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Constructing a virtual leader for sign consensus of heterogeneous multi-agent systems

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Abstract— **This paper studies output sign consensus problem for leaderless heterogeneous linear multi-agent systems (MASs) over switching signed graphs. Established on the assumption that the communication graph is jointly eventually positive, a distributed 'sign observer' is proposed to estimate a virtual leader. This virtual leader is not pre-determined, but induced from the topology of the graph, the initial conditions of the agents and the structure of the 'observer'. Based on the distributed 'sign observer' and output regulation method, a state feedback controller is designed to drive the output signals of the MAS to have the same sign.**

I. Introduction

Distributed control of multi-agent systems (MASs) has drawn great interest in the past two decades, and most of the works focus on MASs over cooperative networks [1]. In other words, the communication graphs of the MASs are non-negative [2].

However, in various practical scenarios, such as social networks [3], antagonistic relationships coexist with cooperative ones. Under such circumstances, signed graphs better characterize the interactions among the agents, in which cooperative links are represented by positive edges, and antagonistic ones are represented by negative edges.

Since the adjacency matrix of a signed graph ceases to be non-negative, how MASs behave over signed communication graphs needs further investigation. A pioneering work [4] uncovered that over certain signed graphs, specifically the structurally balanced graphs [5], all the agents are divided into two antagonistically related subgroups, where agents within the same subgroup are cooperatively related. The output signals of these two subgroups can be controlled to have the same magnitudes but opposite signs, and such behavior is called bipartite consensus. Studies on bipartite consensus include but are not limited to linear [6] and nonlinear [7] MASs.

In practice, being structurally balanced is a rather restrictive condition to be satisfied, since a simple alteration of a few links' sign would break such balance. Further investigations into structurally unbalanced graphs, particularly the eventually positive graphs [8], unveiled another collective bahvior —unanimity of opinoins [9]. Since the output signals of the MAS that achieves such unanimity eventually have the same sign but different magnitudes, this behavior is also referred to as sign consensus [10], [11]. This synchronous behavior has huge potential for applciations in the area of opinion dynamics [9] and social networks [12]. It is worth mentioning that in [10] and [16], the considered MAS is assumed to be homogeneous, and in [11] and [17], the authors utilized a leader-follower scheme to achieve output sign consensus of a heterogeneous MAS. Indeed, the dynamics of the agents within certain MASs are nonidentical, and the leader-follower scheme is a very common method used to deal with heterogeneous MASs.

Nonetheless, the existence of an explicit leadership is sometimes harmful for the MAS. For example, adopting a leader-follower control scheme for space-robot teams could be detrimental if the leader has a failure, whereas a leaderless architecture that only utilizes the local information of each agent's neighbors and the target point could avoid such situation [13]. Therefore, how to deal with leaderless heterogeneous MAS is an interesting problem and needs further study. Our recent work [14] shows that using a properly designed distributed observer and a controller, output sign consensus of a leaderless heterogeneous MAS can be achieved. However, this work requires the communication graph to be of fixed topology.

Moreover, the communication network of some MASs could be time-varying. For instance, in an unmanned aerial vehicle swarm system, the interaction between two vehicles may be interrupted or created due to communication range constraints [15]. Hence, in this paper, we consider a switching communication graph.

Inspired by the abovementioned motivations, we aim to achieve sign consensus of leaderless heterogeneous MASs over switching communication networks. To do so, we utilize a distributed 'sign observer' to estimate a virtual leader. This virtual leader is not pre-specified but completely arises from the topology of the graph, the initial conditions of the agents and the structure of the 'observer'. Then based on output regulation theory, we design a state feedback controller to drive the output signals of all agents to reach sign consensus.

The rest of this paper is organized as follows: Section II introduces some preliminary results and formulates the problem; Section III designs the distributed 'sign observer' and establishes the virtual leader; Section IV proposes a state feedback controller; Section V illustrates the effectiveness of the designed controller with a simulation example and Section VI concludes this paper.

Notations: $\|\cdot\|$ denotes the Euclidian norm. The

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Kronecker product is represented by $\otimes$. For matrices $X_i \in \mathbb{R}^{m\times n}$, $i = 1, 2, \cdots, N$, $\text{col}(X_1, X_2, \cdots, X_N) = [X_1^T, X_2^T, \cdots, X_N^T]^T \in \mathbb{R}^{Nm\times n}$. For a matrix $X \in \mathbb{R}^{m\times n}$, $\text{vec}(X) = \text{col}(X_1, X_2, \cdots, X_n)$, where $X_i \in \mathbb{R}^m$ is the ith column of matrix X. $A \succ 0$ means matrix $A \in \mathbb{R}^{n\times n}$ is positive entrywise, i.e. $a_{ij} > 0$ for all i, j. The notation $v \succ 0$ is defined similarly for a vector $v \in \mathbb{R}^n$. Expression $\Lambda = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N)$ means Λ is a diagonal matrix with diagonal entries λ_i. The spectral radius of a matrix $A \in \mathbb{R}^{n\times n}$ is denoted by $\rho(A)$. The ith eigenvalue of matrix A is $\lambda_i(A)$. The signum function for a vector $x = \text{col}(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ is given as $\text{sgn}(x) = \text{col}(\text{sgn}(x_1), \text{sgn}(x_2), \cdots, \text{sgn}(x_n))$. Function $M_n^p(\cdot) : \mathbb{R}^{pn} \to \mathbb{R}^{n\times p}$, transforms a vector $x = \text{col}(x_1, x_2, \cdots, x_p) \in \mathbb{R}^{pn}$, $x_i \in \mathbb{R}^n$, into a matrix $[x_1, x_2, \cdots, x_p] \in \mathbb{R}^{n\times p}$.

