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Further results on the structure of normal forms of input-affine nonlinear MIMO systems

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Abstract— **In a recent paper it has been shown that the existence, for a MIMO nonlinear system, of normal forms with a special structure that proves to be useful in the design of feedback laws is implied by an assumption introduced a long time ago by Hirschorn in his work on systems invertibility. In this paper, we provide an alternative viewpoint and prove that a necessary and sufficient condition for the existence of such kind of normal forms can be identified in a special feature of the so-called maximal controlled invariant distribution algorithm.**

I. Introduction

The problem of controlling a MIMO system is a fundamental problem in control theory. If the system is nonlinear, and in what follows we refer in particular to the case of the so-called *input-affine* systems, solving major design problems – such as stabilization, disturbance-isolation, non-interacting control, asymptotic tracking/rejection of exogenous inputs – is not terribly difficult if the system has a vector relative degree. As a matter of fact, successful solutions to such problems became available in the 1980's, and appropriate robust versions thereof in the two subsequent decades. However, as it is well-known, having a vector relative degree is a quite restrictive hypothesis. A much broader class of systems would be that of those systems that are *invertible*, from an input-output viewpoint. In fact, invertibility is a fundamental property that makes it possible to solve a variety of design problems for *linear* MIMO systems. Therefore, it makes sense to try to address similar design problems for such broader class of systems. For an input-affine MIMO system, characterization – and exploitation in the context of feedback design – of the property of invertibility reposes on suitable recursive algorithms, known as *structure algorithms*, which can be seen as extensions of an algorithm introduced by Silverman [1] for the construction of inverses of a linear MIMO system, extended by Hirschorn [2] and then further extended by Singh [3], a special case of which is the so-called *zero dynamics* algorithm [7, pp. 294-296] (see also [6] for analysis of various related issues). The specific features of the algorithm of [3] were in fact exploited for the design of globally stabilizing feedback laws by Liberzon in [11], where it is shown that global stabilization can be achieved via state feedback if the system is invertible and has a property that can be viewed as an extension of the property of being minimum-phase. To the best of our knowledge, this seems to be the only available result, to date, dealing with the design of globally stabilizing feedback laws for such broad class of systems. The method in question, though, reposes on auxiliary hypotheses that are not easy to test, requires accurate knowledge of all functions that characterize the model of the system, and full access to the state.

An approach that has proven very successful in feedback design for SISO systems, and for those MIMO systems that have vector relative degree, is the one based on transforming the model of the system into its so-called *normal form*. It is for this reason that, in recent years, the analysis of normal forms for MIMO systems that do not possess a relative degree, but are otherwise invertible, gained interest.

The nonlinear versions of the structure algorithms used to characterize invertibility do in fact provide data that can be used to define a change of coordinates yielding a normal form (see, e.g., [9] and [10, pp. 109-124]). However, such normal forms are not of immediate use in the solution of design problems, unless extra assumptions are made. By taking such assumptions (that will be described in detail in section III), a class of systems that sits between the "broad" class of systems that are invertible and the "narrow" class of systems that have a relative degree is identified. Such class has been recently successfully considered in [15], for the design of observer-based stabilizing laws, and in [16], which extends to MIMO systems an important result of Freidovich and Khalil [12] concerning the design of feedback laws by means of which it is possible to robustly recover the performances achievable by means of feedback-linearization-based methods.

The assumptions in question are specified in terms of properties of various functions that characterize the expression of the system in normal form. Thus, in principle, it is not possible to say whether or not the system belongs to such special class until the normal form has been actually built, and this is a non-negligible issue. Hence, the interest arises of determining conditions (sufficient or, if possible, necessary and sufficient), in terms of the original system's data, ensuring the existence of normal forms with such special properties. This problem has been addressed in [14] for systems having two inputs, and – in a general setting – in the recent paper [17], where it was shown that the existence of normal forms with such properties is actually a *consequence of a stronger property of invertibility*, characterized by an assumption introduced much earlier by Hirschorn in [2]. In the present paper we take an alternative viewpoint and we show that a necessary and sufficient condition for the existence of a normal form having such special properties can be determined by looking at certain features of the *algorithm* for computing the *largest controlled invariant distribution* contained in the kernel of the differential of the output map.

