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Leader-Following Consensus of Multiple Uncertain Rigid Body Systems by a Sampled-Data Adaptive Distributed Observer

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Abstract— **In this paper, we study the leader-following attitude consensus problem for multiple uncertain rigid body systems by a sampled-data adaptive distributed observer. Unlike the existing sampled-data distributed observer, which can only asymptotically estimate the state of the leader, the sampled-data adaptive distributed observer can estimate both the state and the system matrix of the leader exponentially. We synthesize a distributed control law utilizing sampled-data communications to solve the leader-following attitude consensus problem for multiple uncertain rigid body systems based on the sampled-data adaptive distributed observer. Compared with a distributed control law that uses continuous-time communications, the distributed control law utilizing sampled-data communications consumes fewer communication resources and is more robust to communication failures.**

I. Introduction

The cooperative control of multiple rigid body systems has received extensive attention from the research community due to its applications in aerospace engineering. One of the fundamental problems in the cooperative control of multiple rigid body systems is the consensus problem. The consensus of multiple rigid body systems is categorized into the leaderless consensus and the leader-following consensus. The leaderless consensus seeks to design a control law that makes the attitudes of all vehicles synchronize to some common trajectory [1], [16], [24]. The leader-following consensus, on the other hand, aims to design a control law such that the attitudes and the angular velocities of all vehicles track a desired reference attitude and a desired reference angular velocity generated by the so-called leader system [2], [3], [7], [15], [17].

Assuming that the inertia matrices of the vehicles are known, the leader-following attitude consensus problem of multiple rigid body systems was studied in [3], [17]. Specifically, the problem was addressed in [3] over connected static networks and in [17] over jointly connected switching networks. The global finite-time leader-following attitude consensus problem of multiple rigid body systems was studied in [10] over a connected static network. For the case where the inertia matrices of the vehicles are unknown, the leader-following attitude consensus problem of multiple uncertain rigid body systems was further studied in [4] over a connected static network. The result of [4] was extended from a connected static network to a jointly connected switching network in [25].

The results in [3], [4], [10], [17], [25] were established using the so-called distributed observer approach. The establishment of the distributed observer for the leader system is at the heart of the distributed observer based approach. However, the distributed observers in [3], [4], [10], [17], [25] are analog. Since sampled-data communications consume fewer communication resources than continuous-time communications [9], it is desired to design a distributed control law that makes use of sampled-data communications to conserve limited onboard resources of spacecraft systems. Indeed, a number of results on the stability analysis of sampled-data systems have been established [6], [8], [19], [21], [23].

The consensus of multiple rigid body systems utilizing sampled-data communications has also been investigated in, for instance, [12], [20], [26]. In particular, the leaderless attitude consensus problem of multiple rigid body systems was studied over static networks in [20] using periodic sampled-data communications and in [26] using aperiodic sampled-data communications. The leader-following attitude consensus problem of multiple rigid body systems was studied in [12] by a distributed observer utilizing aperiodic sampled-data communications. However, the result of [12] has the following limitations. First, the inertia matrices of the vehicles are assumed known. Second, all vehicles are assumed to know the system matrix of the leader system. Since these two assumptions are restrictive in practice, it is desirable to remove these assumptions.

In this paper, we will further consider the leader-following attitude consensus problem of multiple uncertain rigid body systems utilizing aperiodic sampled-data communications. Compared with the result of [12], this paper offers three distinctive features. First, unlike [12] which requires all vehicles to know the system matrix of the leader, we do not require all vehicles to know the system matrix of the leader. Second, while the sampled-data distributed observer in [12] can only achieve asymptotic convergence, the sampled-data adaptive distributed observer in this paper can achieve exponential convergence. Third, our approach can deal with uncertain rigid body systems that cannot be handled by [12]. It is worth noting that the exponential convergence feature of the adaptive distributed observer is essential in dealing with the uncertain parameters of rigid body systems. A full version of this paper is under review [13].

Notation. $\mathbb{N}$ denotes the set of all natural numbers. $\|A\|_F$

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denotes the Frobenius norm of matrix A. If $A \in \mathbb{R}^{N\times M}$ with A_i its ith column, then $\mathrm{vec}(A) = \begin{bmatrix} A_1 \\ \vdots \\ A_M \end{bmatrix}$. A matrix $A \in \mathbb{R}^{N\times N}$ is called an $\mathcal{M}$-matrix, if all of its off-diagonal elements are non-positive and all of its eigenvalues have positive real parts. For any $q = \mathrm{col}(\hat{q}, \bar{q}) \in \mathbb{R}^4$ with $\hat{q} \in \mathbb{R}^3$ and $\bar{q} \in \mathbb{R}$, let $T : \mathbb{R}^4 \to \mathbb{R}^{4\times 3}$ be defined as $T(q) = \begin{bmatrix} \bar{q}I_3 + \hat{q}^\times \\ -\hat{q}^\top \end{bmatrix}$. For $x = \mathrm{col}(x_1, x_2, x_3) \in \mathbb{R}^3$, $x^\times = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$. $\mathbb{Q} = \{q \mid q = \mathrm{col}(\hat{q}, \bar{q}), \hat{q} \in \mathbb{R}^3, \bar{q} \in \mathbb{R}\}$ denotes the set of all quaternions. $\mathbb{Q}_u = \{q \mid q \in \mathbb{Q}, \|q\| = 1\}$ denotes the set of all unit quaternions. For $q_i, q_j \in \mathbb{Q}$, the quaternion product $q_i \odot q_j = \begin{bmatrix} \bar{q}_i\hat{q}_j + \bar{q}_j\hat{q}_i + \hat{q}_i^\times \hat{q}_j \\ \bar{q}_i\bar{q}_j - \hat{q}_i^\top \hat{q}_j \end{bmatrix}$. For $q \in \mathbb{Q}$, the quaternion conjugate $q^* = \mathrm{col}(-\hat{q}, \bar{q})$. For $q \in \mathbb{Q}_u$, the quaternion inverse $q^{-1} = q^*$. For $x \in \mathbb{R}^3$, $\mathbf{Q}(x) = \mathrm{col}(x, 0) \in \mathbb{R}^4$. For $q \in \mathbb{Q}$, $C(q) = (\bar{q}^2 - \hat{q}^\top\hat{q})I_3 + 2\hat{q}\hat{q}^\top - 2\bar{q}\hat{q}^\times$.