II. Preliminaries & Problem Formulation

A. Graph Theory

The communication network of a MAS is described by a directed graph, where the agents are represented by the nodes, and the interaction between two agents is represented by an edge between the two nodes. A directed graph is denoted by $\mathcal{G} = (\Omega, \mathcal{E})$, where $\Omega = \{1, 2, \cdots, N\}$ is the set of all the nodes, and $\mathcal{E} \subset \Omega \times \Omega$ is the set of all the edges. The adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ captures the topology of a graph, and $a_{ij} \neq 0$ if and only if $(j, i) \in \mathcal{E}$, which indicates that the ith node receives information from the jth node, and these two nodes are neighbors; $a_{ij} = 0$ elsewise. In this paper, no self-loop exists in the graph, i.e. $a_{ii} = 0$, $\forall i \in \Omega$. Graph $\mathcal{G}(\mathcal{A})$ is non-negative if and only if for all i and j, $a_{ij} \geq 0$; otherwise $\mathcal{G}(\mathcal{A})$ is a signed graph.

For switching graphs, there is a sequence of non-overlapping and bounded switching time intervals $[t_k, t_{k+1}), k \in \{0, 1, 2, \cdots\}$, such that $t_0 = 0$ and $\lim_{k\to\infty} t_k = \infty$. We assume that there exists a constant T such that $t_{k+1} - t_k \leq T$ for all k. Each time interval is further divided into m_k sub-intervals $[t_{k_0}, t_{k_1}), [t_{k_1}, t_{k_2}), \cdots, [t_{k_{m_k-1}}, t_{k_{m_k}})$, where $t_{k_0} = t_k$ and $t_{k_{m_k}} = t_{k+1}$. The topology of a switching graph $\mathcal{G}(\mathcal{A}_{\delta(t)})$ is assumed to be fixed in each sub-interval, where $\delta(t) : [t_k, t_{k+1}) \to \{k_0, k_1, \cdots, k_{m_{k-1}}\}$ is the switching signal.

B. Problem Formulation

Consider a heterogeneous MAS over a switching signed graph $\mathcal{G}(\mathcal{A}_{\delta(t)})$ with the following dynamics:

$$\begin{aligned} \dot{x}_i &= A_i x_i + B_i u_i \\ y_i &= C_i x_i, \quad i = 1, 2, \cdots, N, \end{aligned} \tag{1}$$

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$ and $y_i \in \mathbb{R}^p$ are the states, inputs and outputs, respectively. We assume that (A_i, B_i, C_i) is controllable and observable for all i. The goal of this paper is to achieve sign consensus of a leaderless MAS (1), and the problem is defined below.

Definition 1 (Leaderless Output Sign Consensus): [16] Consider the heterogeneous MAS (1) over a signed graph $\mathcal{G}(\mathcal{A}_{\delta(t)})$. Leaderless output sign consensus is reached if for some nontrivial trajectory $z(t) \in \mathbb{R}^p$:

$$\lim_{t\to\infty} [\text{sgn}(y_{i_k}(t)) - \text{sgn}(z_k(t))] = 0, \ \forall k \in K_1;$$
$$\lim_{t\to\infty} (y_{i_k}(t) - z_k(t)) = 0, \ \forall k \in K_2;$$

where $z_k(t)$ and $y_{i_k}(t)$ are the kth entries of $z(t)$ and $y_i(t)$, respectively, $K_1 = \{k \mid \lim_{t\to\infty} z_k(t) \neq 0,\ k = 1, 2, \ldots, p\}$, $K_2 = \{k \mid \lim_{t\to\infty} z_k(t) = 0,\ k = 1, 2, \ldots, p\}$, $K_1 \cup K_2 = \{1, 2, \ldots, p\}$ and $K_1 \cap K_2 = \emptyset$.

C. Preliminary Results

The following preliminary results are crucial for solving the leaderless output sign consensus problem.

Definition 2 (Eventually Positive): [8] A matrix $A \in \mathbb{R}^{n\times n}$ is *eventually positive*, if there exists a positive integer k_0 such that for all positive integer $k \geq k_0$, $A^k \succ 0$.

Definition 3 (Jointly Eventually Positive): [17] A switching signed graph $\mathcal{G}(\mathcal{A}_{\delta(t)})$ is *jointly eventually positive*, if there exists a sequence of bounded and non-overlapping time intervals $[t_k, t_{k+1}), k \in \{0, 1, 2, \cdots\}$, where $t_0 = 0$ and $\lim_{k\to\infty} t_k = \infty$, such that

$$\bar{\mathcal{A}} = \int_{t_k}^{t_{k+1}} \mathcal{A}_{\delta(t)}\, dt$$

is eventually positive for all k.

Definition 4 (Perron-Frobenius Property): [18] A matrix $A \in \mathbb{R}^{n\times n}$ is said to possess *strong Perron-Frobenius property* if its spectral radius $\rho(A)$ is a simple eigenvalue, and the corresponding right eigenvector $v_r \succ 0$.

Proposition 1: [18] For a matrix $A \in \mathbb{R}^{n\times n}$, the following statements are equivalent:

1) Both A and A^T have strong Perron-Frobenius property;
2) A is eventually positive;
3) A^T is eventually positive.

