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An Adaptive Distributed Observer for A Class of Discrete-time Uncertain Linear Systems over Acyclic Digraphs

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Abstract— **This paper proposes an adaptive distributed observer for a class of discrete-time uncertain linear leader systems. The leader system is assumed to be neutrally stable with unknown parameters in the system matrix. Such a leader system can produce multi-tone sinusoidal signals with unknown frequencies, magnitudes, and phases. Under the assumption that the digraph of the communication network is a spanning tree with the leader system as the root, the proposed adaptive distributed observer is shown to be capable of estimating over the communication network not only the leader's state, but also the unknown parameters of the leader's system matrix.**

I. INTRODUCTION

The research of multi-agent control systems is constantly expanding its frontiers [2], [7], [13], [14], [15], [20]. A typical multi-agent system consists of a leader producing a class of command signals and a group of followers whose outputs asymptotically track the command signals of the leader in a coordinated fashion [3], [4], [5], [12]. The leader and the followers are called agents and the communications among different agents are described by a communication network. A control law that satisfies the communication constraints imposed by the communication network is called a distributed control law. An effective approach for designing a distributed control law is called the distributed observer based approach, which contains a distributed dynamic compensator called the distributed observer that is able to estimate the leader's state over the communication network and pass the estimated state of the leader to each follower's controller. The first distributed observer was developed in [16] for studying the cooperative output regulation problem of linear multi-agent systems assuming every follower can access the dynamics of the leader. Later in [1], the distributed observer was further rendered the capability of estimating and transmitting not only the leader's state but also the leader's dynamics assuming only the leader's children know the leader's dynamics. Such a distributed observer is called an adaptive distributed observer for a known leader. In practice, the leader's dynamics may contain some unknown parameters. In this case, none of the followers know the exact dynamics of the leader. Thus, reference [11] proposed a model-free distributed dynamic compensator that can estimate the leader's state without using the exact dynamics of the leader. Such a distributed dynamic compensator is called an adaptive distributed observer for an uncertain leader. Reference [17] further proposed an adaptive distributed observer for an uncertain leader and gave a sufficient condition on the convergence of the estimated unknown parameters to their actual values. Other relevant results can be found in [18] and [19].

All the references mentioned above focus on the design of distributed observers for continuous-time systems. Since 2016, the distributed observers have also been developed for discrete-time systems. For example, the discrete-time counterpart of the continuous-time adaptive distributed observer for a known leader was established in [6]. Nevertheless, the discrete-time counterpart of the adaptive distributed observer for an uncertain leader has not been touched so far.

This paper aims to develop an adaptive distributed observer for a class of discrete-time linear leader systems, which are neutrally stable with unknown parameters in the system matrix. Such a leader system can produce a multi-tone sinusoidal signal with unknown frequencies, magnitudes, and phases. Motivated by our recent study on the exponential estimation problem of the unknown frequencies of discrete-time multi-tone sinusoidal signals in [9] and [10], we propose an adaptive distributed observer for the uncertain leader system. Under the assumption that the digraph of the communication network is a spanning tree with the leader system as the root, we show that the proposed adaptive distributed observer is capable of providing for each follower, not only an exponentially convergent estimate of the leader's state, but also an exponentially convergent estimate of the unknown parameters of the leader's uncertain system matrix. Compared with the continuous-time adaptive distributed observers in [11], [17], [19], which rely on the leader's state, one distinct feature of the proposed discrete-time adaptive distributed observer is that it only relies on the leader's output and thus it works when only the leader's output is available. Compared with the continuous-time output-based adaptive distributed observer in [18], which requires to transmit the estimated states among the followers, the proposed discrete-time adaptive distributed observer only requires to transmit the estimated outputs of the leader among the followers.

The rest of this paper is organized as follows. We first formulate the problem in Section II. Then, we establish a so-called identified leader system in Section III. Our design of the adaptive distributed observer is presented in Section

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IV and some concluding remarks are drawn in Section V.

Notation. $\mathbb{R}$ denotes the set of real numbers and $\mathbb{Z}^+$ denotes the set of nonnegative integers. $\mathrm{col}(x_1,\ldots,x_n)$ denotes a column vector in $\mathbb{R}^n$ whose ith component is $x_i \in \mathbb{R}, i = 1,\ldots,n$. For any $x \in \mathbb{R}^n$, $\|x\|$ denotes the Euclidean norm of x. A discrete-time signal $x(t) \in \mathbb{R}^n$ is said to be bounded over $\mathbb{Z}^+$ if $\|x(t)\| \leq M, \forall\, t \in \mathbb{Z}^+$ for some finite number M, and is said to be persistently exciting (PE), if there exist a positive integer T and a positive constant α such that $\sum_{t=t_0}^{t_0+T-1} x(t)x(t)^T \geq \alpha I_n, \forall\, t_0 \in \mathbb{Z}^+$.

II. Problem Formulation

Consider the following multi-tone sinusoidal signal:

$$y_0(t) = \sum_{i=1}^{n} \Omega_i \sin(\omega_i t + \sigma_i), \quad t \in \mathbb{Z}^+ \tag{1}$$

where, for $i = 1,\ldots,n$, $0 < \omega_i < \pi$ are n distinct unknown frequencies, $\Omega_i > 0$ are unknown amplitudes, and $-\pi \leq \sigma_i \leq \pi$ are unknown phases.