II. Preliminaries and problem formulation

We consider a group of N uncertain rigid body systems described by the following equations [3], [17]:

$$\dot{q}_i = \frac{1}{2} q_i \odot \mathbf{Q}(\Omega_i) \tag{1a}$$

$$J_i\dot{\Omega}_i = -\Omega_i^\times J_i\Omega_i + u_i,\ i = 1, \dots, N \tag{1b}$$

where, for the ith rigid body system, $i = 1, \dots, N$, $q_i \in \mathbb{Q}_u$ is the unit quaternion representation of the attitude of the body-fixed frame $\mathcal{B}_i$ with respect to the inertial frame $\mathcal{I}$; $\Omega_i \in \mathbb{R}^3$ is the angular velocity of $\mathcal{B}_i$ relative to $\mathcal{I}$; $J_i \in \mathbb{R}^{3\times 3}$ is the positive definite inertia matrix whose entries are unknown; and $u_i \in \mathbb{R}^3$ is the control torque. Note that Ω_i, J_i, and u_i are all expressed in the body-fixed frame $\mathcal{B}_i$.

The desired reference angular velocity Ω_0 and the desired reference attitude q_0 are assumed to be generated by the following leader system [17]:

$$\dot{v} = Sv, \quad \Omega_0 = Ev \tag{2a}$$

$$\dot{q}_0 = \frac{1}{2} q_0 \odot \mathbf{Q}(\Omega_0) \tag{2b}$$

where $S \in \mathbb{R}^{m\times m}$ and $E \in \mathbb{R}^{3\times m}$ are constant matrices, $v \in \mathbb{R}^m$, $\Omega_0 \in \mathbb{R}^3$, and $q_0 = \mathrm{col}(\hat{q}_0, \bar{q}_0) \in \mathbb{Q}_u$.

The system composed of (1) and (2) is a multi-agent system of $N + 1$ agents, where system (2) is the leader and the N agents of (1) are followers. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be the digraph associated with the multi-agent system composed of (1) and (2), where $\mathcal{V} = \{0, 1, \dots, N\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. Let node 0 be associated with the leader system (2) and node i, $i = 1, \dots, N$, be associated with the ith agent of system (1). For $i = 1, \dots, N$, $j = 0, 1, \dots, N$, $i \neq j$, $(j, i) \in \mathcal{E}$ if and only if the ith agent of (1) can use the sampled information of $\vartheta_j(t) = (S_j(t), \xi_j(t), \eta_j(t))$ of agent j, which will be defined precisely in (13). If the digraph $\mathcal{G}$ contains a sequence of edges of the form $(i_1, i_2), (i_2, i_3), \dots, (i_{k-1}, i_k)$, then node i_k is said to be reachable from node i_1. The weighted adjacency matrix of the digraph $\mathcal{G}$ is a nonnegative matrix $\mathcal{A} = [a_{ij}]_{i,j=0}^N \in \mathbb{R}^{(N+1)\times(N+1)}$, where $a_{ii} = 0$, and, for $i \neq j$, $a_{ij} > 0 \Leftrightarrow (j, i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. Let $H = [h_{ij}]_{i,j=1}^N \in \mathbb{R}^{N\times N}$, where $h_{ii} = \sum_{j=0}^N a_{ij}$ and $h_{ij} = -a_{ij}$ if $i \neq j$. Let $\mathcal{N}_i$ denote the neighbor set of the ith follower.

We consider a control law of the following form:

$$u_i(t) = f_i\left(q_i(t), \Omega_i(t), \vartheta_i(t)\right) \tag{3a}$$

$$\dot{\hat{\Theta}}_i(t) = g_i\left(q_i(t), \Omega_i(t), \vartheta_i(t)\right) \tag{3b}$$

$$\dot{\vartheta}_i(t) = r_i\left(\vartheta_i(t), \vartheta_j(t_s), j \in \mathcal{N}_i\right),\ i = 1, \dots, N \tag{3c}$$

for any $t \in [t_s, t_{s+1})$, where $t_0 = 0$, $t_{s+1} = t_s + T_s$, $s \in \mathbb{N}$, $T_s \in [\underline{h}, \bar{h}]$ with $\underline{h} \leq \bar{h} < \bar{T}$ being two positive real numbers; $\vartheta_0(t_s) = (S, v(t_s), q_0(t_s))$; $\vartheta_i(t) = (S_i(t), \xi_i(t), \eta_i(t))$ with $S_i(t)$, $\xi_i(t)$, and $\eta_i(t)$ being the estimates of S, $v(t)$, and $q_0(t)$, respectively; and $f_i(\cdot)$, $g_i(\cdot)$, and $r_i(\cdot)$ are some nonlinear functions to be defined.