Lemma 1: [16] For any matrix $A \in \mathbb{R}^{n\times n}$ and a vector $v = [v_1, \cdots, v_n]^T \succ 0$, there exists a diagonal matrix $\Sigma = \text{diag}(\sigma_1, \cdots, \sigma_n) \in \mathbb{R}^{n\times n}$ such that $(\Sigma - A)v = 0$, and matrix $L = \Sigma - A$ can be decomposed as $L = ME$ where $M \in \mathbb{R}^{n\times(n-1)}$ and

$$E = \begin{bmatrix} \frac{1}{v_1} & -\frac{1}{v_2} & 0 & \cdots & 0 \\ 0 & \frac{1}{v_2} & -\frac{1}{v_3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{v_{n-1}} & -\frac{1}{v_n} \end{bmatrix}. \tag{2}$$

If A is eventually positive and v is the eigenvector corresponding to $\rho(A)$, then M is of full column rank and $\text{Re}(\lambda_i(EM)) > 0$, $\forall i$.

The following lemma facilitates the development of the distributed 'sign observer'.

Lemma 2: Consider the following system:

$$\dot{x} = F(t)x + G(t),$$

where $x \in \mathbb{R}^n$, $F(\cdot) : \mathbb{R} \to \mathbb{R}^{n\times n}$, $G(\cdot) : \mathbb{R} \to \mathbb{R}^n$, and $F(t)$ and $G(t)$ are bounded and piecewise continuous. Then

x converges to a bounded vector, if both $F(t)$ and $G(t)$ vanish exponentially.

Proof: Consider a Lyapunov function candidate $V = x^T x$. Take the time derivative of V as:

$$\dot{V} = x^T \left(F^T(t) + F(t)\right) x + 2G^T(t)x.$$

Since both $F(t)$ and $G(t)$ decay to 0 exponentially, there exist positive constants α, β, λ and γ such that $\|F(t)\| \le \alpha e^{-\lambda t}$ and $\|G(t)\| \le \beta e^{-\gamma t}$. Notice that $G^T(t)x \le \|G(t)\|\|x(t)\|$. Then we have

$$\dot{V} \le 2\alpha e^{-\lambda t}\|x\|^2 + 2\beta e^{-\gamma t}\|x\|.$$

Let $W = \sqrt{V} = \|x(t)\|$. Taking the time derivative of W gives

$$\dot{W} = \frac{\dot{V}}{2\sqrt{V}} \le \alpha e^{-\lambda t} W + \beta e^{-\gamma t}$$

Then $\forall t \ge 0$,

$$\begin{aligned} W &\le e^{\int_0^t \alpha e^{-\lambda \tau}\, d\tau}\|x(0)\| + \int_0^t e^{\int_0^\tau \alpha e^{-\lambda s}\, ds} \beta e^{-\gamma \tau}\, d\tau \\ &\le \frac{\alpha}{\lambda}\|x(0)\| + \frac{\beta}{\gamma} e^{\frac{\alpha}{\lambda}}, \end{aligned}$$

which implies that $\|x(t)\|$ eventually converges to a bounded vector for all $x(0)$ and $t \ge 0$. ∎

The communication graph of the MAS satisfies the following assumption.

Assumption 1: The switching graph $\mathcal{G}(\mathcal{A}_{\delta(t)})$ is assumed to be jointly eventually positive in any time interval $[t_k, t_{k+1})$.

III. Distributed 'Sign Observer' Design and Virtual Leader Construction

In this section, we propose a distributed 'sign observer' for heterogeneous MASs over switching topologies, which estimates the system matrix and the state of a virtual leader.

A. Distributed 'Sign Observer' Design

From Assumption 1 and Proposition 1, we have a vector $v_r \succ 0$ such that $\bar{A} v_r = \rho(\bar{A}) v_r$, where $\bar{A}$ is defined as in Definition 3. Then for node i, a distributed 'sign observer' is designed as follows:

$$\dot{S}_i = \mu_1 \left(-\sigma_i(t) S_i + \sum_{j=1}^N a_{ij}(t) S_j \right), \tag{3a}$$

$$\dot{\zeta}_i = \frac{1}{v_{r_i}} S_i \zeta_i + \mu_2 \left(-\sigma_i(t)\zeta_i + \sum_{j=1}^N a_{ij}(t)\zeta_j \right), \tag{3b}$$

where $S_i \in \mathbb{R}^{p\times p}, \zeta_i \in \mathbb{R}^p$, and $\sigma_i(t)$ is a time-varying parameter designed as $\sigma_i(t) = \sum_{j=1}^N a_{ij}(t) v_{r_j}/v_{r_i}$.'Observer' S_i and ζ_i estimate the system matrix and the state of the virtual leader, respectively. A more detailed description of this 'sign observer' is given in Section III B.

The following lemma shows that 'sign observer' (3a) reaches sign consensus:

Lemma 3: Consider 'sign observer' (3a). Under Assumption 1, for any scalar $\mu_1 > 0$ and arbitrary initial conditions $S_i(0)$, $S_i(t)$ achieves sign consensus.

