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# Bounded extremum seeking for single-variable static map with large measurement delay via time-delay approach to averaging

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*Abstract*— **In this paper, we present a time-delay approach to gradient-based bounded extremum seeking (ES) with large measurement constant delay, for an unknown single-input static quadratic map. We assume that the extremum point and the Hessian $H$ belong to known intervals, whereas the sign of $H$ is known. We apply a time-delay approach to the bounded ES system and arrive at the neutral type system with a nominal linear delayed system. We present the latter system as a retarded one and employ variation of constants formula for practical stability analysis. Explicit conditions in terms of simple scalar inequalities depending on tuning parameters and delay are established to guarantee the practical stability of the bounded ES control systems. Given any delay and neighborhood of the extremum point and through the solution of the constructed inequalities, we find lower bounds on the dither period that ensures the practical stability.**

*Index Terms*— **Bounded extremum seeking, averaging, time-delay, practical stability.**

## I. Introduction

ES is a model-free, real-time on-line adaptive optimization control method. Under the premise of the existence of extremum value, the ES control can search the extremum value with an unknown nonlinear map. In 2000, Krstic and Wang gave the first rigorous stability analysis for an ES system by using averaging and singular perturbations in [1]. After that, a large number of theoretical studies on ES have emerged in the literature [2], [3], [4], [5]. Particularly, the bounded ES schemes were proposed in [5], [6], [7], in which the uncertainty is confined to the argument of a sine/cosine function, resulting in guaranteed bounds on update rate in minimum seeking and control effort in stabilization.

Additionally, the delay phenomenon is inevitably encountered in ES due to time needed to measuring and processing of the data, which makes theoretical research very complex and challenging [8]. To address the challenges of delays in extremum seeking, Oliveira et al. in [9] first investigated the design and analysis of multi-variable ES for static maps subject to arbitrarily long time delays. Based on this pioneer work, the case of ES with time-varying delays and uncertain delays were considered in [10] and [11] (also see [8]). Recently, Malisoff et al. in [12] reconsidered the multi-variable ES for static maps with arbitrarily long time constant delays by using a one-stage sequential predictor, which can avoid the interference of the integral term appeared in [9]. The above literature employ the classical averaging theory in infinite dimensions (see [13]) to prove the stability of time-delay ES systems. However, these methods only provide the qualitative analysis, and cannot suggest quantitative upper bounds on the parameter that preserves the stability. The analysis is also a bit complicated.

Recently, a new constructive time-delay approach to the continuous-time averaging was presented in [14] with efficient and quantitative bounds on the small parameter that ensures the stability. This approach to averaging was successfully applied for the quantitative stability analysis of continuous-time ES algorithms in [15], sampled-data ES algorithms in the presence of small constant delay in [16] and bounded ES in the presence of small time-varying delay in [17] for static quadratic maps by constructing appropriate Lyapunov-Krasovskii (L-K) functionals. In [18] and [19], we suggested a robust time-delay approach to ES without delay also with large measurement delays, respectively, where we presented the resulting time-delay model as an averaged one with disturbances and further employed a variation of constants formula. The latter can greatly simplify the stability analysis via L-K method, simplify the conditions and reduce conservatism, particularly, allow large delays.

In the present paper, for the first time, we study bounded ES in the presence of large constant measurement delay. We first transform the original delayed ES system to a neutral type system via the time-delay approach. The practical stability of the original system can be guaranteed by the resulting neutral type system. We further present the neutral type system as a retarded one with disturbances. Finally, we use the variation of constants formula together with tight bounds on the fundamental solutions of the linear systems with delays in [20] to quantitatively analyze the practical stability of the retarded systems (and thus of the original ES systems). Explicit conditions in terms of simple inequalities are established to guarantee the practical stability of the ES control systems. Through the solution of the constructed inequalities, we find upper bounds on the dither period that ensures the practical stability, and also provide quantitative ultimate bound (UB) on estimation error.

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## II. PRELIMINARIES

We will employ the solution representation formula for delay differential equations and some properties of the corresponding fundamental solution as in the following lemma, these results are brought from [20] (see Theorem 2.7, Corollary 2.14 and Lemma 9.1).

