0} \\ C_{j_1}e^{A_{j_0}h_1} \\ C_{j_2}e^{A_{j_1}h_2}e^{A_{j_0}h_1} \\ \vdots \\ C_{j_l}e^{A_{j_{l-1}}h_l}\cdots e^{A_{j_0}h_1} \end{bmatrix} \tag{4}$$

is of full rank for almost all real numbers $h_1, \ldots, h_{l-1}$.

Proof: It has been proven in [10, Theorem 1] that, for any completely controllable switched system, there is a switching path that realizes the full controllability. In addition, the switching path is with not more than $\frac{n(n+1)}{2} - 1$ switches. By the duality principle [15, Corollary 4.8], there is a switching path p with l switches, $l \le \frac{n(n+1)}{2} - 1$, that fully realizes the complete observability. That is, the equations

$$\begin{aligned} y(t_0) &= C_{j_0}x_0 \\ y(t_1) &= C_{j_1}e^{A_{j_0}(t_1-t_0)}x_0 \\ y(t_l) &= C_{j_l}e^{A_{j_{l-1}}(t_l-t_{l-1})}\cdots e^{A_{j_0}(t_1-t_0)}x_0 \end{aligned}$$

always admit a unique x_0 for any $\{y(t_0), \ldots, y(t_l)\}$, where $0 = t_0 < t_1 < \ldots < t_{l-1} < t_l = |p|$ are switching times of p, and $j_k = p(t_k)$, $k = 0, 1, \ldots, l$, are switching indices of p. It follows that the matrix

$$\begin{bmatrix} C_{j_0} \\ C_{j_1}e^{A_{j_0}(t_1-t_0)} \\ \vdots \\ C_{j_l}e^{A_{j_{l-1}}(t_l-t_{l-1})}\cdots e^{A_{j_0}(t_1-t_0)} \end{bmatrix}$$

is of full rank. Let $\rho_k = t_k - t_{k-1}$, $k = 1, \cdots, l$. Re-write the above matrix, we have

$$\text{rank}\begin{bmatrix} C_{j_0} \\ C_{j_1}e^{A_{j_0}\rho_1} \\ \vdots \\ C_{j_l}e^{A_{j_{l-1}}\rho_l}\cdots e^{A_{j_0}\rho_1} \end{bmatrix} = n. \tag{5}$$

Define matrix function

$$r(h_1,\ldots,h_l)=\begin{bmatrix} C_{j_0} \\ C_{j_1}e^{A_{j_0}h_1} \\ \vdots \\ C_{j_l}e^{A_{j_{l-1}}h_l}\ldots e^{A_{j_0}h_1} \end{bmatrix}.$$

It follows from (5) that $r(\rho_1,\ldots,\rho_l)$ is of full rank. Thus, there are n rows that is linearly independent. Examine the corresponding sub-matrix of $r(h_1,\ldots,h_l)$. The determinant of the sub-matrix is non-zero at $\rho_1,\ldots,\rho_l$. This, together with the fact that function r is an analytic function in terms of $h_1,\ldots,h_l$, the sub-matrix is nonsingular for almost all $h_1,\ldots,h_l$. This completes the proof.

III. Main Results

A. Hybrid Observer: Single-output Case

Switched system (1) is said to be of single output if $\sum_{i=1}^m \operatorname{rank} C_i = 1$. Without loss of generality, we assume that the first row of C_1, denoted C_{11}, is nonzero.

For the matrix in (4), by searching linearly independent rows from the top to the bottom, we have n rows that form a nonsingular matrix. To be more specific, there exist the row indices $1=\iota_0<\iota_1<\iota_2<\cdots<\iota_{n-1}\le l$, such that matrix

$$Q=\begin{bmatrix} C_{11} \\ C_{11}\Pi_{k=\iota_1-1}^{1}e^{A_{j_k}h_k} \\ \vdots \\ C_{11}\Pi_{k=\iota_{n-1}-1}^{1}e^{A_{j_k}h_k} \end{bmatrix} \tag{6}$$

is square and nonsingular. Clearly, Q contains only those rows of the matrix in (4) started with C_{11}. Matrix Q is said to be the observability matrix, which plays a key role in designing the gain matrices of the observer. Note that $C_2=\cdots=C_m=0$. Due to the nested structure of Q, we have $j_{\iota_k}=1$, $k=0,1,\ldots,n-1$. Define recursively