Proof: Let $\Sigma_{\delta(t)} = \operatorname{diag}(\sigma_1(t), \cdots, \sigma_N(t))$ and $L_{\delta(t)} = \Sigma_{\delta(t)} - \mathcal{A}_{\delta(t)}$. With $\sigma_i(t) = \sum_{j=1}^N a_{ij}(t) v_{r_j}/v_{r_i}$, straightforward computation gives

$$L_{\delta(t)} v_r = \left(\Sigma_{\delta(t)} - \mathcal{A}_{\delta(t)}\right) v_r = 0. \tag{4}$$

Note that for matrix $\bar{\mathcal{A}}$, there exists a constant nonsingular matrix U such that $U^{-1}\bar{\mathcal{A}}U = \mathcal{J}_{\bar{\mathcal{A}}}$. Let $U = [v_r \;\; v_{N-1}]$, $U^{-1} = \begin{bmatrix} u_l \\ u_{N-1} \end{bmatrix}$ and $S = \operatorname{col}(S_1, \cdots, S_N)$. Introduce transformation $\Phi = (U^{-1} \otimes I_p) S$. Taking the time derivative of Φ gives

$$\dot{\Phi} = -\mu_1 \left(\begin{bmatrix} 0 & l_{\delta(t)} \\ 0_{(N-1)\times 1} & L^*_{\delta(t)} \end{bmatrix} \otimes I_p \right) \Phi, \tag{5}$$

where $l_{\delta(t)} = u_l L_{\delta(t)} v_{N-1}$ and $L^*_{\delta(t)} = u_{N-1} L_{\delta(t)} v_{N-1}$. By integrating $U^{-1} L_{\delta(t)} U$ within time interval $[t_k, t_{k+1})$, one obtains

$$\int_{t_k}^{t_{k+1}} U^{-1} L_{\delta(t)} U \, dt = \begin{bmatrix} 0 & \int_{t_k}^{t_{k+1}} l_{\delta(t)}\, dt \\ 0_{(N-1)\times 1} & \int_{t_k}^{t_{k+1}} L^*_{\delta(t)}\, dt \end{bmatrix}. \tag{6}$$

On the other hand, (4) implies $\int_{t_k}^{t_{k+1}} (\Sigma_{\delta(t)} - \mathcal{A}_{\delta(t)}) v_r \, dt = 0$. It is trivial to show that

$$\int_{t_k}^{t_{k+1}} \Sigma_{\delta(t)} v_r \, dt = \int_{t_k}^{t_{k+1}} \mathcal{A}_{\delta(t)} v_r \, dt = \bar{\mathcal{A}} v_r = \rho(\bar{\mathcal{A}}) v_r,$$

which indicates that $\int_{t_k}^{t_{k+1}} \Sigma_{\delta(t)}\, dt = \rho(\bar{\mathcal{A}}) I_N$. Without loss of generality, let $\mathcal{J}_{\bar{\mathcal{A}}} = \operatorname{diag}\left(\rho(\bar{\mathcal{A}}), \mathcal{J}_{\bar{\mathcal{A}}_{N-1}}\right)$. Then

$$\begin{aligned} &\int_{t_k}^{t_{k+1}} U^{-1} L_{\delta(t)} U \, dt \\ &\qquad = \begin{bmatrix} 0 & 0_{1\times(N-1)} \\ 0_{(N-1)\times 1} & \rho(\bar{\mathcal{A}}) I_{N-1} - \mathcal{J}_{\bar{\mathcal{A}}_{N-1}} \end{bmatrix}. \end{aligned} \tag{7}$$

A comparison of (6) and (7) shows that $\int_{t_k}^{t_{k+1}} L^*_{\delta(t)}\, dt = \rho(\bar{\mathcal{A}}) I_{N-1} - \mathcal{J}_{\bar{\mathcal{A}}_{N-1}}$. Let $\Phi = \operatorname{col}(\Phi_1, \Phi_{N-1})$. Splitting dynamics (5) gives

$$\dot{\Phi}_1 = -\mu_1 \left(l_{\delta(t)} \otimes I_p\right) \Phi_{N-1}, \tag{8a}$$

$$\dot{\Phi}_{N-1} = -\mu_1 \left(L^*_{\delta(t)} \otimes I_p\right) \Phi_{N-1}. \tag{8b}$$

Now we consider the average system of (8b) within time interval $[t_k, t_{k+1})$ as follows:

$$\begin{aligned} \dot{\overline{\Phi}}_{N-1} &= -\mu_1 \frac{\int_{t_k}^{t_{k+1}} \left(L^*_{\delta(t)} \otimes I_p\right) dt}{t_{k+1} - t_k} \overline{\Phi}_{N-1} \\ &= -\mu_1 \frac{\rho(\bar{\mathcal{A}}) I_{N-1} - \mathcal{J}_{\bar{\mathcal{A}}_{N-1}}}{t_{k+1} - t_k} \overline{\Phi}_{N-1}. \end{aligned} \tag{9}$$