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II. NORMAL FORMS OF MIMO SQUARE SYSTEMS

We consider in this paper MIMO systems modeled by equations of the form

$$\begin{array}{rcll} \dot{x} & = & f(x)+g(x)u & x\in\mathbb{R}^n, \quad u\in\mathbb{R}^m \\ y & = & h(x) & y\in\mathbb{R}^m\,, \end{array} \tag{1}$$

which we assume to be *uniformly invertible* in the sense of Singh [3]. It has been shown in various earlier papers (see, e.g., [9] and [10, pages 109-112]) that a consequence of the property of uniform invertibility is the existence of a submersion $\Xi:\mathbb{R}^n\to\mathbb{R}^d$ defined as

$$\xi=\Xi(x):=\mathrm{col}\{X_1(x),X_2(x),\dots,X_{k^*}(x)\}\,, \tag{2}$$

in which the $X_i(x)$'s are vector-valued functions, recursively constructed by means of the so-called structure algorithm (see [1][2][3] and [4]), that can be used to define a (partial if $d<n$) set of new coordinates. The functions in question have the following properties, inherited from the structure algorithm:

- $X_1(x)=h(x)$
- there exist an integer $\ell\le m$, a strictly increasing sequence $0=r_0<r_1<r_2<\ldots<r_\ell=k^*$ and, for each $i=1,2,\dots,k^*$, a splitting
$$X_i(x)=\begin{pmatrix}X_i'(x)\\X_i''(x)\end{pmatrix}$$
in which the upper block X_i' of X_i is empty for
$$\begin{array}{c} i=1,\dots,r_1-1,r_1+1,\dots,r_2-1,r_2+1, \\ \dots,r_3-1,r_3+1,\dots,r_\ell-1, \end{array}$$
while $X_{r_1}',X_{r_2}',\dots,X_{r_\ell}'$ are not empty and $X_{r_\ell}'=X_{r_\ell}$.
- there exists a set of integers $m_1,m_2,\dots,m_\ell$, with $\sum_{j=1}^{\ell}m_j=m$, such that $X_i\in\mathbb{R}^{m_j+\cdots+m_\ell}$, for $r_{j-1}+1\le i\le r_j$ with $j=1,\dots\ell$, and $X_{r_j}'\in\mathbb{R}^{m_j}$.
- the matrix [1]
$$\begin{pmatrix}L_gX_1'(x)\\L_gX_2'(x)\\\cdots\\L_gX_{k^*}'(x)\end{pmatrix}$$
is square and nonsingular for each $x\in\mathbb{R}^n$,
- for each $i=1,2,\dots,k^*-1$, there exists matrices $F_{i,1}(x),F_{i,2}(x),\dots,F_{i,i}(x)$ such that [2]
$$\begin{aligned} &L_gX_i''(x)\\ &=-\begin{pmatrix}F_{i,1}(x) & F_{i,2}(x) & \dots & F_{i,i}(x)\end{pmatrix}\begin{pmatrix}L_gX_1'(x)\\L_gX_2'(x)\\\cdots\\L_gX_i'(x)\end{pmatrix}\end{aligned}$$
- for each $i=1,2,\dots,k^*-1$
$$\begin{aligned} X_{i+1}(x)&=L_fX_i''(x)\\ &+\begin{pmatrix}F_{i,1}(x) & F_{i,2}(x) & \dots & F_{i,i}(x)\end{pmatrix}\begin{pmatrix}L_fX_1'(x)\\L_fX_2'(x)\\\cdots\\L_fX_i'(x)\end{pmatrix}.\end{aligned}$$

By means of simple calculations (see [13, pp. 268-272]), it is found that the sets $X_1,\dots,X_{r_1}$ satisfy equations of the form

$$\begin{array}{rcl} \dot{X}_1 & = & X_2 \\ & & \cdots \\ \dot{X}_{r_1-1} & = & X_{r_1} \\ \dot{X}_{r_1}' & = & a_1(x)+b_1(x)u \\ \dot{X}_{r_1}'' & = & X_{r_1+1}-F_{r_1,r_1}(x)[a_1(x)+b_1(x)u]\,, \end{array} \tag{3}$$

the sets $X_{r_1+1},\dots,X_{r_2}$ satisfy equations of the form

$$\begin{array}{rl} \dot{X}_{r_1+1} & =X_{r_1+2}-F_{r_1+1,r_1}(x)[a_1(x)+b_1(x)u] \\ & \cdots \\ \dot{X}_{r_2-1} & =X_{r_2}-F_{r_2-1,r_1}(x)[a_1(x)+b_1(x)u] \\ \dot{X}_{r_2}' & =a_2(x)+b_2(x)u \\ \dot{X}_{r_2}'' & =X_{r_2+1}-\sum_{j=1}^{2}F_{r_2,r_j}(x)[a_j(x)+b_j(x)u], \end{array} \tag{4}$$

and so on, until it is found that the sets $X_{r_{\ell-1}+1},\dots,X_{r_\ell}$ satisfy equations of the form

$$\begin{array}{rl} \dot{X}_{r_{\ell-1}+1} & =X_{r_{\ell-1}+2}-\sum_{j=1}^{\ell-1}F_{r_{\ell-1}+1,r_j}(x)[a_j(x)+b_j(x)u] \\ & \cdots \\ \dot{X}_{r_\ell-1} & =X_{r_\ell}-\sum_{j=1}^{\ell-1}F_{r_\ell-1,r_j}(x)[a_j(x)+b_j(x)u] \\ \dot{X}_{r_\ell} & =a_\ell(x)+b_\ell(x)u\,. \end{array} \tag{5}$$

where

$$\begin{array}{rcl} a_j(x) & = & L_fX_{r_j}'(x) \\ b_j(x) & = & L_gX_{r_j}'(x). \end{array}$$