$$t_k=t_{k-1}+h_k,\quad k=1,2,\ldots,\iota_{n-1}.$$

Let

$$\tau_{i+1}=t_{\iota_i-1},\quad i=0,1,\ldots,n-1. \tag{7}$$

Note that each row of Q is a multiplication of C_{11} with matrix exponentials. Denote

$$\begin{aligned} D_i &= e^{A_1h_{\iota_{i-1}}},\quad i=1,\ldots,n \\ G_i &= \Pi_{k=\iota_i-1}^{\iota_{i-1}+1}e^{A_{j_k}h_k},\quad i=1,\ldots,n-1. \end{aligned}$$

In terms of D_is and G_is, we re-write matrix Q by

$$Q=\begin{bmatrix} C_{11} \\ C_{11}G_1D_1 \\ \vdots \\ C_{11}G_{n-1}D_{n-1}\cdots G_1D_1 \end{bmatrix}.$$

Let ζ_i be the j-th column of matrix Q^{-1}. Furthermore, define

$$\begin{aligned} H_1 &= \zeta_1 \\ H_2 &= G_1D_1\zeta_2 \\ &\vdots \\ H_n &= G_{n-1}D_{n-1}\cdots G_1D_1\zeta_n. \end{aligned} \tag{8}$$

It can be seen that each H_i is a column vector in $\mathbf{R}^n$.

Define

$$\begin{aligned} E_1 &= (\Pi_{i=n}^2(D_i-D_iH_iC_{11})G_{i-1})D_1H_1 \\ E_2 &= (\Pi_{i=n}^3(D_i-D_iH_iC_{11})G_{i-1})D_2H_2 \\ &\vdots \\ E_{n-1} &= (D_n-D_nH_nC_{11})G_{n-1}D_{n-1}H_{n-1} \\ E_n &= D_nH_n \\ \Psi_0 &= D_nG_{n-1}\cdots D_2G_1D_1. \end{aligned}$$

Furthermore, denote $E=[E_1E_2\cdots E_n]$.

Lemma 3.1: $E=\Psi_0Q^{-1}$.

Proof: As $H_1=\zeta_1$ is orthogonal to the second row of matrix Q, namely, $C_{11}G_1D_1$, we have

$$E_1=(\Pi_{i=n}^3(D_i-D_iH_iC_{11})G_{i-1})D_2G_1D_1\zeta_1.$$

Continue this process gives

$$E_1=(\Pi_{i=n}^2D_iG_{i-1})D_1\zeta_1=\Psi_0\zeta_1.$$

In the same manner, it is computed that

$$E_i=\Psi_0\zeta_i,\ i=2,\cdots,n.$$

Based on the above reasonale, we have

$$E=\Psi_0[\zeta_1\zeta_2\cdots\zeta_n]=\Psi_0Q^{-1}.$$

This completes the proof.

To estimate the state, we propose the following hybrid observer:

$$\begin{aligned} \dot z(t) &= A_{\sigma(t)}z(t),\quad t\notin\{\tau_1,\cdots,\tau_n\} \\ z(t+) &= (I_n-H_iC_{11})z(t-)+H_iy_1(t-) \\ &\qquad t=\tau_i,\ i=1,\cdots,n \\ z(0) &= 0, \end{aligned} \tag{9}$$

where y_1 denotes the first entry of output y, $z(t+)=\lim_{s\searrow t}z(s)$ and $z(t-)=\lim_{s\nearrow t}z(s)$.

Remark 3.1: The observer is a switched dynamical system with state impulse. The impulsive times were given in (7), and the gain columns H_is are given in (8). Note that the switching signal is the same as the original switched system (1). In other words, both the original switched system and the observer shares the same switching signal. As there are only finite impulses, the observer dynamics are well defined.

Remark 3.2: When continuous-time switched linear system (1) is embedded with a switched observer without any impulse, the overall system is still a switched linear system. As a switched autonomous system admit a unique solution for any switching signal, the observer could not be deadbeat due to the fact that the origin is an equilibrium. A motivation of introducing the impulses in the observer is to achieve deadbeat estimation of the unmeasured state in a finite time, as proved later. Another innovation of the proposed observer is that we only need to measure the output in discrete times, which makes the measurement more implementable.