Since $\rho(\bar{\mathcal{A}})$ is the spectral radius of matrix $\bar{\mathcal{A}}$, we have that $\forall i \ge 2$, $\operatorname{Re}\left(\rho(\bar{\mathcal{A}}) - \lambda_i(\bar{\mathcal{A}})\right) > 0$. Thus, the average system (9) is exponentially stable for any positive μ_1. Then by Theorem 2 and equation (30) in [19], if T is small enough, we could select $\alpha^* < 1$, and further $\alpha > \alpha^*$ to be one, which implies that the original system (8b) is exponentially stable, i.e. $\Phi_{N-1} \to 0$ exponentially as $t \to \infty$. Since $\Phi = (U^{-1} \otimes I_p)S$, we have

$$\lim_{t\to\infty} S(t) = \lim_{t\to\infty} (v_r \otimes \Phi_1).$$

From Assumption 1, we have that $\mathcal{A}_{\delta(t)}$ is bounded and so is $L_{\delta(t)}$. Then with (8a), where $l_{\delta(t)} = u_l L_{\delta(t)} v_{N-1}$, we have $\dot{\Phi}_1 \to 0$ as $t \to \infty$. Hence, Φ_1 evetually converges to a constant matrix, denoted as S^*, and

$$\lim_{t\to\infty} S_i(t) = v_{r_i} S^*.$$

Since $v_r \succ 0$, 'observer' (3a) achieves sign consensus. ∎

From the development of Lemma 3, we have

$$\begin{aligned} S^* &= \lim_{t\to\infty} \Phi_1 \\ &= \Phi_1(0) + \lim_{t\to\infty} \int_0^t -\mu_1 (l_{\delta(\tau)} \otimes I_p) \Phi_{N-1}(t)\, d\tau. \end{aligned}$$

Since $l_{\delta(t)}$ is bounded and $\Phi_{N-1} \to 0$ exponentially as $t \to \infty$, there exist positive constants M, a and b such that $\|l_{\delta(t)}\| \leq M$ and $\|\Phi_{N-1}\| \leq a e^{-bt}$. Then we obtain

$$S^* \leq \Phi_1(0) + \frac{\mu_1 M a}{b}. \tag{10}$$

Denote the first row of U^{-1} as $u^T = \mathrm{col}(u_1, u_2, \cdots, u_N)$. With $\Phi = (U^{-1} \otimes I_p)S$, we have $\Phi_1(0) = \sum_{i=1}^N u_i S_i(0)$. Then from (10), we can see that S^* depends on the initial conditions of the 'observer' (3a), the topology of the graph and the nature of the 'observer' itself.

Next, we show sign consensus of 'sign observer' (3b).

Lemma 4: Consider 'sign observer' (3b). Under Assumption 1, if $\mu_1 > 2\|S^*\|/(\rho(\bar{\mathcal{A}}) - \lambda_1)$, where λ_1 is the second largest real part of $\bar{\mathcal{A}}$'s eigenvalues, and there exist positive constants T, S_M, ω and a symmetric positive definite matrix H, such that $\mu_2 > (1 + T^2 S_M^2 \|H\|^2)/w$, then for arbitrary initial condition $\zeta_i(0)$, $\zeta_i(t)$ achieves sign consensus.

Proof: Let $\zeta = \mathrm{col}(\zeta_1, \zeta_2, \cdots, \zeta_N)$ and $S_d = \mathrm{diag}\,(S_1/v_{r_1}, \cdots, S_N/v_{r_N})$. Introduce transformation $\hat{\zeta} = P(t)\zeta$, where $P(t) = e^{Qt}$ and $Q = -I_N \otimes S^*$. Take the time derivative of $\hat{\zeta}$ as

$$\dot{\hat{\zeta}} = e^{Qt} (S_d - I_N \otimes S^*) e^{-Qt} \hat{\zeta} - \mu_2 \left(L_{\delta(t)} \otimes I_p\right) \hat{\zeta}.$$

From Lemma 3, we have $S_i(t) \to S^*$ exponentially at the rate of $\mu_1 \left(\rho(\bar{\mathcal{A}}) - \lambda_1\right)$. It is trivial to show that both $\|e^{Qt}\|$ and $\|e^{-Qt}\|$ are less than $e^{\|S^*\|t}$. Define $\tilde{S}_d = e^{Qt} (S_d - I_N \otimes S^*) e^{-Qt} = \mathrm{diag}\left(\tilde{S}_{d_1}, \cdots, \tilde{S}_{d_N}\right)$. It can be verified that $\tilde{S}_d \to 0$ exponentially as $t \to \infty$ if $\mu_1 > \frac{2\|S^*\|}{\rho(\bar{\mathcal{A}}) - \lambda_1}$, where λ_1 is the maximum real part of $\bar{\mathcal{A}}$'s eigenvalues other than $\rho(\bar{\mathcal{A}})$.

Let $e_{\hat{\zeta}} = (E \otimes I_p)\hat{\zeta}$, where matrix E is defined as in (2). Along with Lemma 1, we obtain

$$\dot{e}_{\hat{\zeta}} = (E \otimes I_p)\tilde{S}_d \hat{\zeta} - \mu_2 (E M_{\delta(t)} \otimes I_p) e_{\hat{\zeta}}. \tag{11}$$

Consider the Lyapunov function candidate $V = e_{\hat{\zeta}}^T (H \otimes I_p) e_{\hat{\zeta}}$, where H is a symmetric positive definite matrix which will be specified later. It is obvious that this selection of V satisfies condition 1 of Theorem 2 in [19]. Manipulate V as follows:

$$\begin{aligned} &\frac{\partial V}{\partial e_{\hat{\zeta}}} \left(e_{\hat{\zeta}}\right) \int_{t_k}^{t_{k+1}} \dot{e}_{\hat{\zeta}}(t)\, dt \\ &\quad = 2 e_{\hat{\zeta}}^T (H \otimes I_p) \int_{t_k}^{t_{k+1}} (E \otimes I_p) \tilde{S}_d \hat{\zeta}\, dt \\ &\quad\quad - 2\mu_2 e_{\hat{\zeta}}^T (H \otimes I_p) \left[\int_{t_k}^{t_{k+1}} \left(E M_{\delta(t)} \otimes I_p\right) dt\right] e_{\hat{\zeta}}. \end{aligned}$$