Remark 2.1: The positive number $\bar{T}$ is called an estimated upper bound of the sampling intervals. It will be made clear later that, for $i = 1, \dots, N$, the dynamics (3b) represents the adaptation law, and the dynamics (3c) represents the adaptive distributed observer for the leader system (2). Since, for each i, the right-hand side of (3c) only depends on the aperiodically sampled state variable $\vartheta_j(t_s)$ if $j \in \mathcal{N}_i$, (3c) can be more precisely called a sampled-data adaptive distributed observer for the leader system (2).

Like in [22] and [27], the attitude tracking error and the angular velocity tracking error between the ith follower and the leader are defined as

$$\epsilon_i = q_0^{-1} \odot q_i \tag{4}$$

$$\check{\Omega}_i = \Omega_i - C(\epsilon_i)\Omega_0,\ i = 1, \dots, N \tag{5}$$

where, for $i = 1, \dots, N$, $\epsilon_i = \mathrm{col}(\hat{\epsilon}_i, \bar{\epsilon}_i) \in \mathbb{Q}_u$ and $\check{\Omega}_i \in \mathbb{R}^3$. By [27, Proposition 1], $\mathcal{B}_i$ and $\mathcal{B}_0$ coincide if and only if $\hat{\epsilon}_i = 0$, where $\mathcal{B}_0$ is the body-fixed frame associated with the reference attitude $q_0(t)$.

The leader-following attitude consensus problem of multiple uncertain rigid body systems is described as follows.

Problem 1: Given systems (1), (2), and a digraph $\mathcal{G}$, design a distributed control law of the form (3) together with a positive number $\bar{T}$ such that, for any $\underline{h} \leq \bar{h} < \bar{T}$ and any sampling intervals T_s, $s \in \mathbb{N}$, satisfying $T_s \in [\underline{h}, \bar{h}]$, and for any initial conditions $\Omega_i(0)$, $\hat{\Theta}_i(0)$, $\vartheta_i(0)$, $i = 1, \dots, N$, and $q_i(0) \in \mathbb{Q}_u$, $i = 0, 1, \dots, N$, the solution of the closed-loop system exists and satisfies

$$\lim_{t\to\infty} \hat{\epsilon}_i(t) = 0, \quad \lim_{t\to\infty} \check{\Omega}_i(t) = 0,\ i = 1, \dots, N. \tag{6}$$

For the solvability of the problem, we make the following assumptions.

Assumption 1: All the eigenvalues of the matrix S are semi-simple with zero real parts.

Assumption 2: The initial condition of system (2a) satisfies $v(0) \in V_0$ with $V_0 \subset \mathbb{R}^m$ being an arbitrary known compact set containing the origin of $\mathbb{R}^m$.

Assumption 3: Every node i, $i = 1, \dots, N$, is reachable from node 0 in the digraph $\mathcal{G}$.

Remark 2.2: Under Assumption 1, the reference angular velocity $\Omega_0(t) = \mathrm{col}(\Omega_{01}(t), \Omega_{02}(t), \Omega_{03}(t))$ is a multi-tone sinusoidal signal whose components are given by $\Omega_{0i}(t) = A_{i0} + \sum_{j=1}^{p_i} A_{ij}\sin(\omega_{ij}t + \phi_{ij})$, $i = 1, 2, 3$, where p_i is some finite integer; A_{i0} is the step magnitude; and A_{ij}, ω_{ij}, and ϕ_{ij} are the amplitudes, frequencies, and phase angles of the sinusoidal functions, respectively. Under Assumptions 1 and 2, there exists a known positive constant $\bar{v}$ such that

$$\sup_{t\geq 0}\|v(t)\| < \bar{v}. \tag{7}$$

Under Assumption 3, the digraph $\mathcal{G}$ is connected with node 0 as the root.

III. Key Lemmas

We recall a technical result from [18] as follows.

Lemma 3.1 (Lemma 2 of [18]): Suppose $W(t) : [0,\infty) \to [0,\infty)$ is continuous, and there exists a sequence $\{t_s \mid s \in \mathbb{N}, t_s \in [0,\infty)\}$ satisfying $t_{s+1} - t_s \geq h$ for all $s \in \mathbb{N}$ and for some positive real number h such that $W(t)$ is differentiable on each interval $[t_s, t_{s+1})$ and

$$\dot{W}(t) \leq -\beta_1 W(t) + \beta_2 W_M(t),\ \forall t \in [t_s, t_{s+1}) \tag{8}$$

where $W_M(t) = \max_{\tau\in[t_s,t]} W(\tau)$ for any $t \in [t_s, t_{s+1})$, and β_1 and β_2 are two positive real numbers with $\beta_2 < \beta_1$. Then $\lim_{t\to\infty} W(t) = 0$.