Define a switching path

$$p_0(t)=j_k,\quad \text{when } t\in[t_{k-1},t_k),\quad k=1,2,\ldots,\iota_{n-1}. \tag{10}$$

It can be seen that $|p_0| = t_{\iota_{n-1}}$.

Theorem 3.1: Suppose that the single-output switched system (1) is completely observable. For any initial state x_0, we have

$$z(|p_0|) = \phi(|p_0|; 0, x_0, p_0). \quad (11)$$

Proof: Let $e(t) = x(t) - z(t)$, $t \in [0, |p_0|)$. It can be verified that

$$\dot{e}(t) = A_{p_0(t)} e(t), \quad t \notin \{\tau_1, \cdots, \tau_n\}$$
$$e(\tau_i+) = (I - H_i C_{11}) e(\tau_i-), \ i = 1, \cdots, n.$$

As a result, we have

$$\begin{array}{rcl} e(|p_0|) & = & D_n(I - H_n C_{11}) G_{n-1} D_{n-1} (I - H_{n-1} C_{11}) \\ & & \cdots G_1 D_1 (I - H_1 C_{11}) e(0) \stackrel{def}{=} \Psi e(0). \end{array}$$

Simple calculation gives

$$\Psi = -[E_1 E_2 \cdots E_n] Q + \Psi_0 = -EQ + \Psi_0.$$

It follow from Lemma 3.1 that $E = \Psi_0 Q^{-1}$, which directly leads to $\Psi = 0$. This completes the proof.

Remark 3.3: Theorem 3.1 shows that the proposed hybrid observer could exactly estimate the state. Note that $|p_0|$ could be arbitrarily small for achieving the observability via switching path p_0 [10]. This means that we could design the deadbeat observer to achieve exact state estimate in any pre-assigned time interval. On the other hand, the design of the observer gains is of high-gain in nature, which might produce large system overshoot. Indeed, the transient performance improvement is an interesting issue for further investigation.

B. Hybrid Observer: Multi-output Case

For general multi-output switched systems, the design procedure for single-output case could be extended in two ways. The first way is direct extension: starting from (4), searching properly to produce a nonsingular observability matrix, and designing the hybrid observer in form (9) with suitable impulsive times and gain matrices. The design procedure is essentially the same yet the notational burden is high. Alternatively, an indirect way is to transform the multi-output system into a single-output system that preserves the observability property, and design the observer for the transformed single-output system. To avoid symbolic burden, we briefly present the indirect design approach accordingly.

Lemma 3.2: Suppose that switched system (1) is completely observable and the first row of C_1, denoted C_{11}, is non-zero. Then, there exists matrices $L_i \in \mathbf{R}^{n \times p}$, such that switched linear system

$$\begin{array}{rcl} \dot{x}(t) & = & (A_{\sigma(t)} - L_{\sigma(t)} C_{\sigma(t)}) x(t) \\ y_1(t) & = & \bar{C}_{\sigma(t)} x(t), \end{array} \quad (12)$$

is completely observable, here $\bar{C}_1 = C_{11}$, $\bar{C}_i = 0$, $i = 2, \ldots, m$.

Proof: By the duality principle and Theorem 4.51 in [15], the lemma follows straightforwardly.

As system (12) is completely observable, we have positive integers $1 = \bar{\iota}_0 < \bar{\iota}_1 < \ldots < \bar{\iota}_n$ and index sequence $\bar{j}_1, \ldots, \bar{j}_{\bar{\iota}_n}$, such that $\bar{j}_{\bar{\iota}_k} = 1$, $k = 1, \ldots, n$, and

$$\bar{Q} \stackrel{def}{=} \begin{bmatrix} \bar{C}_1 \\ \bar{C}_1 \Pi_{k=\bar{\iota}_1 - 1}^{1} e^{\bar{A}_{\bar{j}_k} \bar{h}_k} \\ \vdots \\ \bar{C}_1 \Pi_{k=\bar{\iota}_{n-1} - 1}^{1} e^{\bar{A}_{\bar{j}_k} \bar{h}_k} \end{bmatrix} \quad (13)$$

is nonsingular for almost all $\bar{h}_1, \ldots, \bar{h}_{\bar{\iota}_{n-1}}$, where $\bar{A}_i = A_i - L_i C_i$, $i \in M$.