Let $V_1 = 2 e_{\hat{\zeta}}^T (H \otimes I_p) \int_{t_k}^{t_{k+1}} (E \otimes I_p) \tilde{S}_d \hat{\zeta}\, dt$ and $V_2 = -2\mu_2 e_{\hat{\zeta}}^T (H \otimes I_p) \left[\int_{t_k}^{t_{k+1}} \left(E M_{\delta(t)} \otimes I_p\right) dt\right] e_{\hat{\zeta}}$. Since $\tilde{S}_d \to 0$ exponentially as $t \to \infty$, there exists $S_M > 0$ such that $\|\tilde{S}_d\| \leq S_M$. Then

$$\begin{aligned} V_1 &\leq e_{\hat{\zeta}}^T e_{\hat{\zeta}} + S_M^2 \|H\|^2 \left(\int_{t_k}^{t_{k+1}} e_{\hat{\zeta}}\, dt\right)^T \left(\int_{t_k}^{t_{k+1}} e_{\hat{\zeta}}\, dt\right) \\ &= \left(1 + T^2 S_M^2 \|H\|^2\right) \|e_{\hat{\zeta}}\|^2. \end{aligned}$$

Notice that $\bar{L} = \int_{t_k}^{t_{k+1}} (\Sigma_{\delta(t)} - \mathcal{A}_{\delta(t)})\, dt = \rho(\bar{\mathcal{A}}) I_N - \bar{\mathcal{A}}$, and it can be decomposed as $\bar{L} = \int_{t_k}^{t_{k+1}} L_{\delta(t)}\, dt = \int_{t_k}^{t_{k+1}} M_{\delta(t)} E\, dt = \bar{M} E$. Assumption 1 ensures that $\bar{\mathcal{A}}$ is eventually positive. Then from Lemma 1, we know that $\mathrm{Re}(\lambda_i(E\bar{M})) > 0$ for all i; thus, $\int_{t_k}^{t_{k+1}} \left(E M_{\delta(t)} \otimes I_p\right) dt$ is Hurwitz and $\exists H > 0$ such that

$$\begin{aligned} &\left[\int_{t_k}^{t_{k+1}} \left(E M_{\delta(t)} \otimes I_p\right) dt\right]^T (H \otimes I_p) \\ &\quad + (H \otimes I_p) \left[\int_{t_k}^{t_{k+1}} \left(E M_{\delta(t)} \otimes I_p\right) dt\right] \\ &\quad\quad \leq -w I_{(N-1)p \times (N-1)p}, \end{aligned}$$

for some positive scalar w, which allows $V_2 \leq -\mu_2 w \|e_{\hat{\zeta}}\|^2$. Then we have $\frac{\partial V}{\partial e_{\hat{\zeta}}} \left(e_{\hat{\zeta}}\right) \int_{t_k}^{t_{k+1}} \dot{e}_{\hat{\zeta}}(t)\, dt \leq -w^* \|e_{\hat{\zeta}}\|^2$ for some positive scalar w^* as long as $\mu_2 > \frac{1 + T^2 S_M^2 \|H\|^2}{w}$, which satisfies condition 2 of Theorem 2 in [19]. Thus, $e_{\hat{\zeta}} \to 0$ exponentially as $t \to \infty$.

Next, we study the boundedness of $\hat{\zeta}$. Notice that

$$(L_{\delta(t)} \otimes I_p)\hat{\zeta} = (M_{\delta(t)} \otimes I_p) e_{\hat{\zeta}}.$$

Since we already have that $e_{\hat{\zeta}} \to 0$ exponentially as $t \to \infty$, $(L_{\delta(t)} \otimes I_p)\hat{\zeta} \to 0$ exponentially as $t \to \infty$. Also, $\tilde{S}_d \to 0$ exponentially with $\mu_1 > \frac{2\|S^*\|}{\rho(\bar{\mathcal{A}}) - \lambda_1}$. Then, by Lemma 2, $\hat{\zeta}$ eventually converges to a bounded vector.

Now that $e_{\hat{\zeta}}$ decays to 0, and $\hat{\zeta}$ eventually converges to a bounded vector. There exists a bounded vector $\hat{\zeta}^* \in \mathbb{R}^p$ such that $\lim_{t\to\infty} \hat{\zeta} = v_r \otimes \hat{\zeta}^*$. Thus, according to the definition of $\hat{\zeta}$, we have $\lim_{t\to\infty} \zeta_i = v_{r_i} e^{S^* t} \hat{\zeta}^*$. In other words, sign consensus of ζ is achieved. ∎

B. Virtual Leader Construction

By Lemma 4, we have that $\zeta_i(t) \to v_{r_i} e^{S^* t} \hat{\zeta}^*$ as $t \to \infty$. In analogy to the solution of a general linear system $\dot{x} = Ax$, which takes the form of $x(t) = e^{At}x(0)$, the virtual leader is given as

$$\dot{\eta} = S^* \eta, \tag{12}$$

where $\eta \in \mathbb{R}^p$ and $S^* \in \mathbb{R}^{p\times p}$ are the state and system matrix, respectively. From Lemma 3 and Lemma 4, we know that the value of S^* depends on the initial conditions of each agent, the topology of the switching graph and the nature of 'observer' (3a); besides, the initial condition of η is given as $\hat{\zeta}^*$. Each 'observer' (3a) and (3b) is able to estimate the system matrix and the state of a virtual leader, respectively.