In the next lemma, we strengthen Lemma 3.1 from asymptotic convergence to exponential convergence.

Lemma 3.2: Suppose $W(t) : [0,\infty) \to [0,\infty)$ is continuous, and there exists a sequence $\{t_s \mid s \in \mathbb{N}, t_s \in [0,\infty)\}$ satisfying $\underline{h} \leq t_{s+1} - t_s \leq \bar{h}$ for all $s \in \mathbb{N}$ and for some positive real numbers $\underline{h}$ and $\bar{h}$ such that $W(t)$ is differentiable on each interval $[t_s, t_{s+1})$ and

$$\dot{W}(t) \leq -\beta_1 W(t) + \beta_2 W_M(t),\ \forall t \in [t_s, t_{s+1}) \tag{9}$$

where $W_M(t) = \max_{\tau\in[t_s,t]} W(\tau)$ for any $t \in [t_s, t_{s+1})$, and β_1 and β_2 are two positive real numbers satisfying $\beta_2 < \beta_1$. Let

$$\rho = \left(1 - \frac{\beta_2}{\beta_1}\right)\mathbf{e}^{-\beta_1\underline{h}} + \frac{\beta_2}{\beta_1}. \tag{10}$$

Then, we have

$$W(t) \leq \frac{1}{\rho}\mathbf{e}^{-\ln(\frac{1}{\rho})\frac{1}{\bar{h}}t}W(0),\ \forall t \geq 0. \tag{11}$$

Remark 3.1: Lemma 3.2 can be viewed as a variant of Halanay inequality (see [11, Section 4.5]), which also establishes the exponential convergence of the function $W(t)$. Compared with Halanay inequality, Lemma 3.2 further provides an explicit estimate of the exponential convergence rate, that is, $\ln(\frac{1}{\rho})\frac{1}{\bar{h}}$. The proof of Lemma 3.2 is omitted due to space limitations.

IV. Sampled-Data Adaptive Distributed Observer

Let us first recall the continuous-time adaptive distributed observer for the leader system (2) from [5] as follows:

$$\dot{S}_i(t) = \mu_1 \sum_{j=0}^{N} a_{ij}\left(S_j(t) - S_i(t)\right) \tag{12a}$$

$$\dot{\xi}_i(t) = S_i(t)\xi_i(t) + \mu_2 \sum_{j=0}^{N} a_{ij}\left(\xi_j(t) - \xi_i(t)\right) \tag{12b}$$

$$\dot{\eta}_i(t) = \frac{1}{2}\eta_i(t) \odot \mathbf{Q}\left(\zeta_i(t)\right) + \mu_3 \sum_{j=0}^{N} a_{ij}\left(\eta_j(t) - \eta_i(t)\right) \tag{12c}$$

where μ_1, μ_2, and μ_3 are positive real numbers; $S_0 = S$, $\xi_0 = v$, and $\eta_0 = q_0$; and, for $i = 1, \ldots, N$, $S_i \in \mathbb{R}^{m\times m}$, $\xi_i \in \mathbb{R}^m$, $\eta_i \in \mathbb{R}^4$, and $\zeta_i = E\xi_i \in \mathbb{R}^3$. It was shown in [5] that, under Assumptions 1 and 3, for any $\mu_1, \mu_2, \mu_3 > 0$ and any initial conditions $S_i(0)$, $\xi_i(0)$, $\eta_i(0)$, and $v(0)$, the solution of (12) satisfies $\lim_{t\to\infty}(S_i(t) - S) = 0$, $\lim_{t\to\infty}(\xi_i(t) - v(t)) = 0$, and $\lim_{t\to\infty}(\eta_i(t) - q_0(t)) = 0$ exponentially. For this reason, system (12) can estimate the system matrix S, the state $v(t)$ of (2a), and the state $q_0(t)$ of (2b) based on the neighboring information. It is thus a continuous-time adaptive distributed observer for the leader system (2).

Based on the continuous-time adaptive distributed observer (12), one can synthesize the sampled-data version of the adaptive distributed observer for the leader system (2) as follows:

$$\dot{S}_i(t) = \mu_1 \sum_{j=0}^{N} a_{ij}\left(S_j(t_s) - S_i(t_s)\right) \tag{13a}$$

$$\dot{\xi}_i(t) = S_i\xi_i(t) + \mu_2 \sum_{j=0}^{N} a_{ij}\left(\xi_j(t_s) - \xi_i(t_s)\right) \tag{13b}$$

$$\dot{\eta}_i(t) = \frac{1}{2}\eta_i(t) \odot \mathbf{Q}\left(\zeta_i(t)\right) + \mu_3 \sum_{j=0}^{N} a_{ij}\left(\eta_j(t_s) - \eta_i(t_s)\right) \tag{13c}$$

for $t \in [t_s, t_{s+1})$, where $t_0 = 0$, $t_{s+1} = t_s + T_s$, $s \in \mathbb{N}$, $T_s \in [\underline{h}, \bar{h}]$ with $\underline{h} \leq \bar{h} < \bar{T}$ being two positive real numbers to be specified; and μ_1, μ_2, and μ_3 are positive observer gains to be designed.