First, let $\omega = \mathrm{col}(\omega_1, \omega_2, \ldots, \omega_n)$. It is easy to show that $y_0(t)$ can be generated by the following system:

$$\tau(t+1) = \bar{S}(\omega)\tau(t), \quad y_0(t) = \bar{\mathbf{c}}\tau(t), \quad t \in \mathbb{Z}^+ \tag{2}$$

with $\bar{\mathbf{c}} = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots & 1 & 0 \end{bmatrix} \in \mathbb{R}^{1\times 2n}$,

$$\bar{S}(\omega) = \text{block diag}\left\{ \begin{bmatrix} \cos\omega_i & \sin\omega_i \\ -\sin\omega_i & \cos\omega_i \end{bmatrix}, i = 1,\ldots,n \right\}$$

and the following initial condition:

$$\tau(0) = \mathrm{col}\left(\Omega_1 \sin\sigma_1, \Omega_1\cos\sigma_1, \ldots, \Omega_n \sin\sigma_n, \Omega_n \cos\sigma_n\right).$$

Next, notice that the characteristic polynomial of the system matrix $\bar{S}(\omega)$ of (2) is given by

$$\prod_{i=1}^{n}\left(s^2 - (2\cos\omega_i)s + 1\right) =: s^{2n} + \theta_1 s^{2n-1} + \theta_2 s^{2n-2} + \cdots + \theta_n s^n + \cdots + \theta_2 s^2 + \theta_1 s + 1$$

which establishes a one-to-one correspondence between the n unknown frequencies ω and the n unknown parameters $\theta = \mathrm{col}(\theta_1, \theta_2, \ldots, \theta_n)$.

Since system (2) is observable, there exists a nonsingular matrix $P \in \mathbb{R}^{2n\times 2n}$ that defines a coordinate transformation $v_0(t) = P\tau(t)$, and, under which, system (2) is transformed into the following observable canonical form:

$$v_0(t+1) = S(\theta)v_0(t), \quad y_0(t) = \mathbf{c}v_0(t), \quad t \in \mathbb{Z}^+ \tag{3}$$

with $\mathbf{c} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{1\times 2n}$ and

$$S(\theta) = \begin{bmatrix} \mathbf{0} & I_{2n-1} \\ 0 & \mathbf{0} \end{bmatrix} - \begin{bmatrix} \theta_1 & \cdots & \theta_n & \cdots & \theta_1 & 1 \end{bmatrix}^T \mathbf{c} \in \mathbb{R}^{2n\times 2n}.$$

As pointed out in the introduction, the distributed observer arises in studying the cooperative output regulation problem of linear multi-agent systems with one leader and N followers [16]. Like in [16], we can describe the communication network for this leader-follower multi-agent system of $(N+1)$ agents by a digraph $\bar{\mathcal{G}} = \{\bar{\mathcal{V}}, \bar{\mathcal{E}}\}$, where $\bar{\mathcal{V}} = \{0, 1, \ldots, N\}$ is called the node set and $\bar{\mathcal{E}} \subseteq \bar{\mathcal{V}} \times \bar{\mathcal{V}}$ is called the edge set. We associate node 0 with the leader system (3) and node $i, i = 1,\ldots,N$, with the ith follower. Then, $(j,i) \in \bar{\mathcal{E}}, i \neq j$, indicates that ith agent can access the information of the jth agent, and agent j is called a neighbor of agent i. We define $\bar{\mathcal{N}}_i := \{j \in \bar{\mathcal{V}} : (j,i) \in \bar{\mathcal{E}}\}$ as the neighbor set of agent i.

Now we are ready to formulate the problem.

Problem 1: Given the digraph $\bar{\mathcal{G}}$ and the leader system (3) generating the output $y_0(t)$ of the form (1), design, for each follower a dynamic compensator of the following form:

$$\begin{aligned}
\zeta_i(t+1) &= \mathbf{f}_i\left(\zeta_i(t), \{y_j(t) : j \in \bar{\mathcal{N}}_i\}\right) \\
\hat{\theta}_i(t+1) &= \mathbf{g}_i\left(\hat{\theta}_i(t), \zeta_i(t), \{y_j(t) - y_i(t) : j \in \bar{\mathcal{N}}_i\}\right) \\
y_i(t) &= \mathbf{h}_i\left(\zeta_i(t), \hat{\theta}_i(t)\right) \\
v_i(t) &= \mathbf{l}_i\left(\zeta_i(t), \hat{\theta}_i(t)\right), \qquad i = 1,\ldots,N
\end{aligned} \tag{4}$$

where, for $i = 1,\ldots,N$, $\zeta_i(t) \in \mathbb{R}^{n_\zeta}$ with n_ζ being some positive integer, $\hat{\theta}_i(t) \in \mathbb{R}^n$, $y_i(t) \in \mathbb{R}$, $v_i(t) \in \mathbb{R}^{2n}$, and $\mathbf{f}_i(\cdot), \mathbf{g}_i(\cdot)$, $\mathbf{h}_i(\cdot)$, $\mathbf{l}_i(\cdot)$ are some globally defined smooth functions, such that for any initial conditions $\left(\zeta_i(0), \hat{\theta}_i(0)\right) \in \mathbb{R}^{n_\zeta} \times \mathbb{R}^n, i = 1,\ldots,N$, the solution of system (4) is bounded over $\mathbb{Z}^+$ and satisfies

$$\begin{aligned}
\lim_{t\to\infty}\left(\hat{\theta}_i(t) - \theta\right) &= 0, \quad i = 1,\ldots,N. \\
\lim_{t\to\infty}\left(y_i(t) - y_0(t)\right) &= 0, \quad i = 1,\ldots,N \\
\lim_{t\to\infty}\left(v_i(t) - v_0(t)\right) &= 0, \quad i = 1,\ldots,N
\end{aligned}$$

all exponentially.