If $d<n$, a complementary set of $n-d$ coordinates z can be found, so as to obtain a locally defined diffeomorphism. Under appropriate hypotheses,[3] it can be shown (see [9] and also [13, pp. 274-275]) that a smooth map $Z:\mathbb{R}^n\to\mathbb{R}^{n-d}$ exists, that makes the map

$$\tilde{x}=\begin{pmatrix}z\\\xi\end{pmatrix}=\begin{pmatrix}Z(x)\\\Xi(x)\end{pmatrix}:=\Phi(x)$$

a globally defined diffeomorphism. Accordingly, the normal form can be completed by adding the dynamics

$$\dot{z}=f_0(z,\xi)+g_0(z,\xi)u \tag{6}$$

in which $f_0(z,\xi)=L_fZ(x)|_{x=\Phi^{-1}(\tilde{x})}$ and $g_0(z,\xi)=L_gZ(x)|_{x=\Phi^{-1}(\tilde{x})}$.

[1]It must be borne in mind that the only nonempty blocks of the matrix below are the $L_gX_{r_j}'(x)$'s, with $j=1,\dots,\ell$. Since $L_gX_{r_j}'(x)\in\mathbb{R}^{m_j\times m}$, the matrix in question is square.

[2]Clearly, the matrix $F_{i,h}$ is empty if the corresponding block L_gX_h' is empty. Thus, for $i=1,\dots,r_1-1$ all $F_{i,h}$'s are empty and, for each $r_j\le i\le r_{j+1}-1$, with $j=1,\dots,\ell-1$, the only non-empty blocks are $F_{i,r_1},F_{i,r_2},\dots,F_{i,r_j}$. The notation used here may seem redundant. However, it facilitates some passages in the subsequent analysis.

[3]The assumption in question considers the vector fields (11), and requires that certain linear combinations of such vector fields, with coefficient that are entries of the matrices $F_{i,h}(x)$, be *complete*.

III. A RELEVANT CLASS OF INVERTIBLE SYSTEMS

As shown in the recent papers [15] and [16], the structure of the normal form thus described can be fruitfully exploited in the design of feedback laws if the coefficient matrices $F_{i,h}(\Phi^{-1}(\tilde{x}))$ are independent of the component z of $\tilde{x}$ and depend on the components X_k's of $\tilde{x}$ in a special "triangular" fashion. More precisely, [15] and [16] considered the case in which the normal form has the property indicated below.

Property P. For all $r_j \leq i \leq r_{j+1} - 1$, $h = r_1, \ldots r_j$ and $j = 1, \ldots, \ell - 1$,

$$\frac{\partial F_{i,h}(\Phi^{-1}(\tilde{x}))}{\partial z} = 0\,, \tag{7}$$

$$\frac{\partial F_{i,h}(\Phi^{-1}(\tilde{x}))}{\partial X_k} = 0 \qquad \text{if } k > i\,. \tag{8}$$

Indeed, not all invertible systems have a normal form in which the properties indicated above hold. Nevertheless, the properties in question make it possible to successfully address various major problems of analysis and design. For instance, it can be shown that, if $d = n$ and property (8) holds, the state x of such systems can be expressed as a function of its output and of a suitable set of its higher-order time-derivatives, that is the system is uniformly completely observable (UCO) in the sense of [8]. For such systems it is possible to design a dynamical extension yielding an (extended) system having vector relative degree $\{r_\ell, \ldots, r_\ell\}$, a property that can be exploited in the design of observer-based dynamic output-feedback stabilizers [15]. In case $d > n$, if both properties (7) and (8) hold and the system is strongly minimum phase, in the sense of [17], it is possible to robustly recover, by means of appropriate output-feedback control laws, the dynamic performances that would have been achieved by means of the classical (non-robust) feedback-linearization methods and to solve problems of robust output regulation [16][17].

In view of the convenience of dealing with systems for which such property holds, the problem arose of determining conditions, on the data $f(x), g(x), h(x)$ that characterize the model (1), to the purpose of directly identifying the class of systems that possess a normal form of this kind, without explicit computation of the normal form itself. A preliminary answer was given in the paper [14], but this was limited only to the case of system with two inputs. In the recent paper [18], assuming (7), it was shown that the special dependence on the X_k's indicated in (8) is a consequence of an hypothesis introduced earlier by Hirschorn in his seminal work [2] on invertibility of nonlinear, input-affine, systems. In this paper we present an alternative set of hypothesis, implying – for the $F_{i,h}(\Phi^{-1}(\tilde{x}))$'s – the properties indicated in (7) and (8), by showing that this is a consequence of certain properties of the algorithm for computing the so-called maximal controlled invariant distribution contained in $\ker(dh)$ (see [7, section 6.3]).