Define recursively

$$\bar{t}_k = \bar{t}_{k-1} + \bar{h}_k, \quad k = 1, \ldots, \bar{\iota}_{n-1}.$$

Let

$$\bar{\tau}_{i+1} = \bar{t}_{\bar{\iota}_i - 1}, \quad i = 0, 1, \ldots, n-1. \quad (14)$$

Furthermore, denote

$$\begin{array}{rcl} \bar{D}_i & = & e^{\bar{A}_1 \bar{h}_{\bar{\iota}_{i-1}}}, \quad i = 1, \ldots, n \\ \bar{G}_i & = & \Pi_{k=\bar{\iota}_i - 1}^{\bar{\iota}_{i-1}+1} e^{\bar{A}_{j_k} \bar{h}_k}, \quad i = 1, \ldots, n-1. \end{array}$$

Re-write matrix $\bar{Q}$ by

$$\bar{Q} = \begin{bmatrix} \bar{C}_1 \\ \bar{C}_1 \bar{G}_1 \bar{D}_1 \\ \vdots \\ \bar{C}_1 \bar{G}_{n-1} \bar{D}_{n-1} \cdots \bar{G}_1 \bar{D}_1 \end{bmatrix}.$$

Let $\bar{\zeta}_i$ be the j-th column of matrix $\bar{Q}^{-1}$. Furthermore, define

$$\begin{array}{rcl} \bar{H}_1 & = & \bar{\zeta}_1 \\ \bar{H}_2 & = & \bar{G}_1 \bar{D}_1 \bar{\zeta}_2 \\ & \vdots & \\ \bar{H}_n & = & \bar{G}_{n-1} \bar{D}_{n-1} \cdots \bar{G}_1 \bar{D}_1 \bar{\zeta}_n. \end{array} \quad (15)$$

The hybrid observer is

$$\begin{array}{rcl} \dot{\bar{z}}(t) & = & \bar{A}_{\sigma(t)} \bar{z}(t) + L_{\sigma(t)} y(t), \quad t \notin \{\bar{\tau}_1, \cdots, \bar{\tau}_n\} \\ \bar{z}(t+) & = & (I_n - \bar{H}_i \bar{C}_1) \bar{z}(t-) + \bar{H}_i y_1(t-) \\ & & t = \bar{\tau}_i, \ i = 1, \cdots, n \\ \bar{z}(0) & = & 0. \end{array} \quad (16)$$

Define a switching path

$$\bar{p}_0(t) = j_k, \quad \text{when } t \in [\bar{t}_{k-1}, \bar{t}_k), \quad k = 1, 2, \ldots, \bar{\iota}_{n-1}. \quad (17)$$

It can be seen that $|p_0| = \bar{t}_{\iota_{n-1}}$.

Theorem 3.2: Suppose that switched system (1) is completely observable. For any initial state x_0, we have

$$\bar{z}(|\bar{p}_0|) = \phi(|\bar{p}_0|; 0, x_0, \bar{p}_0). \quad (18)$$

Proof: Let $\bar{e}(t) = x(t) - \bar{z}(t)$, $t \in [0, |\bar{p}_0|)$. It can be verified that

$$\dot{\bar{e}}(t) = \bar{A}_{\bar{p}_0(t)} \bar{e}(t), \quad t \notin \{\bar{\tau}_1, \cdots, \bar{\tau}_n\}$$
$$\bar{e}(\bar{\tau}_i+) = (I_n - \bar{H}_i \bar{C}_1) \bar{e}(\bar{\tau}_i-), \ i = 1, \cdots, n.$$

Utilizing the method in the proof of Theorem 3.1, we could prove that the state transition matrix of the error system is null along $\bar{p}_0$. This directly leads to (18).

C. Observer-driven Stabilizing Design

Based on the discussion in the previous subsections, we assume that the switched system admits deadbeat observer (16) along switching path $\bar{p}_0$. Furthermore, suppose that $\theta_1, \ldots, \theta_\kappa$ form a set of λ-norm-contractive switching paths, where $\lambda \in (0,1)$. By definition, path $\bar{p}_0$ is with not more than $\frac{n(n+1)}{2} - 1$ switches, and the switching times of $\bar{p}_0$ could be (almost) arbitrarily chosen. Therefore, we could design $\bar{p}_0$ such that

$$|\bar{p}_0| \leq -\ln(\frac{1+\lambda}{2})(\max_{i=1}^{m} \|A_i\|). \tag{19}$$

For the augmented system (1)-(16), we are to design a common switching law that stabilizes the overall system. By extending the pathwise state feedback switching law developed in [16, §4.4.1], we propose the pathwise observer-driven switching law in the following manner.