Remark 1: If Problem 1 is solvable, then the distributed dynamic compensator (4) is called an adaptive distributed observer for the uncertain linear leader system (3).

III. The Identified Leader System

Since the system matrix $S(\theta)$ of the leader system (3) contains the unknown parameters θ and hence cannot be used, the design methodology of an adaptive distributed observer for an uncertain leader system is entirely different from that of a distributed observer for a known leader system. Thus, we will first establish, in this section, another system that is capable of recovering the unknown parameters θ and the state $v_0(t)$ of the leader system (3) by using only the output $y_0(t)$ of the leader system (3). We call this system the identified leader system and will carry out our design of the adaptive distributed observer based on this identified leader system in the next section. The main results of this section are summarized from [9] and [10].

First, let $A = \begin{bmatrix} \mathbf{0} & I_{2n-1} \\ 0 & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{2n\times 2n}, \mathbf{b} = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} \in \mathbb{R}^{2n}$ and let $\mathbf{d} \in \mathbb{R}^{2n}$ be such that the matrix $D := A - \mathbf{dc}$ is Schur. Design two filters for the output $y_0(t)$ of the leader system (3) as follows:

$$\eta_0(t+1) = D\eta_0(t) + (\mathbf{d} - \mathbf{b})\, y_0(t) \tag{5}$$

$$\xi_0(t+1) = D\xi_0(t) - \mathbf{b}y_0(t). \quad (6)$$

Then, based on the state $\xi_0(t)$ of the filter (6), further define

$$\Xi_0(t) := \left[\begin{array}{cccc} \Xi_{01}(t) & \Xi_{02}(t) & \cdots & \Xi_{0n}(t) \end{array} \right] \in \mathbb{R}^{2n\times n}$$

$$\Xi_{0j}(t) := \begin{cases} \left(D^j + D^{2n-j}\right)\xi_0(t), & j = 1,\ldots,n-1 \\ D^n\xi_0(t), & j = n. \end{cases}$$

Lemma 1 (Lemma 2 of [9]): Given system (3), design two filters (5) and (6), and define

$$\tilde{v}_0(t) := \eta_0(t) + \Xi_0(t)\theta - v_0(t) \in \mathbb{R}^{2n}. \quad (7)$$

Then, for any initial conditions $(v_0(0), \eta_0(0), \xi_0(0)) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n}$, $\lim_{t\to\infty} \tilde{v}_0(t) = 0$ exponentially.

Motivated by Lemma 1, we propose to estimate $v_0(t)$ by $\hat{v}_0(t)$ as follows:

$$\hat{v}_0(t) := \eta_0(t) + \Xi_0(t)\hat{\theta}_0(t) \in \mathbb{R}^{2n} \quad (8)$$

and propose to estimate $y_0(t)$ by $\hat{y}_0(t)$ as follows:

$$\hat{y}_0(t) := \mathbf{c}\left(\eta_0(t) + \Xi_0(t)\hat{\theta}_0(t)\right) \in \mathbb{R}^n \quad (9)$$

where, in (8) and (9), $\hat{\theta}_0(t) \in \mathbb{R}^n$ is an estimate of the unknown parameters θ, and is updated according to the following adaptation law:

$$\hat{\theta}_0(t+1) = \hat{\theta}_0(t) + \frac{\gamma_1\phi_0(t)}{1+\gamma_2\phi_0(t)^T\phi_0(t)}\left(y_0(t) - \hat{y}_0(t)\right) \quad (10)$$

in which, $0 < \gamma_1 \le 2\gamma_2$ are two constants and

$$\phi_0(t) := \left(\mathbf{c}\Xi_0(t)\right)^T \in \mathbb{R}^n. \quad (11)$$

Now we draw the main conclusion on the system composed of (5), (6), and (10), into a proposition, which follows from Theorem 2 of [10] and the above Lemma 1.

Proposition 1: Consider system (3) generating the output $y_0(t)$ of the form (1) and the system composed of (5), (6), and (10). For any initial conditions $\left(\eta_0(0), \xi_0(0), \hat{\theta}_0(0)\right) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^n$, the solution of the system composed of (5), (6), and (10), is bounded over $\mathbb{Z}^+$ and satisfies

$$\lim_{t\to\infty}\left(\hat{\theta}_0(t) - \theta\right) = 0$$
$$\lim_{t\to\infty}\left(\hat{y}_0(t) - y_0(t)\right) = 0$$
$$\lim_{t\to\infty}\left(\hat{v}_0(t) - v_0(t)\right) = 0$$

all exponentially, where $\hat{y}_0(t)$ is given in (9) and $\hat{v}_0(t)$ is given in (8).

From Proposition 1, we see that the system composed of (5), (6), and (10), has the capability of recovering the unknown parameters, the output, and the state of the leader system (3). Thus, we call this system the identified leader system for the uncertain leader system (3).