IV. EXPLOITING CONTROLLED INVARIANCE

We assume in what follows that the reader is familiar with the notion of controlled invariance for a nonlinear, input-affine, system and on the consequence of such notion on the internal structure of a system.[4]

Since the map Ξ is a submersion, the d rows of $d\Xi$ are linearly independent at each $x \in \mathbb{R}^n$. Thus, the codistribution $\mathrm{span}\{d\Xi\}$ is a codistribution of dimension d and the distribution

$$\Delta = (\mathrm{span}\{d\Xi\})^\perp$$

is an involutive distribution of dimension $n - d$. In what follows, we discuss conditions under which this distribution is *controlled invariant* and the consequences of such property on the dependence of the coefficient matrices $F_{i,h}(\Phi^{-1}(\tilde{x}))$ on the components z and $X_1, X_2, \ldots, X_{k^*}$ of $\tilde{x}$.

To this end, observe that the vectors $a_i(x)$ and the matrices $b_i(x)$, $i = 1, \ldots, \ell$, appearing in the equations that characterize the normal form can be arranged into a single vector $A(x) \in \mathbb{R}^m$ and a single matrix $B(x) \in \mathbb{R}^{m \times m}$ defined as

$$\begin{aligned} A(x) &= \begin{pmatrix} a_1(x) \\ \cdots \\ a_\ell(x) \end{pmatrix} = \begin{pmatrix} L_f X'_{r_1} \\ \cdots \\ L_f X'_{r_\ell} \end{pmatrix} \\ B(x) &= \begin{pmatrix} b_1(x) \\ \cdots \\ b_\ell(x) \end{pmatrix} = \begin{pmatrix} L_g X'_{r_1} \\ \cdots \\ L_g X'_{r_\ell} \end{pmatrix} \end{aligned} \tag{9}$$

where, as a consequence of the property of uniform invertibility, the matrix $B(x)$ is nonsingular for all $x \in \mathbb{R}^n$. Hence, for any choice of $v = \mathrm{col}(v_1, \ldots, v_\ell)$ with $v_j \in \mathbb{R}^{m_j}$, there exists a unique u that makes

$$A(x) + B(x)u = v,$$

in which $v \in \mathbb{R}^m$ is seen as a new input. The feedback transformation implicitly defined by such equation,

$$\begin{aligned} u &= \alpha(x) + \beta(x)v \\ &= -B^{-1}(x)A(x) + B^{-1}(x)v \end{aligned} \tag{10}$$

which changes the vector fields of (1) into vector fields defined as

$$\begin{aligned} \tilde{f}(x) &= f(x) + g(x)\alpha(x) \\ \tilde{g}(x) = \big(\tilde{g}_1(x) \ \cdots \ \tilde{g}_m(x)\big) &= \big(g_1(x) \ \cdots \ g_m(x)\big)\,\beta(x)\,, \end{aligned} \tag{11}$$

[4]See, e.g., [7]. In particular, the following notations, borrowed from [7], are used here. For a real-valued function $\lambda : \mathbb{R}^n \to \mathbb{R}$ we denote by $d\lambda$ its differential

$$d\lambda = \begin{pmatrix} \frac{\partial \lambda}{\partial x_1} & \cdots & \frac{\partial \lambda}{\partial x_n} \end{pmatrix}$$

viewed as a covector field on $\mathbb{R}^n$. For a vector-valued function $\Lambda : \mathbb{R}^n \to \mathbb{R}^p$ we denote by $d\Lambda$ the $p \times n$ matrix whose i-th row is the differential $d\lambda_i$ of the i-th component λ_i of Λ. By $\mathrm{span}\{d\Lambda\}$ we mean the codistribution spanned by the d covector fields $d\lambda_1, \ldots, d\lambda_p$. The annihilator of a distribution Δ is the codistribution $\Delta^\perp = \{\omega \in (\mathbb{R}^n)^* : <\omega, v> = 0, \text{for all } v \in \Delta\}$. Likewise, the annihilator of a codistribution Ω is the distribution $\Omega^\perp = \{v \in \mathbb{R}^n : <\omega, v> = 0, \text{for all } \omega \in \Omega\}$. For consistency with the notations of [7], we use $\ker(dh)$ to denote the annihilator of the codistribution $\mathrm{span}\{dh\}$.