Starting from initial time $t_0 = 0$, we pick switching path $\bar{p}_0$, which is designed for the observer to estimate the state. Let $\bar{T}_1 = |\bar{p}_0|$. The state of observer (16) at $\bar{T}_1$ is

$$z_1 = \bar{\phi}(|\bar{p}_0|; 0, 0, \bar{p}_0),$$

where $\bar{\phi}$ denotes the solution of observer system (16). Next, we are to pick a switching path that makes z_1 λ-norm-contractive along the path. For this, choose ϑ_1 from the set $\{\theta_1, \ldots, \theta_\kappa\}$ such that

$$\|\bar{\phi}(|\vartheta_1|; 0, z_1, \vartheta_1)\| = \min_{i=1}^{\kappa}\{\|\bar{\phi}(|\theta_i|; 0, z_1, \theta_i)\|\}.$$

If there are two or more switching paths that achieve the minimum, then just pick one that satisfies the equation. Let $z_2 = \bar{\phi}(|\vartheta_1|; 0, z_1, \vartheta_1)$. Continue the process, and define recursively that

$$\begin{aligned} \bar{T}_k &= \bar{T}_{k-1} + |\vartheta_{k-1}| \\ \vartheta_k &= \{\theta \in \{\theta_1, \ldots, \theta_\kappa\} : \\ &\quad \|\bar{\phi}(|\theta|; 0, z_k, \theta)\| = \min_{i=1}^{\kappa}\{\|\bar{\phi}(|\theta_i|; 0, z_k, \theta_i)\|\}\}. \\ z_{k+1} &= \bar{\phi}(|\vartheta_k|; 0, z_k, \vartheta_k), k = 2, 3, \ldots. \end{aligned} \tag{20}$$

Finally, let

$$\sigma = \bar{p}_0 \sqcup \vartheta_1 \sqcup \vartheta_2 \sqcup \cdots. \tag{21}$$

Note that the switching law depends on the observer measurement, but does not depend on the state measurement. Note also that, the switching law is concatenating $\bar{p}_0, \vartheta_1, \ldots$ recursively to form a well-posed switching signal over time interval $[0, +\infty)$. In the switching law, the observer path, $\bar{p}_0$, is used to estimate the state, and the observer-driven mechanism presented in (20) applies when the state is exactly reconstructed.

Theorem 3.3: Suppose that switched system (1) is both stabilizable and completely observable. The observer-driven switching law (21) steers the overall system (1)-(16) exponentially convergent.

Proof: It can be verified that

$$\begin{aligned} \|\phi(\bar{T}_1; 0, x_0, \sigma)\| &\leq \exp(\max_{i=1}^{m} \|A_i\| |\bar{p}_0|)\|x_0\| \\ &\leq \frac{2}{1+\lambda}\|x_0\|. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} \|\phi(\bar{T}_k; 0, x_0, \sigma)\| &\leq \lambda\|\phi(\bar{T}_{k-1}; 0, x_0, \sigma))\| \\ &\leq \lambda^2\|\phi(\bar{T}_{k-2}; 0, x_0, \sigma)\| \\ &\ddots \\ &\leq \lambda^{k-2}\|x_0\|, \end{aligned}$$

which implies that

$$\|\phi(t; 0, x_0, \sigma)\| \leq \beta e^{-\alpha t}\|x_0\|,$$

where

$$\begin{aligned} \alpha &= -\frac{\ln(\lambda)}{\max\{|\bar{p}_0|, |\theta_1|, \ldots, |\theta_\kappa|\}} \\ \beta &= \frac{\exp(\max_{i=1}^{m} \|A_i\| \max\{|\bar{p}_0|, |\theta_1|, \ldots, |\theta_\kappa|\})}{\lambda^2}. \end{aligned}$$

This means that any state trajectory is exponentially convergent.

On the other hand, for the observer system, as $x(|\bar{p}_0|) = \bar{z}(|\bar{p}_0|)$, we have

$$\bar{A}_i\bar{z}(|\bar{p}_0|) + L_i y(|\bar{p}_0|) = A_i x(|\bar{p}_0|),$$

which further implies that

$$\bar{z}(t) = x(t), \quad \forall\ t \geq |\bar{p}_0|.$$

This means that the observer is also exponentially convergent.