IV. Solvability of the Problem

Having obtained the so-called identified leader system, we will first show, in this section, that the solvability of Problem 1 reduces to the solvability of a leader-following consensus problem with the identified leader system composed of (5), (6), and (10), as the leader and the proposed adaptive distributed observer as followers. Then, we will further solve this leader-following consensus problem that leads to the solution of Problem 1. As the identified leader system is a known system, we have converted a seemingly intractable problem into a tractable problem.

To begin with, we make the following assumption on the digraph $\bar{\mathcal{G}}$ describing the communication network of the leader-follower multi-agent system.

Assumption 1: The digraph $\bar{\mathcal{G}}$ is a spanning tree with node 0 as the root.

Let $\bar{\mathcal{A}} := [a_{ij}]_{i,j=0}^N \in \mathbb{R}^{(N+1)\times(N+1)}$ denote the weighted adjacency matrix of the digraph $\bar{\mathcal{G}}$, where $a_{ii} = 0$, and, for $i \neq j, a_{ij} > 0$ if and only if $(j,i) \in \bar{\mathcal{E}}$. Then, under Assumption 1, we have $\sum_{j=0}^N a_{ij} > 0, i = 1,\ldots,N$.

Define $w_{ij} := \frac{a_{ij}}{\sum_{j=0}^N a_{ij}}, i = 1,\ldots,N,\ j = 0,1,\ldots,N$. Then, for $i = 1,\ldots,N$, we construct two distributed filters of the following forms:

$$\eta_i(t+1) = D\eta_i(t) + (\mathbf{d}-\mathbf{b})\sum_{j=0}^N w_{ij}y_j(t) \quad (12)$$

$$\xi_i(t+1) = D\xi_i(t) - \mathbf{b}\sum_{j=0}^N w_{ij}y_j(t) \quad (13)$$

where, for $i = 1,\ldots,N$, $y_i(t)$ is an estimate of $y_0(t)$ by the ith follower and is defined as follows:

$$y_i(t) := \mathbf{c}\left(\eta_i(t) + \Xi_i(t)\hat{\theta}_i(t)\right) \in \mathbb{R} \quad (14)$$

in which,

$$\Xi_i(t) := \left[\begin{array}{cccc} \Xi_{i1}(t) & \Xi_{i2}(t) & \cdots & \Xi_{in}(t) \end{array} \right] \in \mathbb{R}^{2n\times n}$$

$$\Xi_{ij}(t) := \begin{cases} \left(D^j + D^{2n-j}\right)\xi_i(t), & j = 1,\ldots,n-1 \\ D^n\xi_i(t), & j = n. \end{cases}$$

Moreover, for $i = 1,\ldots,N$, $\hat{\theta}_i(t) \in \mathbb{R}^n$ in the definition of $y_i(t)$ in (14) is an estimate of the unknown parameters θ by the ith follower, and is updated according to the following distributed adaptation law:

$$\hat{\theta}_i(t+1) = \hat{\theta}_i(t) + \frac{\gamma_1\phi_i(t)}{1+\gamma_2\phi_i(t)^T\phi_i(t)}\sum_{j=0}^N w_{ij}\left(y_j(t) - y_i(t)\right) \quad (15)$$

where $0 < \gamma_1 \le 2\gamma_2$ are two constants and $\phi_i(t) := \left(\mathbf{c}\Xi_i(t)\right)^T \in \mathbb{R}^n$. Also, for $i = 1,\ldots,N$, define

$$v_i(t) := \eta_i(t) + \Xi_i(t)\hat{\theta}_i(t) \in \mathbb{R}^{2n} \quad (16)$$

as an estimate of $v_0(t)$ by the ith follower.

Remark 2: We call the system composed of (12), (13), and (15), an adaptive distributed observer candidate for the leader system (3), which is in the form of (4) with $n_\zeta = 4n$,

$\zeta_i(t) = \mathrm{col}(\eta_i(t), \xi_i(t))$, $y_i(t)$ given by (14), and $v_i(t)$ given by (16), $i = 1, \ldots, N$. If Problem 1 is solvable, then the adaptive distributed observer candidate is further called an adaptive distributed observer.

Next, we show that the solvability of Problem 1 can be converted to that of a leader-following consensus problem with the identified leader system composed of (5), (6), and (10), as the leader and the adaptive distributed observer candidate composed of (12), (13), and (15), as N followers.

Proposition 2: Under Assumption 1, Problem 1 is solvable if

$$\lim_{t\to\infty} (\eta_i(t) - \eta_0(t)) = 0, \quad i = 1, \ldots, N \tag{17}$$

$$\lim_{t\to\infty} (\xi_i(t) - \xi_0(t)) = 0, \quad i = 1, \ldots, N \tag{18}$$

$$\lim_{t\to\infty} \left(\hat{\theta}_i(t) - \hat{\theta}_0(t)\right) = 0, \quad i = 1, \ldots, N \tag{19}$$

all exponentially.

Proof: First, by Proposition 1, we have

$$\lim_{t\to\infty} \left(\hat{\theta}_0(t) - \theta\right) = 0 \tag{20}$$

$$\lim_{t\to\infty} (\hat{y}_0(t) - y_0(t)) = 0 \tag{21}$$

both exponentially. Hence, from (19) and (20), we have

$$\lim_{t\to\infty} \left(\hat{\theta}_i(t) - \theta\right) = 0, \quad i = 1, \ldots, N \tag{22}$$

exponentially.