is such that

$$\dot{X}'_{r_j} = L_{\tilde{f}+\tilde{g}v} X'_{r_j} = v_j$$

for $j = 1, \ldots, \ell$ and hence

$$L_{\tilde{f}} X'_{r_j} = 0 \qquad \text{for } j = 1, \ldots, \ell$$

$$\begin{pmatrix} L_{\tilde{g}_1} X'_{r_1} & \cdots & L_{\tilde{g}_m} X'_{r_1} \\ \cdot & \cdots & \cdot \\ L_{\tilde{g}_1} X'_{r_\ell} & \cdots & L_{\tilde{g}_m} X'_{r_\ell} \end{pmatrix} = I\,.$$

Changing $f, g_1, \ldots, g_m$ into the vector fields $\tilde{f}, \tilde{g}_1, \ldots, \tilde{g}_m$ has the effect, on the normal form (3), (4), ... , (5), of changing $a_j(x) + b_j(x)u$ into v_j, for $i = 1, \ldots, \ell$. A consequence of this is that, in the new coordinates (z, ξ), the vector field $\tilde{f}$ appears decomposed as

$$\tilde{f}(z,\xi) = \begin{pmatrix} f_0(z,\xi) \\ f_1(\xi) \end{pmatrix}$$

from which it is seen that the distribution Δ is invariant under $\tilde{f}$.

This is not the case, though, for the vector fields $\tilde{g}_i$. Invariance under (all) the $\tilde{g}_i$'s would occur if all $F_{i,h}(\Phi^{-1}(\tilde{x}))$'s were independent of z. In this case, in fact, a decomposition of the form

$$\tilde{g}_i(z,\xi) = \begin{pmatrix} g_{0,i}(z,\xi) \\ g_{1,i}(\xi) \end{pmatrix}$$

would occur for all i.

Since we are interested in seeking conditions for the $F_{i,h}(\Phi^{-1}(\tilde{x}))$'s to be independent of z, it is important to establish conditions under which the distribution Δ is invariant also under all $\tilde{g}_i$'s. Motivated by this, we exploit some features of the algorithm for computing the largest controlled invariant distribution in contained in $\ker(dh)$, known as the *Controlled Invariant Distribution Algorithm*.[5] Having set

$$G = \text{span}\{g_1, \ldots, g_m\},$$

the algorithm in question begins with the codistribution

$$\Omega_1 = \text{span}\{dh\},$$

and generates, at each $i > 1$, a codistribution defined as

$$\Omega_i = \Omega_{i-1} + L_f(\Omega_{i-1} \cap G^\perp) + \sum_{j=1}^{m} L_{g_j}(\Omega_{i-1} \cap G^\perp)\,. \tag{12}$$

The following (classical) result holds (see, e.g., [7, Lemma 6.6.3]).

Lemma 1: Suppose there is an integer k^* such that $\Omega_{k^*} = \Omega_{k^*+1}$ and set $\Delta^* = \Omega_{k^*}^\perp$. Suppose Δ^* and $\Delta^* + G$ are nonsingular. Then $\Delta^\star$ is involutive and is the largest locally controlled invariant distribution contained in $\ker(dh)$.

The following result presents a sufficient condition for $(\text{span}\{d\Xi\})^\perp$ to be the largest controlled invariant distribution contained in $\ker(dh)$.

[5]The analysis that follows is based on and extends to the present setting the analysis carried out in [7, pp. 325-329].

Proposition 1: Assume system (1) is uniformly invertible and the map Φ is a globally defined diffeomorphism. Suppose

$$L_{g_j}(\Omega_i \cap G^\perp) \subset \Omega_i \quad \text{for all } j = 1, \ldots, m \text{ and all } i \geq 0. \tag{13}$$

Then

$$\Omega_i = \text{span}\{dX_1\} + \text{span}\{dX_2\} + \cdots + \text{span}\{dX_i\}. \tag{14}$$

As a consequence, $(\text{span}\{d\Xi\})^\perp$ is the largest controlled invariant distribution in $\ker(dh)$.

Proof: By definition, (14) is true for $i = 1$. We proceed by induction. Suppose (14) holds for some $i \geq 1$. It is known that the right-hand side of (12) is invariant under a feedback transformation of the form (10) with invertible β. Hence Ω_{i+1} can be computed via the formula

$$\Omega_{i+1} = \Omega_i + L_{\tilde{f}}(\Omega_i \cap G^\perp) + \sum_{j=1}^{m} L_{\tilde{g}_j}(\Omega_i \cap G^\perp)$$

in which the vector fields $\tilde{f}, \tilde{g}_1, \ldots, \tilde{g}_m$ are the vector fields defined in (11).