*Lemma 1:* Consider the following scalar delay differential equation:

$$\dot{x}(t)+ax(t-D)=f(t),\ t\geq t_0 \tag{1}$$

with the initial value

$$x(t)=\varphi(t),\ t\in[t_0-D,t_0),\ x(t_0)=x_0, \tag{2}$$

where $f:[t_0,\infty)\to\mathbf{R}$ is a Lebesgue measurable locally essentially bounded function and $\varphi:[t_0-D,t_0)\to\mathbf{R}$ is a piecewise continuous and bounded function. Then there exists one and only one solution for (1)-(2) as in the following form

$$\begin{aligned}x(t)=X(t-t_0)x_0+\textstyle\int_{t_0}^{t}X(t-s)f(s)\mathrm{d}s\\ -\textstyle\int_{t_0-D}^{t_0}X(t-s-D)a\varphi(s)\mathrm{d}s,\end{aligned} \tag{3}$$

where the fundamental solution $X(t)$ is the solution of

$$\dot{x}(t)+ax(t-D)=0,\ x(t)=0,\ t<0,\ x(0)=1.$$

Let $a>0$ and

$$Da\leq\tfrac{1}{\mathrm{e}}.$$

Then for $t\geq 0$,

$$0<X(t)\leq\begin{cases}1, & 0\leq t\leq D,\\ \mathrm{e}^{-a(t-D)}, & t\geq D.\end{cases}$$

## III. BOUNDED ES WITH LARGE DISTINCT DELAYS VIA A TIME-DELAY APPROACH AVERAGING

Consider a single-input static map $Q(\theta)$ of the following quadratic form:

$$Q(\theta(t))=Q^*+\tfrac{H}{2}[\theta(t)-\theta^*]^2, \tag{4}$$

where $\theta(t)\in\mathbf{R}$ is the scalar input, $Q^*$ and $\theta^*$ are constants, and $H$ is the gradient which is a non-zero constant. It is clear that the quadratic map (4) has a maximum or minimum value $Q^*$ at $\theta(t)=\theta^*$ such that

$$\left.\tfrac{\partial Q}{\partial\theta}\right|_{\theta=\theta^*}=0,\ \left.\tfrac{\partial^2 Q}{\partial\theta^2}\right|_{\theta=\theta^*}=H<0\text{ or }>0.$$

Usually, the cost function (4) is unknown, but the sign of $H$ is known. In this paper, in order to derive efficient conditions, we assume that:

**A1** The extremum point $\theta^*$ to be sought is uncertain from a known interval $\theta^*\in[\underline{\theta}^*,\bar{\theta}^*]$ with $\left|\bar{\theta}^*-\underline{\theta}^*\right|=\sigma_0$.

**A2** The sign of $H$ is known, whereas $H$ is unknown and subject to $H_{\mathrm{m}}\leq|H|\leq H_{\mathrm{M}}$ with $H_{\mathrm{m}}$ and $H_{\mathrm{M}}$ being known.

In this paper, we consider the bounded ES of static quadratic map in the presence of large and known constant measurement delay $D>0$. Let the delayed measurement has a form

$$y(t)=Q(\theta(t-D)),\ t\geq D. \tag{5}$$

Define the estimation error as

$$\tilde{\theta}(t)=\theta(t)-\theta^*. \tag{6}$$

Then it follows from (4), (5) and (6) that

$$y(t)=Q^*+\tfrac{H}{2}\tilde{\theta}^2(t-D),\ t\geq D.$$

Inspired by [5], we consider the gradient-based bounded ES as follows

$$\dot{\tilde{\theta}}(t)=\begin{cases}0, & t\in[0,D),\\ \sqrt{\alpha\omega}\cos(\omega t+ky(t)), & t\geq D,\end{cases} \tag{7}$$

namely,

$$\begin{aligned}\dot{\tilde{\theta}}(t)&=\sqrt{\alpha\omega}\cos(\omega t)\cos(ky(t))\\ &\quad-\sqrt{\alpha\omega}\sin(\omega t)\sin(ky(t)),\ t\geq D,\\ \tilde{\theta}(t)&=\tilde{\theta}(0),\ t\in[0,D],\end{aligned} \tag{8}$$

where $\omega$ is the frequency of the dither signal whose magnitude is proportional to $\alpha$, $k$ is the adaptation gain whose sign is selected to be identical with that of $H$.