A. Error Systems

In what follows, we define the error systems of (13). For this purpose, for $t \in [t_s, t_{s+1})$, let

$$\begin{aligned}\bar{S}_i(t) &= S_i(t) - S, \quad \bar{S}(t) = \mathrm{col}(\bar{S}_1(t), \ldots, \bar{S}_N(t))\\ \bar{\theta}(t) &= \mathrm{vec}(\bar{S}(t)), \quad \tilde{\bar{\theta}}_i(t) = \bar{\theta}_i(t_s) - \bar{\theta}_i(t)\\ \tilde{\bar{\theta}}(t) &= \mathrm{col}(\tilde{\bar{\theta}}_1(t), \ldots, \tilde{\bar{\theta}}_N(t))\end{aligned} \tag{14}$$

where, for $i = 1, \ldots, N$, $\bar{S}_i(t)$ denotes the estimation error of $S_i(t)$ and $\tilde{\bar{\theta}}$ denotes the sampling error of $\bar{\theta}(t)$. Then, for $t \in [t_s, t_{s+1})$, $\bar{\theta}(t)$ is governed by

$$\begin{aligned}\dot{\bar{\theta}}(t) &= -\mu_1\left(I_m \otimes H \otimes I_m\right)\bar{\theta}(t_s)\\ &= -\mu_1\left(I_m \otimes H \otimes I_m\right)\bar{\theta}(t) - \mu_1\left(I_m \otimes H \otimes I_m\right)\tilde{\bar{\theta}}(t).\end{aligned} \tag{15}$$

Also, let

$$\begin{aligned} S_d(t) &= \text{block diag}(S_1(t),\ldots,S_N(t)) \\ \bar{S}_d(t) &= \text{block diag}(\bar{S}_1(t),\ldots,\bar{S}_N(t)) \\ \bar{\xi}(t) &= \text{col}(\bar{\xi}_1(t),\ldots,\bar{\xi}_N(t)) \\ \tilde{\bar{\xi}}(t) &= \text{col}(\tilde{\bar{\xi}}_1(t),\ldots,\tilde{\bar{\xi}}_N(t)). \end{aligned} \tag{16}$$

Then, for $t \in [t_s, t_{s+1})$, $\bar{\xi}(t)$ is governed by

$$\begin{aligned} \dot{\bar{\xi}}(t) &= S_d(t)\xi(t) - \mu_2(H\otimes I_m)\bar{\xi}(t_s) - (I_N\otimes S)\left(\mathbf{1}_N\otimes v(t)\right) \\ &= A_c\bar{\xi}(t) + \bar{S}_d(t)\left(\bar{\xi}(t) + \mathbf{1}_N\otimes v(t)\right) - \mu_2(H\otimes I_m)\tilde{\bar{\xi}}(t) \end{aligned} \tag{17}$$

where $A_c = I_N \otimes S - \mu_2(H\otimes I_m)$.

To define the error system of (13c), for $t \in [t_s, t_{s+1})$, let

$$\begin{aligned} \bar{\eta}_i(t) &= \eta_i(t) - q_0(t), \quad \tilde{\bar{\eta}}_i(t) = \bar{\eta}_i(t_s) - \bar{\eta}_i(t) \\ \bar{\eta} &= \text{col}(\bar{\eta}_1,\ldots,\bar{\eta}_N), \quad \tilde{\bar{\eta}} = \text{col}(\tilde{\bar{\eta}}_1,\ldots,\tilde{\bar{\eta}}_N) \end{aligned} \tag{18}$$

where, for $i = 1,\ldots,N$, $\bar{\eta}_i(t)$ denotes the estimation errors of $\eta_i(t)$ and $\tilde{\bar{\eta}}_i(t)$ denotes the sampling errors of $\bar{\eta}_i(t)$.

Let $M : \mathbb{R}^3 \to \mathbb{R}^{4\times 4}$ be defined such that, for any $y = \text{col}(y_1,y_2,y_3) \in \mathbb{R}^3$, $M(y) = \begin{bmatrix} 0 & y_3 & -y_2 & y_1 \\ -y_3 & 0 & y_1 & y_2 \\ y_2 & -y_1 & 0 & y_3 \\ -y_1 & -y_2 & -y_3 & 0 \end{bmatrix}$. Then, from (2b) and (13c), as shown in [12], for $t \in [t_s, t_{s+1})$ and $i = 1,\ldots,N$, $\bar{\eta}_i(t)$ is governed by

$$\begin{aligned} \dot{\bar{\eta}}_i(t) &= \frac{1}{2}M(\zeta_i(t))\bar{\eta}_i(t) + \frac{1}{2}T(q_0(t))E\bar{\xi}_i(t) \\ &\quad + \mu_3\sum_{j=0}^{N} a_{ij}\left(\bar{\eta}_j(t) - \bar{\eta}_i(t)\right) \\ &\quad + \mu_3\sum_{j=0}^{N} a_{ij}\left(\tilde{\bar{\eta}}_j(t) - \tilde{\bar{\eta}}_i(t)\right) \end{aligned} \tag{19}$$

which can be put in the following compact form:

$$\begin{aligned} \dot{\bar{\eta}}(t) &= \left(\mathcal{M}(\zeta(t)) - \mu_3\left(H\otimes I_4\right)\right)\bar{\eta}(t) - \mu_3\left(H\otimes I_4\right)\tilde{\bar{\eta}}(t) \\ &\quad + \frac{1}{2}\left(I_N\otimes T(q_0(t))E\right)\bar{\xi}(t) \end{aligned} \tag{20}$$

where $\mathcal{M}(\zeta(t)) = \frac{1}{2}\text{block diag}(M(\zeta_1(t)),\ldots,M(\zeta_N(t)))$ with $\zeta(t) = \text{col}(\zeta_1(t),\ldots,\zeta_N(t))$ and $\zeta_i(t) = E\xi_i(t)$.