Using assumption (13), it is seen[6] that $L_{\tilde{g}_j}(\Omega_i \cap G^\perp) \subset \Omega_i$ and hence Ω_{i+1} can be computed as

$$\Omega_{i+1} = \Omega_i + L_{\tilde{f}}(\Omega_i \cap G^\perp)\,.$$

Any covector field in $\Omega_i \cap G^\perp$ is a linear combination $\omega = \sum_{k=1}^{i} \gamma_k dX_k$, with $\gamma = \begin{pmatrix}\gamma_1 & \cdots & \gamma_i\end{pmatrix}$ such that

$$0 = \gamma \begin{pmatrix} dX_1 \\ \cdots \\ dX_i \end{pmatrix} g = \gamma \begin{pmatrix} L_g X_1 \\ \cdots \\ L_g X_i \end{pmatrix}\,. \tag{15}$$

Bearing in mind the properties of the various $X_i(x)$'s recalled in section II, observe that

$$\begin{aligned} &F_{11} L_g X'_1 + L_g X''_1 = 0 \\ &F_{21} L_g X'_1 + F_{22} L_g X'_2 + L_g X''_2 = 0 \\ \\ &F_{i1} L_g X'_1 + F_{i2} L_g X'_2 + \cdots + F_{ii} L_g X'_i + L_g X''_i = 0 \end{aligned}$$

from which it can be seen that the space of solutions γ of (15) is spanned by the rows of a matrix of the form

$$\begin{pmatrix} F_{11} & M_1 & 0 & 0 & \cdots & 0 & 0 \\ F_{21} & 0 & F_{22} & M_2 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ F_{i1} & 0 & F_{i2} & 0 & \cdots & F_{ii} & M_i \end{pmatrix} \tag{16}$$

in which, for $1 \leq 1 \leq r_1 - 1$, all $F_{i,h}$'s are empty and M_i is an identity matrix of dimension m, while, for $r_j \leq i \leq r_{j+1} - 1$ with $j = 1, \ldots, \ell - 1$, the only non-empty blocks are $F_{i,r_1}, F_{i,r_2}, \ldots, F_{i,r_j}$ and M_i is an identity matrix of dimension $\overline{m}_j := m - m_1 - \ldots - m_j$.[7] Clearly,

[6]If ω is a covector field in $\Omega_i \cap G^\perp$, we have $L_{\tilde{g}_j}\omega = \sum_{k=1}^{m}(L_{g_k}\omega)\beta_{kj} + \sum_{k=1}^{m} <\omega, g_k> d\beta_{kj} = \sum_{k=1}^{m}(L_{g_k}\omega)\beta_{kj}$ because $<\omega, g_k> = 0$. Hence $L_{\tilde{g}_j}(\Omega_i \cap G^\perp) \subset \sum_{k=1}^{m} L_{g_k}(\Omega_i \cap G^\perp)$. Since β is invertible, also the reverse inclusion holds.

[7]An easy, but a bit tedious, calculation shows that number of rows of the matrix (16), linearly independent by construction, is exactly equal to the dimension of the space of solutions of (15). In fact, for $1 \leq i \leq r_1 - 1$, the space of solutions of (15) has dimension $m \cdot i$, while, for $r_j \leq i \leq r_{j+1} - 1$ with $j = 1, \ldots, \ell - 1$, the space of solutions of (15) has dimension $(m - m_1 - \ldots - m_j) \cdot i + (r_1 - 1)m_1 + \ldots + (r_j - 1)m_j$. Such dimensions coincide, for each i, with the number of rows of the matrix (16).

linear combinations with coefficients taken from the first $i-1$ block-rows of the matrix (16) generate covector fields that are in $\sum_{k=1}^{i-1} \mathrm{span}\{dX_k\} = \Omega_{i-1}$ and hence not relevant for the computation of Ω_{i+1}. Thus, the only covector fields of interest in $\Omega_i \cap G^\perp$ are those resulting from coefficients taken in the last block-row. Such covector fields span a codistribution

$$\Omega = \mathrm{span}\{\omega_1, \omega_2, \ldots, \omega_{\overline{m}_j}\} \tag{17}$$

in which [8]

$$\omega_p = e_p^\top \left[F_{i1} dX_1' + F_{i2} dX_2' + F_{ii} dX_i' + dX_i''\right]$$

where e_p is a vector whose p-th entry is 1 while all others are 0. That is, any covector $\omega \in \Omega$ can be expressed as $\omega = \sum_{p=1}^{\overline{m}_j} c_p \omega_p$. Recalling the formula $L_{\tilde{f}}(c_p\omega_p) = (L_{\tilde{f}} c_p)\omega_p + c_p L_{\tilde{f}} \omega_p$ we see that

$$\begin{aligned} L_{\tilde{f}}(c_p\omega_p) &= (L_{\tilde{f}} c_p)\omega_p + c_p L_{\tilde{f}} \left[\textstyle\sum_{k=1}^{i} e_p^\top F_{ik} dX_k' + e_p^\top dX_i''\right] \\ &= (L_{\tilde{f}} c_p)\omega_p + \textstyle\sum_{k=1}^{i} c_p e_p^\top (L_{\tilde{f}} F_{ik}) dX_k' \\ &\quad + \textstyle\sum_{k=1}^{i} c_p e_p^\top F_{ik} dL_{\tilde{f}} X_k' + c_p e_p^\top dL_{\tilde{f}} X_i''. \end{aligned}$$