Next, note that (18) implies

$$\lim_{t\to\infty} (\Xi_i(t) - \Xi_0(t)) = 0, \quad i = 1, \ldots, N \tag{23}$$

exponentially. Since the matrix D is Schur and $y_0(t)$ is bounded over $\mathbb{Z}^+$, the state $\xi_0(t)$ of the filter (6) remains bounded over $\mathbb{Z}^+$ and, as a linear function of $\xi_0(t)$, $\Xi_0(t)$ also remains bounded over $\mathbb{Z}^+$. Then, by (23), $\Xi_i(t), i = 1, \ldots, N$, are all bounded over $\mathbb{Z}^+$.

From the definitions of $y_i(t)$ in (14) and $\hat{y}_0(t)$ in (9), we have, for $i = 1, \ldots, N$,

$$\begin{aligned} & y_i(t) - \hat{y}_0(t) \\ &= \mathbf{c}\Big(\eta_i(t) - \eta_0(t) + \Xi_i(t)\left(\hat{\theta}_i(t) - \hat{\theta}_0(t)\right) \\ &\qquad + (\Xi_i(t) - \Xi_0(t))\,\hat{\theta}_0(t)\Big). \end{aligned} \tag{24}$$

By (17), (19), (23), and the fact that $\hat{\theta}_0(t)$ and $\Xi_i(t), i = 1, \ldots, N$, are bounded over $\mathbb{Z}^+$, we have

$$\lim_{t\to\infty} (y_i(t) - \hat{y}_0(t)) = 0, \quad i = 1, \ldots, N \tag{25}$$

exponentially. Thus, from (21) and (25), we have

$$\lim_{t\to\infty} (y_i(t) - y_0(t)) = 0, \quad i = 1, \ldots, N \tag{26}$$

exponentially.

Now, from the definitions of $v_i(t), i = 1, \ldots, N$, in (16) and $\tilde{v}_0(t)$ in (7), we have, for $i = 1, \ldots, N$,

$$\begin{aligned} & v_i(t) - v_0(t) \\ &= \eta_i(t) - \eta_0(t) + \Xi_i(t)\left(\hat{\theta}_i(t) - \theta\right) \\ &\quad + (\Xi_i(t) - \Xi_0(t))\,\theta + \tilde{v}_0(t). \end{aligned}$$

In particular, by Lemma 1, $\lim_{t\to\infty} \tilde{v}_0(t) = 0$ exponentially. Thus, from (17), (22), (23), and the fact that $\Xi_i(t), i = 1, \ldots, N$, are bounded over $\mathbb{Z}^+$, we have

$$\lim_{t\to\infty} (v_i(t) - v_0(t)) = 0, \quad i = 1, \ldots, N \tag{27}$$

exponentially.

Finally, combining (22), (26), and (27) shows the solvability of Problem 1. □

As a result of Proposition 2, in what follows, we will focus on solving the leader-following consensus problem of the identified leader system and the adaptive distributed observer candidate. For this purpose, we first define, for $i = 1, \ldots, N$, the following errors between the adaptive distributed observer candidate and the identified leader system:

$$\begin{aligned} \tilde{\eta}_i(t) &= \eta_i(t) - \eta_0(t), & \tilde{\xi}_i(t) &= \xi_i(t) - \xi_0(t) \\ \tilde{\theta}_i(t) &= \hat{\theta}_i(t) - \hat{\theta}_0(t), & \tilde{\Xi}_i(t) &= \Xi_i(t) - \Xi_0(t). \end{aligned} \tag{28}$$

Then, from (24), we have

$$y_i(t) - \hat{y}_0(t) = \mathbf{c}\left(\tilde{\eta}_i(t) + \Xi_i(t)\tilde{\theta}_i(t) + \tilde{\Xi}_i(t)\hat{\theta}_0(t)\right).$$

It is noted that, under Assumption 1, by appropriately labeling the nodes $i, i = 1, \ldots, N$, we have, $\sum_{j=0}^{N} w_{ij} = \sum_{j=0}^{i-1} w_{ij} = 1$. Thus, for $i = 1, \ldots, N$, the error dynamics of the distributed filter (12) are governed by

$$\begin{aligned} & \tilde{\eta}_i(t+1) \\ &= D\tilde{\eta}_i(t) + (\mathbf{d} - \mathbf{b})\,(1 - w_{i0})(\hat{y}_0(t) - y_0(t)) \\ &\quad + (\mathbf{d} - \mathbf{b}) \sum_{j=1}^{i-1} w_{ij}(y_j(t) - \hat{y}_0(t)) \\ &= D\tilde{\eta}_i(t) + (\mathbf{d} - \mathbf{b})\,(1 - w_{i0})(\hat{y}_0(t) - y_0(t)) \\ &\quad + (\mathbf{d} - \mathbf{b})\,\mathbf{c} \sum_{j=1}^{i-1} w_{ij}\left(\tilde{\eta}_j(t) + \Xi_j(t)\tilde{\theta}_j(t) + \tilde{\Xi}_j(t)\hat{\theta}_0(t)\right), \end{aligned} \tag{29}$$

the error dynamics of the distributed filter (13) are governed by

$$\begin{aligned} & \tilde{\xi}_i(t+1) \\ &= D\tilde{\xi}_i(t) - \mathbf{b}(1 - w_{i0})(\hat{y}_0(t) - y_0(t)) \\ &\quad - \mathbf{b} \sum_{j=1}^{i-1} w_{ij}(y_j(t) - \hat{y}_0(t)) \\ &= D\tilde{\xi}_i(t) - \mathbf{b}(1 - w_{i0})(\hat{y}_0(t) - y_0(t)) \\ &\quad - \mathbf{b}\mathbf{c} \sum_{j=1}^{i-1} w_{ij}\left(\tilde{\eta}_j(t) + \Xi_j(t)\tilde{\theta}_j(t) + \tilde{\Xi}_j(t)\hat{\theta}_0(t)\right), \end{aligned} \tag{30}$$