IV. Controller Design and Analysis

Based on the 'sign observer' designed in the previous section, we propose a state feedback controller, which drives the output signals of a leaderless heterogeneous MAS to reach sign consensus. Before proceeding, the following technical lemma is presented.

Lemma 5: [11] Consider the following systems

$$\dot{\theta} = -\mu_3 P_i^T(t)\left[P_i(t)\theta_i - b_i\right], \quad i = 1, \cdots, N, \tag{13}$$

where

$$b_i = \text{vec}\left(\begin{bmatrix} 0_{n_i\times p} \\ -I_p \end{bmatrix}\right)$$

and

$$P_i(t) = S_i^T(t) \otimes \begin{bmatrix} I_{n_i} & 0 \\ 0 & 0 \end{bmatrix} - I_p \otimes \begin{bmatrix} v_{r_i}A_i & B_i \\ C_i & 0 \end{bmatrix}$$

and μ_3 is sufficiently large. Let

$$P_i = v_{r_i}(S^*)^T \otimes \begin{bmatrix} I_{n_i} & 0 \\ 0 & 0 \end{bmatrix} - I_p \otimes \begin{bmatrix} v_{r_i}A_i & B_i \\ C_i & 0 \end{bmatrix}.$$

If rank(P_i) =rank($[P_i, b_i]$), and the following regulator equations

$$\begin{aligned} v_{r_i}\Pi_i S^* &= v_{r_i}A_i\Pi_i + B_i\Gamma_i \\ C_i\Pi_i &= I_p, \end{aligned} \tag{14}$$

are solvable, then for any initial conditions $\theta_i(t_0)$, $\theta_i(t)$ exists and has a unique solution. With the following transformation

$$\Theta_i(t) = M_{n_i+m_i}^p(\theta_i(t)) = \text{col}\left(\Theta_{1i}(t), \Theta_{2i}(t)\right), \tag{15}$$

we have that $\Theta_i(t)$ converges to col(Π_i, Γ_i) exponentially as $t \to \infty$ with $\mu_3 > \frac{\omega}{\kappa}$, where ω is the rate at which $S_i(t)$ converges to $v_{r_i}S^*$ and κ is the minimal nonzero singular value of P_i.

With Lemma 5, we propose the following state feedback controller

$$u_i = K_i x_i + K_{\zeta_i}(t)\zeta, \tag{16a}$$

$$K_{\zeta_i}(t) = \frac{1}{v_{r_i}}\Theta_{2i}(t) - K_i\Theta_{1i}(t), \tag{16b}$$

where K_i is chosen such that $A_i + B_iK_i$ is Hurwitz, Θ_{1i} and Θ_{2i} are generated by (13) and (15). The following theorem shows that the designed state feedback controller achieves leaderless sign consensus of system (1).

Theorem 1: Consider MAS (1). Suppose Assumption 1 holds. For large enough positive scalar μ_1, μ_2 and μ_3, leaderless output sign consensus is reached by control law (3), (13) and (16).

Proof: Consider (1) and (12), and let $\tilde{x}_i = x_i - v_{r_i}\Pi_i\eta_i$, $\tilde{u}_i = u_i - \Gamma_i\eta$, $e_i = y_i - v_{r_i}\eta$, $\epsilon_i = \zeta_i - v_{r_i}\eta$, $K_{\zeta_i} = \frac{1}{v_{r_i}}\Gamma_i - K_i\Pi_i$, $\tilde{K}_{\zeta_i}(t) = K_{\zeta_i}(t) - K_{\zeta_i}$. Straightforward computation gives

$$\begin{aligned} \dot{\tilde{x}}_i &= A_i\tilde{x}_i + B_i\tilde{u}_i, \\ \tilde{u}_i &= K_i\tilde{x}_i + K_{\zeta_i}(t)\epsilon_i + v_{r_i}\tilde{K}_{\zeta_i}(t)\eta, \end{aligned}$$

which leads to

$$\dot{\tilde{x}}_i = (A_i + B_iK_i)\tilde{x}_i + B_iK_{\zeta_i}(t)\epsilon_i + v_{ri}B_i\tilde{K}_{\zeta_i}(t)\eta. \tag{17}$$

By Lemma 4, we have $\epsilon_i \to 0$ as $t \to \infty$ if μ_1 and μ_2 are large enough. Also, Lemma 5 enables $\tilde{K}_{\zeta_i}(t)$ to vanish as $t \to \infty$ if μ_3 is sufficiently large. With $A_i + B_iK_i$ being Hurwitz, system (17) is input-to-state stable with input $B_iK_{\zeta_i}(t)\epsilon_i + v_{ri}B_i\tilde{K}_{\zeta_i}(t)\eta$. Thus, $\tilde{x}_i(t) \to 0$ as $t \to \infty$, and the error

$$e_i = C_i\tilde{x}_i,$$

decays to 0 as well. In other words, $y_i \to v_{r_i}\eta$ as $t \to \infty$. Since $v_r \succ 0$ by Proposition 1, along with Definition 1 we have

$$\lim_{t\to\infty}\left[\text{sgn}\left(y_{i_k}(t)\right) - \text{sgn}\left(\eta_k(t)\right)\right] = 0,\ k \in K_1;$$
$$\lim_{t\to\infty}\left(y_{i_k}(t) - \eta_k(t)\right) = 0,\ k \in K_2.$$