The covector fields in the first two terms are in Ω_i and hence not relevant for the computation of Ω_{i+1}. So long as the covector fields in the third term are concerned, observe that the choice of α is such that $L_{\tilde{f}} X_k' = 0$ for $k = 1, \ldots, i$.[9] As a consequence, we see that

$$\Omega_{i+1} = \Omega_i + L_{\tilde{f}}(\Omega_i \cap G^\perp) = \Omega_i + \mathrm{span}\{dL_{\tilde{f}} X_i''\}\,.$$

Finally, observe that

$$\begin{aligned} L_{f+g\alpha} X_i'' &= L_f X_i'' + L_g X_i'' \alpha \\ &= L_f X_i'' - \textstyle\sum_{k=1}^{i} F_{ik} L_g X_k' \alpha \\ &= L_f X_i'' + \textstyle\sum_{k=1}^{i} F_{ik} L_f X_k' \\ &\quad - \textstyle\sum_{k=1}^{i} F_{ik} L_f X_k' - \sum_{k=1}^{i} F_{ik} L_g X_k' \alpha \\ &= X_{i+1} - \textstyle\sum_{k=1}^{i} F_{ik} L_{f+g\alpha} X_k' \\ &= X_{i+1} \end{aligned} \tag{18}$$

because the choice of α yields $L_{f+g\alpha} X_k' = 0$ for all $k = 1, \ldots, i$. Thus, it is seen that, if assumption (13) holds,

$$\Omega_{i+1} = \Omega_i + \mathrm{span}\{dX_{i+1}\}$$

and this completes the proof. ■

We have show in this way that, if assumption (13) holds, the distribution Δ is the largest controlled invariant distribution in $\ker(dh)$. We are not done yet, though, because we have not formally shown that the $F_{i,h}(\Phi^{-1}(\tilde{x}))$'s are independent of z nor that the $F_{i,h}(\Phi^{-1}(\tilde{x}))$'s are independent of X_k for $k > i$, which are the properties indicated in (7) and (8). However, it is possible to prove that both such properties are again consequences of assumption (13).

Proposition 2: Assume system (1) is uniformly invertible and the map Φ is a globally defined diffeomorphism. Property P holds if, and only if, (13) holds.

Proof: The "if" part. We know from the previous proposition that (13) implies (14). Hence we can evaluate $\Omega_i \cap G^\top$ as done in the proof of this proposition. Arguing as in this proof, it is seen that assumption (13) implies

$$L_{\tilde{g}_j} \Omega \subset \Omega_i \tag{19}$$

where Ω is the codistribution (17). Now

$$\begin{aligned} &L_{\tilde{g}_j}(c_p\omega_p) \\ &\quad = (L_{\tilde{g}_j} c_p)\omega_p + c_p L_{\tilde{g}_j} \left[\textstyle\sum_{k=1}^{i} e_p^\top F_{ik} dX_k' + e_p^\top dX_i''\right] \\ &\quad = (L_{\tilde{g}_j} c_p)\omega_p + \textstyle\sum_{k=1}^{i} c_p e_p^\top (L_{\tilde{g}_j} F_{ik}) dX_k' \\ &\qquad + \textstyle\sum_{k=1}^{i} c_p e_p^\top F_{ik} dL_{\tilde{g}_j} X_k' + c_p e_p^\top dL_{\tilde{g}_i} X_i''. \end{aligned}$$

The covector fields in the first two terms are in Ω_i and hence not relevant in the fulfillment of (19). The vector fields in the third term are all zero, because the choice of β is such that

$$\begin{pmatrix} L_{\tilde{g}} X_1' \\ L_{\tilde{g}} X_2' \\ \cdots \\ L_{\tilde{g}} X_\ell' \end{pmatrix} = I. \tag{20}$$

Thus, (19) holds if and only if

$$dL_{\tilde{g}_j} X_i'' \in \Omega_i = \sum_{k=1}^{i} \mathrm{span}\{dX_k\}\,. \tag{21}$$

Moreover, using again the property (20) it is seen that

$$\begin{aligned} L_{\tilde{g}} X_i'' &= (L_g X_i'')\beta = -\textstyle\sum_{k=1}^{i} F_{ik} (L_g X_k')\beta \\ &= -\textstyle\sum_{k=1}^{i} F_{ik} L_{\tilde{g}} X_k' \\ &= -\begin{pmatrix} F_{i1} & F_{i2} & \cdots & F_{ii} & 0 & \cdots & 0 \end{pmatrix}. \end{aligned} \tag{22}$$

Condition (21) constraints the differentials of $F_{i1}, \ldots, F_{ii}$ to be in $\sum_{k=1}^{i} \mathrm{span}\{dX_k\}$. Hence, such functions cannot depend on z and $X_{i+1}, \ldots, X_{k^*}$, and this completes the proof of the "if" part.