D. Numerical Example

Examine the fourth-order two-form switched linear system (1) with

$$\begin{aligned} A_1 &= \begin{bmatrix} 1.5 & 0 & 0 & 0 \\ 1 & -6 & 0.5 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -0.2 & 0.5 & -1.5 \end{bmatrix} \\ A_2 &= \begin{bmatrix} 1 & -4 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & 0 & -1.5 & -1 \\ -1 & 0.3 & 1 & 1 \end{bmatrix} \\ C_1 &= [1\ \ 0\ \ 0\ \ 0] \\ C_2 &= [0\ \ 0\ \ 0\ \ 0]. \end{aligned}$$

It can be verified that neither subsystem is stable and observable. It is clear that

$$\text{rank}\begin{bmatrix} C_1 \\ C_1A_2 \\ C_1A_2A_1 \\ C_1A_2A_1A_2 \end{bmatrix} = 4,$$

which implies that the switched system is completely observable. By applying the constructive procedure in [16, §4.4.2], we could compute a set of norm-contractive switching paths *w.r.t.* the L_1-norm. This implies that the switched system is state feedback stabilizable.

Utilizing the complete observability property, we calculate the full rank matrix

$$Q = \begin{bmatrix} C_1 \\ C_1 e^{A_2 h_2} e^{A_1 h_1} \\ C_1 e^{A_2 h_4} e^{A_1 h_3} e^{A_2 h_2} e^{A_1 h_1} \\ C_1 e^{A_2 h_6} e^{A_1 h_5} e^{A_2 h_4} e^{A_1 h_3} e^{A_2 h_2} e^{A_1 h_1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1.1012 & -0.1547 & -0.0046 & 0 \\ 1.1706 & -0.2885 & -0.0131 & 0.0002 \\ 1.2106 & -0.4013 & -0.0248 & 0.0008 \end{bmatrix},$$

where $h_i = 0.05$, $i = 1, \dots, 6$. Accordingly, the gain matrices are computed to be

$$H_1 = \begin{bmatrix} 1 \\ 1.88 \\ -390.01 \\ -4121.46 \end{bmatrix}, H_2 = \begin{bmatrix} 1 \\ -4.93 \\ 402.57 \\ 12705.16 \end{bmatrix},$$

$$H_3 = \begin{bmatrix} 1 \\ -4.93 \\ 402.57 \\ -13040.67 \end{bmatrix}, H_4 = \begin{bmatrix} 1 \\ -4.93 \\ -420.38 \\ 4454.67 \end{bmatrix}.$$

As discussed in Remark 3.3, the gains are high in nature.

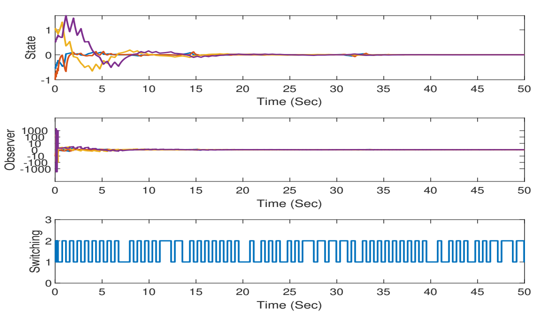

Fig. 1. System dynamics

With these preparations, we proceed to simulate the system behavior. Let $x_0 = [-0.5 \quad -1 \quad 1 \quad 0.5]^T$. Fig. 1 depicts the state, the observer, and the switching signal of the extended system. It can be seen that both the system state and the observer state are exponentially convergent.

IV. Conclusion

In this work, the problem of stabilization has been addressed for general continuous-time switched linear autonomous systems with measured outputs. We proposed a hybrid observer with both switches and impulses, and proved that the observer could achieve exact reconstruction of the state in any pre-assigned time interval. By incorporating the observer into the pathwise feedback switching law, we developed the pathwise observer-driven switching strategy that could achieve exponential stability of the augmented system. A numerical example was presented to show the effectiveness of the proposed design scheme.

Acknowledgements

The authors are grateful to the anonymous reviewers and the AE for their constructive comments. This work was supported by the National Key R&D Program of China (2018YFA0703800), and the Natural Science Foundation of Shandong Province (ZR2020ZD26).

References

[1] L. I. Allerhand and U. Shaked, "Robust stability and stabilization of linear switched systems with dwell time," *IEEE Trans. Automat. Contr.*, vol. 56, no. 2, pp. 381–386, 2010.