and the error dynamics of the distributed adaptation law (15) are governed by

$$\begin{aligned} & \tilde{\theta}_i(t+1) \\ &= \tilde{\theta}_i(t) + \frac{\gamma_1 \phi_i(t)}{1 + \gamma_2 \phi_i(t)^T \phi_i(t)} \sum_{j=0}^{i-1} w_{ij}\,(y_j(t) - y_i(t)) \end{aligned}$$

$$
\begin{aligned}
&- \frac{\gamma_1 \phi_0(t)}{1+\gamma_2 \phi_0(t)^T \phi_0(t)} (y_0(t) - \hat{y}_0(t)) \\
&= \tilde{\theta}_i(t) - \frac{\gamma_1 \phi_i(t)}{1+\gamma_2 \phi_i(t)^T \phi_i(t)} (y_i(t) - \hat{y}_0(t)) \\
&+ \frac{\gamma_1 \phi_i(t)}{1+\gamma_2 \phi_i(t)^T \phi_i(t)} \sum_{j=0}^{i-1} w_{ij} (y_j(t) - \hat{y}_0(t)) \\
&- \frac{\gamma_1 \phi_0(t)}{1+\gamma_2 \phi_0(t)^T \phi_0(t)} (y_0(t) - \hat{y}_0(t)) \\
&= \left(I_n - \frac{\gamma_1 \phi_i(t) \phi_i(t)^T}{1+\gamma_2 \phi_i(t)^T \phi_i(t)} \right) \tilde{\theta}_i(t) \\
&- \frac{\gamma_1 \phi_i(t)}{1+\gamma_2 \phi_i(t)^T \phi_i(t)} \mathbf{c} \left(\tilde{\eta}_i(t) + \tilde{\Xi}_i(t) \hat{\theta}_0(t) \right) \\
&+ \frac{\gamma_1 \phi_i(t)}{1+\gamma_2 \phi_i(t)^T \phi_i(t)} \sum_{j=1}^{i-1} w_{ij} (y_j(t) - \hat{y}_0(t)) \\
&+ \frac{\gamma_1 \phi_i(t) w_{i0}}{1+\gamma_2 \phi_i(t)^T \phi_i(t)} (y_0(t) - \hat{y}_0(t)) \\
&- \frac{\gamma_1 \phi_0(t)}{1+\gamma_2 \phi_0(t)^T \phi_0(t)} (y_0(t) - \hat{y}_0(t)). \qquad (31)
\end{aligned}
$$

Before proceeding, we present another technical lemma whose proof is omitted due to the space limit.

Lemma 2: Suppose that $\phi(t) \in \mathbb{R}^n$ and $\bar{\phi}(t) \in \mathbb{R}^n$ are bounded over $\mathbb{Z}^+$ and are such that $\lim_{t\to\infty} \left(\bar{\phi}(t) - \phi(t)\right) = 0$. Then, $\bar{\phi}(t)$ is PE if and only if $\phi(t)$ is PE.

Proposition 3: Given the digraph $\bar{\mathcal{G}}$ and the leader system (3) generating the output $y_0(t)$ of the form (1), under Assumption 1, the identified leader system composed of (5), (6), and (10), and the adaptive distributed observer candidate composed of (12), (13), and (15), have the property that, for any initial conditions $\left(\eta_i(0), \xi_i(0), \hat{\theta}_i(0)\right) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^n, i = 0, 1, \ldots, N$, their solutions satisfy

$$
\begin{aligned}
\lim_{t\to\infty} (\eta_i(t) - \eta_0(t)) &= 0, \quad i = 1, \ldots, N \\
\lim_{t\to\infty} (\xi_i(t) - \xi_0(t)) &= 0, \quad i = 1, \ldots, N \\
\lim_{t\to\infty} \left(\hat{\theta}_i(t) - \hat{\theta}_0(t)\right) &= 0, \quad i = 1, \ldots, N.
\end{aligned}
$$

all exponentially.

Proof: First, we note that the filter (6) is a stable linear system subject to a bounded input $y_0(t)$. Thus, for any $\xi_0(0) \in \mathbb{R}^{2n}$, its state $\xi_0(t)$ remains bounded over $\mathbb{Z}^+$. Since $\phi_0(t)$ in (11) is a linear function of $\xi_0(t)$, $\phi_0(t)$ is also bounded over $\mathbb{Z}^+$. In addition, by Proposition 1, we have

$$
\lim_{t\to\infty} \left(\hat{\theta}_0(t) - \theta\right) = 0 \text{ and } \lim_{t\to\infty} (\hat{y}_0(t) - y_0(t)) = 0 \qquad (32)
$$

both exponentially. Hence, $\hat{\theta}_0(t)$ is also bounded over $\mathbb{Z}^+$.