The "only if" part. We show that, if Property P holds and $\Omega_i = \mathrm{span}\{dX_1\} + \cdots + \mathrm{span}\{dX_i\}$, then (13) holds and hence, as shown in Proposition 1, $\Omega_{i+1} = \mathrm{span}\{dX_1\} + \cdots + \mathrm{span}\{dX_{i+1}\}$. Since $\Omega_1 = \mathrm{span}\{dX_1\}$ by definition, this provides the proof of the " only if" part.

We know from the proof of Proposition 1 that any covector in $\omega \in \Omega_i \cap G^\perp$ can be written as $\omega = \omega' + \omega''$, with $\omega' \in \Omega_{i-1}$ and $\omega'' \in \Omega$ where Ω is the codistribution (17). By construction, $L_{\tilde{g}_j}\omega' \in \Omega_i$. On the other hand, to evaluate $L_{\tilde{g}_j}\omega''$ we can use the same arguments used above in the proof of the "if" part, and conclude that

$$L_{\tilde{g}_j}\omega'' = \tilde{\omega}' + \tilde{\omega}''$$

where $\tilde{\omega}' \in \Omega_i$ and $\tilde{\omega}'' \in \mathrm{span}\{dL_{\tilde{g}_j} X_i''\}$. Thus, in summary $L_{\tilde{g}_j}\omega$ has a component $\overline{\omega}' \in \Omega_i$ and a component $\overline{\omega}'' \in \mathrm{span}\{dL_{\tilde{g}_j} X_i''\}$. If Property P holds, the functions

[8] We tacitly assume $i \geq r_1$, because for $i < r_1$ we simply have $\Omega_i \cap G^\perp = \Omega_i$, and we observe that, for $r_j \leq i \leq r_{j+1} - 1$ with $j = 1, \ldots, \ell - 1$, the the number of rows of dX_i'' is $\overline{m}_j$.

[9] Strictly speaking, we have shown before that the choice of α is such that $L_{\tilde{f}} X_{r_j}' = 0$. However, the only $X_k(x)$'s for which the component $X_k'(x)$ is nonempty are precisely the $X_{r_j}(x)$'s and thus we can claim that $L_{\tilde{f}} X_k' = 0$ for all k.

$F_{i1}, \ldots, F_{ii}$ only depend on $X_1, \ldots, X_i$. Hence, looking at (22) we observe that also $\mathrm{span}\{dL_{\tilde{g}_j} X_i''\} \in \Omega_i$. Thus $L_{\tilde{g}_j}\omega \in \Omega_i$ and this concludes the proof of the claim. ■

V. CONCLUSIONS

In some recent papers, it has been shown that various relevant feedback design problems can be successfully addressed, for an invertible MIMO input-affine nonlinear system, if the systems in question have a normal forms in which certain coefficients depend on the components of the state in a special way. In view of this, it is important to be able to determine whether or not a system possesses a normal form of this kind, by means of direct tests on the data that characterize the model of the system, skipping the actual construction of a normal form. If fact, in the design of some robust feedback laws, such as the one proposed in [16], it is only needed to know that the system possesses a normal form of this kind and the values the certain "structural" integers (related to the structure algorithm)[10] but not the specific functions that appear in the normal form. Hence, for the design of such laws, the explicit construction of the normal form is not needed.

In a recent paper [17], it was shown that the existence of a normal form of this kind is implied by certain hypotheses introduced in [2]. Specifically, it was shown that such normal for exists if the vector fields $\tilde{g}_1, \ldots, \tilde{g}_m$ defined in (11) are such that $[\tilde{g}_i, \tau] \in \Delta$ for all $\tau \in \Delta = \mathrm{span}\{d\Xi\}^\perp$ and

$$L_{g_i} L_f^q L_{g_j} L_f^s h(x) = 0 \qquad \begin{array}{l} i, j = 1, \ldots, m \\ 0 \leq q + s \leq r_\ell \,. \end{array}$$

In the present paper, we present an alternative approach and we identify a necessary and sufficient condition for the existence of a normal form of this kind. Specifically, the condition in question is that the sequence of codistributions $\Omega_1, \ldots, \Omega_i, \ldots$ generated by means of the so-called controlled invariant distribution algorithm (12) is such that the property indicated in the *crucial assumption* (13) holds.

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[10] Actually, the integers $r_1, r_2, \ldots, r_\ell$, which play in the present setting a role similar to that of the so-called "zero-structure at infinity" of a linear system. See also [5] for various related issues.

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