Remark 4.1: To show that (13) is indeed a sampled-data adaptive distributed observer for the leader system (2), it suffices to show that the solutions of the error systems (15), (17), and (20) converge to the origin as time goes to infinity. To this end, we will show that the solutions of these three error systems indeed converge to the origin as desired in the subsequent three subsections.

B. *Convergence Analysis of Error System* (15)

We will start with studying the convergence property of the error system (15). For this purpose, note that the matrix H is an $\mathcal{M}$-matrix under Assumption 3. Thus, by [14, Theorem 2.5.3], there exists a positive definite diagonal matrix $D \in \mathbb{R}^{N\times N}$ such that $\bar{H} = DH + H^\top D$ is positive definite.

Let

$$\begin{aligned} &d_m = \lambda_{\min}(D), \quad d_M = \lambda_{\max}(D), \quad \lambda_1 = \lambda_{\min}(\bar{H}) \\ &\alpha_1 = \frac{\mu_1\|H\|}{\sqrt{d_m}}, \quad c_0 = \frac{2\mu_1\|DH\|^2}{\lambda_1}\alpha_1^2 \\ &\bar{T}_1 = \sqrt{\frac{\mu_1\lambda_1}{2d_Mc_0}}. \end{aligned} \tag{21}$$

Then we obtain the following result, whose proof is omitted due to space limitations.

Lemma 4.1: Consider system (13a). For all sampling intervals $T_s \in [\underline{h}, \bar{h}]$, $s \in \mathbb{N}$, with $0 < \underline{h} \le \bar{h} < \bar{T}_1$, any $\mu_1 > 0$, and any initial condition $S_i(0)$, the solution of (13a) exists for all $t \ge 0$ and satisfies

$$\lim_{t\to\infty}\left(S_i(t) - S\right) = 0,\ i = 1,\ldots,N \tag{22}$$

exponentially.

C. *Convergence Analysis of Error System* (17)

In what follows, we will further show that the system composed of (13a)–(13b) is a sampled-data adaptive distributed observer for (2a) by studying the convergence property of the error system (17). For this purpose, we define some quantities that will be useful in the proof of Lemma 4.2 as follows.

Under Assumption 3, by Theorem 3.3 of [5], for any $\mu_2 > 0$, the matrix A_c is Hurwitz. Then, there exists a positive definite matrix $P \in \mathbb{R}^{Nm\times Nm}$ satisfying the following Lyapunov equation:

$$A_c^\top P + PA_c = -I_{Nm}. \tag{23}$$

Let

$$\begin{aligned} &p_m = \lambda_{\min}(P), \quad p_M = \lambda_{\max}(P) \\ &c_1 = \min\{d_m, p_m\}, \quad c_2 = \max\{d_M, p_M\} \\ &\lambda_2 = \|P(H\otimes I_m)\| \\ &\alpha_2 = \frac{\|S\| + \sqrt{N}\bar{v} + \mu_2\|H\|}{\sqrt{c_1}} + \frac{1}{4p_M\sqrt{c_1}} \\ &c_3 = \frac{2\mu_1\|DH\|^2}{\lambda_1}\alpha_1^2 + 4\mu_2^2\lambda_2^2\alpha_2^2. \end{aligned} \tag{24}$$

Let

$$Q = \frac{1}{4}\begin{bmatrix} 2\mu_1\lambda_1 & -4\sqrt{N}p_M\bar{v} \\ -4\sqrt{N}p_M\bar{v} & 1 \end{bmatrix} \tag{25}$$

and let μ_1 be such that

$$\mu_1 > \frac{8Np_M^2\bar{v}^2}{\lambda_1}. \tag{26}$$

Then, it can be verified that Q is positive definite. Also, let

$$\bar{T}_2 = \min\left\{\bar{T}_1, \sqrt{\frac{\lambda_{\min}(Q)}{c_2c_3}}\right\} \tag{27}$$

where $\bar{T}_1$ is defined in (21).

The following lemma ascertains that the system composed of (13a)–(13b) is indeed a sampled-data adaptive distributed observer for (2a). Due to space limitations, the proof is omitted.

Lemma 4.2: Consider the system composed of (13a)–(13b). Under Assumptions 1 to 3, if the sampling intervals satisfy $T_s \in [\underline{h}, \bar{h}]$ with $0 < \underline{h} \leq \bar{h} < \bar{T}_2$ for all $s \in \mathbb{N}$, and if μ_1 satisfies (26) and $\mu_2 > 0$, then for any initial conditions $S_i(0)$, $\xi_i(0)$, $i = 1, \ldots, N$, and $v(0) \in V_0$, the solution of the system composed of (13a)–(13b) exists and satisfies (22) and

$$\lim_{t\to\infty} (\xi_i(t) - v(t)) = 0, \; i = 1, \ldots, N \tag{28}$$

exponentially.