For $i = 1, \ldots, N$, define $\tilde{\eta}_i(t), \tilde{\xi}_i(t), \tilde{\theta}_i(t)$, and $\tilde{\Xi}_i(t)$ as in (28), and put the error dynamics in (29), (30), and (31) into the following compact forms, respectively:

$$
\begin{aligned}
\tilde{\eta}_i(t+1) &= D\tilde{\eta}_i(t) + \eta^d_{i,0}(t) + \eta^d_{i,i-1}(t) \qquad (33) \\
\tilde{\xi}_i(t+1) &= D\tilde{\xi}_i(t) - \xi^d_{i,0}(t) - \xi^d_{i,i-1}(t) \qquad (34) \\
\tilde{\theta}_i(t+1) &= \left(I_n - \frac{\gamma_1 \phi_i(t) \phi_i(t)^T}{1+\gamma_2 \phi_i(t)^T \phi_i(t)} \right) \tilde{\theta}_i(t) \\
&+ \theta^d_{i,0}(t) + \theta^d_{i,i-1}(t) - \theta^d_{i,i}(t). \qquad (35)
\end{aligned}
$$

where, for $i = 1, \ldots, N$,

$$
\begin{aligned}
\eta^d_{i,0}(t) &= (\mathbf{d} - \mathbf{b})(1 - w_{i0})(\hat{y}_0(t) - y_0(t)) \\
\eta^d_{i,i-1}(t) &= (\mathbf{d} - \mathbf{b})\, \mathbf{c} \sum_{j=1}^{i-1} w_{ij} \Big(\tilde{\eta}_j(t) + \Xi_j(t) \tilde{\theta}_j(t) \\
&\qquad + \tilde{\Xi}_j(t) \hat{\theta}_0(t) \Big) \\
\xi^d_{i,0}(t) &= \mathbf{b}(1 - w_{i0})(\hat{y}_0(t) - y_0(t)) \\
\xi^d_{i,i-1}(t) &= \mathbf{bc} \sum_{j=1}^{i-1} w_{ij} \left(\tilde{\eta}_j(t) + \Xi_j(t) \tilde{\theta}_j(t) + \tilde{\Xi}_j(t) \hat{\theta}_0(t) \right) \\
\theta^d_{i,0}(t) &= \frac{\gamma_1 \phi_i(t) w_{i0}}{1+\gamma_2 \phi_i(t)^T \phi_i(t)} (y_0(t) - \hat{y}_0(t)) \\
&\quad - \frac{\gamma_1 \phi_0(t)}{1+\gamma_2 \phi_0(t)^T \phi_0(t)} (y_0(t) - \hat{y}_0(t)) \\
\theta^d_{i,i-1}(t) &= \frac{\gamma_1 \phi_i(t)}{1+\gamma_2 \phi_i(t)^T \phi_i(t)} \sum_{j=1}^{i-1} w_{ij} (y_j(t) - \hat{y}_0(t)) \\
\theta^d_{i,i}(t) &= \frac{\gamma_1 \phi_i(t)}{1+\gamma_2 \phi_i(t)^T \phi_i(t)} \mathbf{c} \left(\tilde{\eta}_i(t) + \tilde{\Xi}_i(t) \hat{\theta}_0(t) \right).
\end{aligned}
$$

Consider the base case with $i = 1$. Systems (33) to (35) become

$$
\begin{aligned}
\tilde{\eta}_1(t+1) &= D\tilde{\eta}_1(t) + \eta^d_{1,0}(t) \qquad (36) \\
\tilde{\xi}_1(t+1) &= D\tilde{\xi}_1(t) - \xi^d_{1,0}(t) \qquad (37) \\
\tilde{\theta}_1(t+1) &= \left(I_n - \frac{\gamma_1 \phi_1(t) \phi_1(t)^T}{1+\gamma_2 \phi_1(t)^T \phi_1(t)} \right) \tilde{\theta}_1(t) \\
&+ \theta^d_{1,0}(t) - \theta^d_{1,1}(t). \qquad (38)
\end{aligned}
$$

In particular, since $w_{10} = 1$, $\eta^d_{1,0}(t) \equiv 0$ and $\xi^d_{1,0}(t) \equiv 0$. Then, we have

$$
\lim_{t\to\infty} \tilde{\eta}_1(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} \tilde{\xi}_1(t) = 0 \qquad (39)
$$

both exponentially, which further implies that

$$
\lim_{t\to\infty} \tilde{\Xi}_1(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} (\phi_1(t) - \phi_0(t)) = 0 \qquad (40)
$$

both exponentially. Since $\phi_0(t)$ is bounded over $\mathbb{Z}^+$, so is $\phi_1(t)$, i.e., there exists $\varphi_1 > 0$ such that $\|\phi_1(t)\|^2 \leq \varphi_1, \forall\, t \in \mathbb{Z}^+$. Moreover, by Lemma 3 of [9], $\phi_0(t)$ is PE. Hence, by Lemma 2, $\phi_1(t)$ is also PE.

Now we consider system (38). By (32), (39), (40), and the fact that $\hat{\theta}_0(t)$ and $\phi_0(t)$ are bounded over $\mathbb{Z}^+$, we have $\lim_{t\to\infty} \theta^d_{1,0}(t) = 0$ and $\lim_{t\to\infty} \theta^d_{1,1}(t) = 0$ both exponentially. Further, since $\phi_1(t)$ is PE and satisfies $\|\phi_1(t)\|^2 \leq \varphi_1, \forall\, t \in \mathbb{Z}^+$ and $\gamma_1 - 2\gamma_2 \leq 0 < \frac{2}{\varphi_1}$, by Lemma 1 of [10], the following system:

$$
\tilde{\theta}_1(t+1) = \left(I_n - \frac{\gamma_1 \phi_1(t) \phi_1(t)^T}{1+\gamma_2 \phi_1(t)^T \phi_1(t)} \right) \tilde{\theta}_1(t)
$$

is exponentially stable. Thus, system (38) is seen to be an exponentially stable linear system subject to a bounded and exponentially decaying input. By invoking Lemma 1 of [8], we have $\lim_{t\to\infty}\tilde{\theta}_1(t) = 0$ exponentially, which together with (39) proves the base case.