[2] P. Bernard and R. G. Sanfelice, "Observers for hybrid dynamical systems with linear maps and known jump times," In : *IEEE Conf. Decis. Contr.*, 2018, pp. 3140–3145.

[3] F. Blanchini, S. Miani, and F. Mesquine, "A separation principle for linear switching systems and parametrization of all stabilizing controllers," *IEEE Trans. Automat. Contr.*, vol. 54, no. 2, pp. 279–292, 2009.

[4] F. Blanchini and C. Savorgnan, "Stabilizability of switched linear systems does not imply the existence of convex Lyapunov functions," *Automatica*, vol. 44, no. 4, pp. 1166–1170, 2008.

[5] M. S. Branicky, "Multiple Lyapunov functions and other analysis tools for switched and hybrid systems," *IEEE Trans. Automat. Contr.*, vol. 43, no. 4, pp. 475–482, 1998.

[6] Z. Fei, C. Guan, and X. Zhao, "Event-triggered dynamic output feedback control for switched systems with frequent asynchronism," *IEEE Trans. Automat. Contr.*, vol. 65, no. 7, pp. 3120–3127, 2019.

[7] M. Fiacchini, M. Jungers, and A. Girard, "Stabilization and control Lyapunov functions for language constrained discrete-time switched linear systems," *Automatica*, vol. 93, no. 1, pp.64–74, 2018.

[8] J. Fu and R. Ma, *Stabilization and H_∞ Control of Switched Dynamic Systems,* Switzerland: Springer, 2021.

[9] T. Hu, L. Ma, and Z. Lin, "Stabilization of switched systems via composite quadratic functions," *IEEE Trans. Automat. Contr.*, vol. 53, no. 11, pp. 2571–2585, 2008.

[10] Z. Ji, L. Wang, and X. Guo, "Design of switching sequences for controllability realization of switched linear systems," *Automatica*, vol. 43, no. 4, pp. 662–668, 2007.

[11] X. Li and S. Song, *Impulsive Systems with Delays*, Berlin: Springer, 2022.

[12] H. Lin and P. J. Antsaklis, *Hybrid Dynamical Systems: Fundamentals and Methods*, Berlin: Springer, 2022.

[13] L. Long, "Multiple Lyapunov functions-based small-gain theorems for switched interconnected nonlinear systems," *IEEE Trans. Automat. Contr.*, vol. 62, no. 8, pp. 3943–3958, 2017.

[14] Z. Sun, "Feedback stabilization of third-order switched linear control systems," *IEEE Contr. Syst. Lett.*, vol. 4, no. 4, pp. 857–861, 2020.

[15] Z. Sun and S. S. Ge, *Switched Linear Systems: Control and Design*, London: Springer, 2005.

[16] Z. Sun and S. S. Ge, *Stability Theory of Switched Dynamical Systems*, London: Springer, 2011.

[17] A. Tanwani, S. Hyungbo, and D. Liberzon, "Observability for switched linear systems: Characterization and observer design." *IEEE Trans. Automat. Contr.*, vol. 58, no. 4, pp. 891–904, 2014.

[18] X. Xu and P. J. Antsaklis, "Stabilization of second-order LTI switched systems," *Int. J. Contr.*, vol. 73, no. 14, pp. 1261–1279, 2000.

[19] G. Yang and D. Liberzon, "A Lyapunov-based small-gain theorem for interconnected switched systems," *Syst. Contr. Lett.,* vol. 78, pp. 47–54, 2015.

[20] W. Zhang, A. Abate, J. H. Hu, et al., "Exponential stabilization of discrete-time switched linear systems," *Automatica*, vol. 45, no. 11, pp. 2526–2536, 2009.

[21] J. Zhao and D. J. Hill, "Dissipativity theory for switched systems," *IEEE Trans. Automat. Contr.*, vol. 53, no. 4, pp. 941–953, 2008.

[22] X. Zhao, H. Liu, J. Zhang, *et al.*, "Multiple-mode observer design for a class of switched linear systems," *IEEE Trans. Autom. Sci. Engin.,* vol. 12, no. 1, pp. 272–280, 2013.

[23] Y. Zhu and Z. Sun, "Stabilizing design for reversible discrete-time switched linear control systems: A deadbeat control approach," *Automatica*, vol. 129, 109617, 2021.