D. Convergence Analysis of Error System (20)

In the sequel, we will show that (13) is a sampled-data adaptive distributed observer for the leader (2) by investigating the convergence property of system (20). For this purpose, we define some quantities that will be useful in the proof of Theorem 4.1 as follows.

Let

$$\begin{aligned} \alpha_3 &= \frac{\|E\|\bar{v} + 2\mu_3\|H\| + \|E\|}{2\sqrt{c_1}} \\ c_4 &= \frac{2\mu_1\|DH\|^2}{\lambda_1}\alpha_1^2 + 4\mu_2^2\lambda_2^2\alpha_2^2 + \frac{2\mu_3\|DH\|^2}{\lambda_1}\alpha_3^2. \end{aligned} \tag{29}$$

Let

$$\bar{Q} = \frac{1}{4}\begin{bmatrix} 2\mu_1\lambda_1 & -4\sqrt{N}p_M\bar{v} & 0 \\ -4\sqrt{N}p_M\bar{v} & 1 & -2d_M\|E\| \\ 0 & -2d_M\|E\| & 2\mu_3\lambda_1 \end{bmatrix} \tag{30}$$

which can be rewritten as $\bar{Q} = \begin{bmatrix} Q & B \\ B^\top & C \end{bmatrix}$, where Q is defined in (25), and $B = \text{col}(0, -\frac{d_M\|E\|}{2})$ and $C = \frac{\mu_3\lambda_1}{2}$.

Let μ_1 satisfy (26). Then, the matrix Q is positive definite. In addition, let μ_3 be such that

$$\mu_3 > \frac{2}{\lambda_1}B^\top Q^{-1}B. \tag{31}$$

Then, by Schur complement, the matrix $\bar{Q}$ is also positive definite. Also, let

$$\bar{T} = \min\left\{\bar{T}_2, \sqrt{\frac{\lambda_{\min}(\bar{Q})}{c_2c_4}}\right\} \tag{32}$$

where $\bar{T}_2$ is defined in (27).

The main result of this section is summarized as follows, whose proof is omitted due to space limitations.

Theorem 4.1: Consider system (13). Under Assumptions 1 to 3, if the sampling intervals satisfy $T_s \in [\underline{h}, \bar{h}]$ with $0 < \underline{h} \leq \bar{h} < \bar{T}$ for all $s \in \mathbb{N}$, and if μ_1 satisfies (26), $\mu_2 > 0$, and μ_3 satisfies (31), then, for any initial conditions $v(0) \in V_0$, $q_0(0) \in \mathbb{Q}_u$, $S_i(0)$, $\xi_i(0)$, and $\eta_i(0)$, $i = 1, \ldots, N$, the solution of (13) exists for all $t \geq 0$ and satisfies (22), (28), and

$$\lim_{t\to\infty} (\eta_i(t) - q_0(t)) = 0, \; i = 1, \ldots, N \tag{33}$$

exponentially.

Remark 4.2: The sampled-data distributed observer in [12] is of the form

$$\dot{\xi}_i(t) = S\xi_i(t) + \mu_\xi \sum_{j=0}^{N} a_{ij}\left(\xi_j(t_s) - \xi_i(t_s)\right) \tag{34a}$$

$$\dot{\eta}_i(t) = \frac{1}{2}\eta_i(t) \odot \mathbf{Q}\left(\zeta_i(t)\right) + \mu_\eta \sum_{j=0}^{N} a_{ij}\left(\eta_j(t_s) - \eta_i(t_s)\right) \tag{34b}$$

where μ_ξ and μ_η are some positive real numbers. Since (34) can be viewed as the last two equations of the sampled-data adaptive distributed observer (13) with $S_i = S$, $i = 1, \ldots, N$, as a special case of Theorem 4.1, the solution (ξ_i, η_i) of (34) also converges to (v, q_0) exponentially.

V. Solvability of Problem 1

In this section, we employ the sampled-data adaptive distributed observer to synthesize a distributed control law utilizing sampled-data communications to solve the leader-following attitude consensus problem of multiple uncertain rigid body systems.

As in [3], [12], [17], the estimated attitude tracking error and the estimated angular velocity tracking error are defined as follows:

$$e_i = \eta_i^* \odot q_i \tag{35a}$$

$$\tilde{\Omega}_i = \Omega_i - C(e_i)\zeta_i, \; i = 1, \ldots, N \tag{35b}$$

where, for $i = 1, \ldots, N$, $e_i = \text{col}(\hat{e}_i, \bar{e}_i) \in \mathbb{Q}$ and $\tilde{\Omega}_i \in \mathbb{R}^3$. Then, for any $t \in [t_s, t_{s+1})$, the error variables e_i and $\tilde{\Omega}_i$ are governed by

$$\dot{e}_i = \frac{1}{2}e_i \odot \mathbf{Q}(\tilde{\Omega}_i) + e_{di} \tag{36a}$$