Therefore, leaderless output sign consensus of MAS (1) is reached by control law (3), (13) and (16). ■

V. Simulation Example

We consider a MAS with 6 agents, whose dynamics are specified as follows:

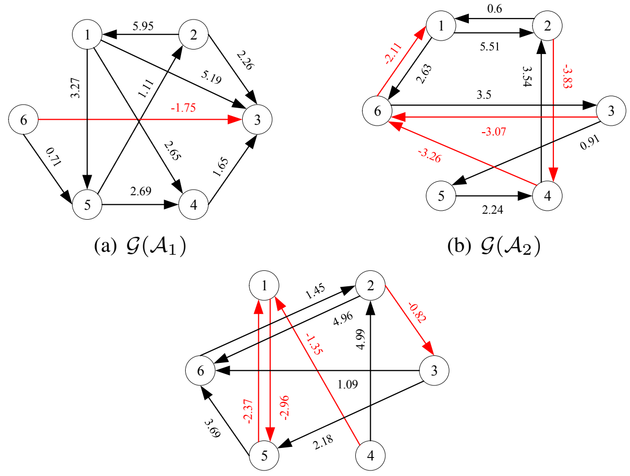
$$A_i = \begin{bmatrix} 0 & 4.1 \\ -17.9 & 0.2 \end{bmatrix}, B_i = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, C_i = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, i = 1, 2, 3;$$
$$A_i = \begin{bmatrix} 0.2 & -17.9 \\ 4.1 & 0 \end{bmatrix}, B_i = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, C_i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, i = 4, 5, 6.$$

The initial conditions of 'sign observer' (3a) take the following values:

$$S_1(0) = \begin{bmatrix} 0 & 4 \\ -2 & 0.02 \end{bmatrix},\ S_2(0) = \begin{bmatrix} -0.01 & 6 \\ -3 & 0.01 \end{bmatrix},$$
$$S_3(0) = \begin{bmatrix} 0 & 4 \\ -2 & 0.02 \end{bmatrix},\ S_4(0) = \begin{bmatrix} -0.01 & 5 \\ -4 & 0.01 \end{bmatrix},$$
$$S_5(0) = \begin{bmatrix} 0 & 6 \\ -1 & 0.02 \end{bmatrix},\ S_6(0) = \begin{bmatrix} -0.02 & 6 \\ -2.5 & 0.01 \end{bmatrix}.$$

The communication graph of the MAS switches among the three signed graphs shown in Fig. 1 with the switching signal:

$$\delta(t) = \begin{cases} 1, & kT \le t < kT + 5\times 10^{-4}, \\ 2, & kT + 5\times 10^{-4} \le t < kT + 8\times 10^{-4}, \\ 3, & kT + 8\times 10^{-4} \le t < (k+1)T, \end{cases}$$



(a) $\mathcal{G}(\mathcal{A}_1)$ (b) $\mathcal{G}(\mathcal{A}_2)$

(c) $\mathcal{G}(\mathcal{A}_3)$

Fig. 1. Communication graphs

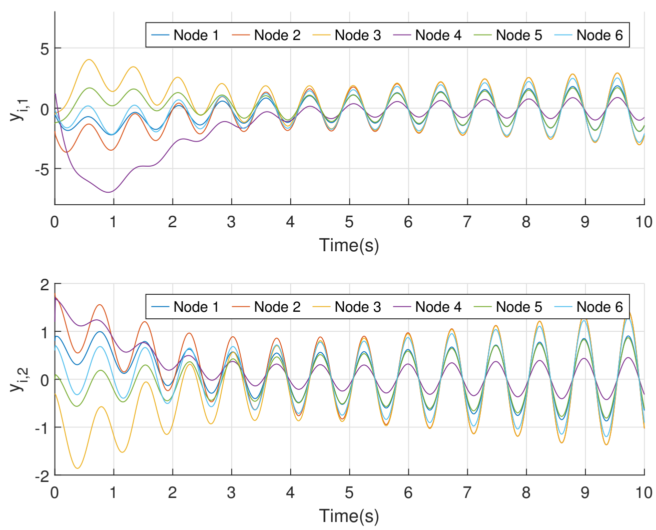


Fig. 2. Output trajectories

where $k = 0, 1, 2, \cdots$ and $T = 1.2\times 10^{-3}s$. Pick $\mu_1 = 1000$, $\mu_2 = 100$ and $\mu_3 = 1000$. Choose K_i with

$$K_i = \begin{bmatrix} 2.52 & -2.86 \end{bmatrix}, \text{for } i = 1, 2, 3;$$
$$K_i = \begin{bmatrix} -1 & -0.05 \end{bmatrix}, \text{for } i = 4, 5, 6.$$

such that $A_i + B_i K_i$ is Hurwitz. The output trajectories $y_{i,1}$ and $y_{i,2}$ are plotted in Fig. 2. From these two figures, we can see that the output signals of the MAS eventually have the same sign but different magnitudes, i.e., sign consensus of the MAS is achieved.

VI. Conclusions

In this paper, we investigated output sign consensus problem for leaderless heterogeneous linear MAS. We removed a mandatory condition that a practical leader should exist for heterogeneous MAS to achieve any synchronous behavior, and allow the communication network to be dynamically changing. However, the selection of the switching signed graphs is quite demanding, and further investigation regarding this limitation is needed.

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