Next, suppose for some integer k with $2 \leq k \leq N-1$,

$$\lim_{t\to\infty}\tilde{\eta}_j(t) = 0, \quad \lim_{t\to\infty}\tilde{\xi}_j(t) = 0, \quad \lim_{t\to\infty}\tilde{\theta}_j(t) = 0 \quad (41)$$

all exponentially, for $j = 1, \ldots, k-1$. Then, consider systems (33) to (35) specified to $i = k$ as follows:

$$\tilde{\eta}_k(t+1) = D\tilde{\eta}_k(t) + \eta^d_{k,0}(t) + \eta^d_{k,k-1}(t) \quad (42)$$

$$\tilde{\xi}_k(t+1) = D\tilde{\xi}_k(t) - \xi^d_{k,0}(t) - \xi^d_{k,k-1}(t) \quad (43)$$

$$\begin{aligned}\tilde{\theta}_k(t+1) = &\left(I_n - \frac{\gamma_1\phi_k(t)\phi_k(t)^T}{1+\gamma_2\phi_k(t)^T\phi_k(t)}\right)\tilde{\theta}_k(t) \\ &+ \theta^d_{k,0}(t) + \theta^d_{k,k-1}(t) - \theta^d_{k,k}(t). \quad (44)\end{aligned}$$

Similar to the base case, from (32), we first have $\lim_{t\to\infty}\eta^d_{k,0}(t) = 0$ and $\lim_{t\to\infty}\xi^d_{k,0}(t) = 0$ both exponentially. Then that $\lim_{t\to\infty}\eta^d_{k,k-1}(t) = 0$ and $\lim_{t\to\infty}\xi^d_{k,k-1}(t) = 0$ both exponentially follows from (41) and the fact that $\xi_0(t)$ and $\hat{\theta}_0(t)$ are bounded over $\mathbb{Z}^+$. Thus, we obtain from (42) and (43) that

$$\lim_{t\to\infty}\tilde{\eta}_k(t) = 0 \quad \text{and} \quad \lim_{t\to\infty}\tilde{\xi}_k(t) = 0 \quad (45)$$

both exponentially, and hence

$$\lim_{t\to\infty}\tilde{\Xi}_k(t) = 0 \quad \text{and} \quad \lim_{t\to\infty}(\phi_k(t) - \phi_0(t)) = 0 \quad (46)$$

both exponentially. By the boundedness of $\phi_0(t)$, $\phi_k(t)$ is bounded and there exists $\varphi_k > 0$ such that $\|\phi_k(t)\|^2 \leq \varphi_k, \forall\, t \in \mathbb{Z}^+$. In addition, since $\phi_0(t)$ is PE by Lemma 3 of [9], by Lemma 2, $\phi_k(t)$ is also PE.

Now we consider system (44). By (32), (41), (45), (46), and the fact that $\hat{\theta}_0(t)$ and $\phi_0(t)$ are bounded over $\mathbb{Z}^+$, we have $\lim_{t\to\infty}\theta^d_{k,0}(t) = 0, \lim_{t\to\infty}\theta^d_{k,k-1}(t) = 0$, and $\lim_{t\to\infty}\theta^d_{k,k}(t) = 0$ all exponentially. Further, since $\phi_k(t)$ is PE and satisfies $\|\phi_k(t)\|^2 \leq \varphi_k, \forall\, t \in \mathbb{Z}^+$ and $\gamma_1 - 2\gamma_2 \leq 0 < \frac{2}{\varphi_k}$, by Lemma 1 of [10], the following system:

$$\tilde{\theta}_k(t+1) = \left(I_n - \frac{\gamma_1\phi_k(t)\phi_k(t)^T}{1+\gamma_2\phi_k(t)^T\phi_k(t)}\right)\tilde{\theta}_k(t)$$

is exponentially stable. Thus, again, we can view system (44) as an exponentially stable linear system subject to a bounded and exponentially decaying input. Then, it follows from Lemma 1 of [8] that $\lim_{t\to\infty}\tilde{\theta}_k(t) = 0$ exponentially, which together with (45) establishes the induction step.

The overall proof is completed by induction. □

Finally, the conjunction of Propositions 2 and 3 gives the following main result of this paper.

Theorem 1: Under Assumption 1, Problem 1 is solvable by an adaptive distributed observer composed of (12), (13), and (15), with the estimated output $y_i(t)$ given by (14) and the estimated state $v_i(t)$ given by (16).

V. Conclusion

We have proposed an adaptive distributed observer for a discrete-time linear leader system, whose system matrix is uncertain and neutrally stable. By assuming that the digraph of the communication network is a spanning tree with the leader system as the root, we have shown that the proposed adaptive distributed observer is able to provide for each follower not only an estimate of the leader's state, but also an estimate of the unknown parameters of the leader's uncertain system matrix. An extension of the result of this paper is to consider the same problem over general digraphs.

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