$$\begin{aligned} J_i\dot{\tilde{\Omega}}_i = &-\Omega_i^\times J_i\Omega_i + J_i\left(\tilde{\Omega}_i^\times C(e_i)\zeta_i - C(e_i)ES_i\xi_i\right) \\ &- J_{di} + u_i, \; i = 1, \ldots, N \end{aligned} \tag{36b}$$

where, for $i = 1, \ldots, N$, $e_{di} = \text{col}(\hat{e}_{di}, \bar{e}_{di}) \in \mathbb{Q}$ and $J_{di} \in \mathbb{R}^3$ are given by

$$e_{di} = \frac{1}{2}\left(e_i^\top e_i - 1\right)\mathbf{Q}(\zeta_i) \odot e_i + \eta_{di}^*(t_s) \odot q_i \tag{37}$$

$$J_{di} = J_i\left(C_{di}\zeta_i + C(e_i)E\xi_{di}(t_s)\right) \tag{38}$$

with $C_{di} = 2(\bar{e}_i\bar{e}_{di} - \hat{e}_i^\top\hat{e}_{di})I_3 + 2(\hat{e}_i\hat{e}_{di}^\top + \hat{e}_{di}\hat{e}_i^\top - \bar{e}_i\hat{e}_{di}^\times - \bar{e}_{di}\hat{e}_i^\times)$, $\xi_{di}(t_s) = \mu_2\sum_{j=0}^N a_{ij}\left(\xi_j(t_s) - \xi_i(t_s)\right)$, and $\eta_{di}(t_s) = \mu_3\sum_{j=0}^N a_{ij}\left(\eta_j(t_s) - \eta_i(t_s)\right)$.

Let

$$z_i = \tilde{\Omega}_i + k_{i1}\hat{e}_i \tag{39}$$

where k_{i1} is a positive real number. Then, (36b) becomes

$$\begin{aligned} J_i\dot{z}_i = &-\Omega_i^\times J_i\Omega_i + J_i\left(\tilde{\Omega}_i^\times C\left(e_i\right)\zeta_i - C\left(e_i\right)ES_i\xi_i\right. \\ &\left.+\frac{1}{2}k_{i1}\left(\hat{e}_i^\times + \bar{e}_iI_3\right)\tilde{\Omega}_i\right) - J_{di}' + u_i \end{aligned} \tag{40}$$

where $J_{di}' = J_{di} - k_{i1}J_i\hat{e}_{di}$.

To handle the uncertain inertia matrix J_i, for $i = 1, \ldots, N$, let J_i be denoted by $J_i = \begin{bmatrix} J_{i11} & J_{i12} & J_{i13} \\ J_{i12} & J_{i22} & J_{i23} \\ J_{i13} & J_{i23} & J_{i33} \end{bmatrix}$, let $\Theta_i = \operatorname{col}(J_{i11}, J_{i22}, J_{i33}, J_{i23}, J_{i13}, J_{i12})$, and, for $x = \operatorname{col}(x_1, x_2, x_3) \in \mathbb{R}^3$, let $L(x) = \begin{bmatrix} x_1 & 0 & 0 & 0 & x_3 & x_2 \\ 0 & x_2 & 0 & x_3 & 0 & x_1 \\ 0 & 0 & x_3 & x_2 & x_1 & 0 \end{bmatrix}$. Then, as noted in [4], for any $x \in \mathbb{R}^3$, it holds that

$$J_i x = L(x)\Theta_i. \tag{41}$$

Using (41), equation (40) can be rewritten as

$$J_i \dot{z}_i = \chi_i \Theta_i - J'_{di} + u_i \tag{42}$$

where

$$\chi_i = -\Omega_i^\times L(\Omega_i) + L\Big(\tilde{\Omega}_i^\times C(e_i)\zeta_i - C(e_i) E S_i \xi_i + \frac{1}{2} k_{i1} \left(\hat{e}_i^\times + \hat{e}_i I_3\right)\tilde{\Omega}_i\Big). \tag{43}$$

As in [4], consider the distributed control law for the ith follower, $i = 1, \ldots, N$, as follows:

$$u_i = -\chi_i \hat{\Theta}_i - k_{i2} z_i \tag{44}$$

$$\dot{\hat{\Theta}}_i = \Lambda_i^{-1} \chi_i^\top z_i \tag{45}$$

where k_{i2} is a positive real number and $\Lambda_i \in \mathbb{R}^{6\times 6}$ is positive definite.

The solvability of Problem 1 is summarized as follows.

Theorem 5.1: Given systems (1), (2), and a digraph $\mathcal{G}$, under Assumptions 1 to 3, Problem 1 is solvable by the distributed control law composed of (13), (44), and (45) with $\bar{T}$ being defined in (32).

Remark 5.1: The success of the proof critically relies on the fact that J_{di} converges to zero exponentially which in turn relies on the exponential convergence property of the sampled-data adaptive distributed observer (13).

VI. Conclusion

In this paper, we have designed a sampled-data adaptive distributed observer for the rigid body leader system and established the exponential convergence property of this sampled-data adaptive distributed observer. By virtue of the exponential convergence property of the sampled-data adaptive distributed observer, we have synthesized a distributed control law to solve the leader-following attitude consensus problem for multiple uncertain rigid body systems.

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