For the stability analysis of the ES control system (8), inspired by [18], [19], we first apply the time-delay approach to averaging of (8). For averaging, we choose

$$\omega=\tfrac{2\pi}{\varepsilon},\ \varepsilon=\tfrac{D}{l},\ l\in\mathbf{N}. \tag{9}$$

Here $l\geq D/\varepsilon^*$ is large enough with $\varepsilon^*>0$ to be found from conditions of Theorem 1 below. Integrating in $t\geq 2D+\varepsilon$ from $t-\varepsilon$ to $t$ and dividing by $\varepsilon$ on both sides of (8), we get

$$\begin{aligned}&\tfrac{1}{\varepsilon}\textstyle\int_{t-\varepsilon}^{t}\dot{\tilde{\theta}}(\tau)\mathrm{d}\tau=\tfrac{1}{\varepsilon}\sqrt{\tfrac{2\pi\alpha}{\varepsilon}}\textstyle\int_{t-\varepsilon}^{t}\cos\left(\tfrac{2\pi}{\varepsilon}\tau\right)\cos\left(ky(\tau)\right)\mathrm{d}\tau\\ &-\tfrac{1}{\varepsilon}\sqrt{\tfrac{2\pi\alpha}{\varepsilon}}\textstyle\int_{t-\varepsilon}^{t}\sin\left(\tfrac{2\pi}{\varepsilon}\tau\right)\sin(ky(\tau))\mathrm{d}\tau,\ t\geq 2D+\varepsilon.\end{aligned} \tag{10}$$

From (9), we have

$$\cos\left(\tfrac{2\pi}{\varepsilon}(s-D)\right)=\cos\left(\tfrac{2\pi}{\varepsilon}s\right),\ \sin\left(\tfrac{2\pi}{\varepsilon}(s-D)\right)=\sin\left(\tfrac{2\pi}{\varepsilon}s\right). \tag{11}$$

Define $x\pm y\triangleq x+y-y$. By (11), for the first term on the right-hand of (10), we have

$$\begin{aligned}&\tfrac{1}{\varepsilon}\sqrt{\tfrac{2\pi\alpha}{\varepsilon}}\textstyle\int_{t-\varepsilon}^{t}\cos\left(\tfrac{2\pi}{\varepsilon}\tau\right)\cos\left(ky(\tau)\right)\mathrm{d}\tau\\ &=\tfrac{1}{\varepsilon}\sqrt{\tfrac{2\pi\alpha}{\varepsilon}}\textstyle\int_{t-\varepsilon}^{t}\cos\left(\tfrac{2\pi}{\varepsilon}\tau\right)\left[\cos\left(ky(\tau)\right)\pm\cos\left(ky(t)\right)\right]\mathrm{d}\tau\\ &=\tfrac{1}{\varepsilon}\sqrt{\tfrac{2\pi\alpha}{\varepsilon}}\textstyle\int_{t-\varepsilon}^{t}\cos\left(\tfrac{2\pi}{\varepsilon}\tau\right)\mathrm{d}\tau\cdot\cos\left(ky(t)\right)\\ &\quad-\tfrac{1}{\varepsilon}\sqrt{\tfrac{2\pi\alpha}{\varepsilon}}\textstyle\int_{t-\varepsilon}^{t}\cos\left(\tfrac{2\pi}{\varepsilon}\tau\right)\left[\cos\left(ky(t)\right)-\cos\left(ky(\tau)\right)\right]\mathrm{d}\tau\\ &=\tfrac{k}{\varepsilon}\sqrt{\tfrac{2\pi\alpha}{\varepsilon}}\textstyle\int_{t-\varepsilon}^{t}\cos\left(\tfrac{2\pi}{\varepsilon}\tau\right)\textstyle\int_{\tau}^{t}\sin(ky(s))\dot{y}(s)\mathrm{d}s\mathrm{d}\tau\\ &=\tfrac{kH}{\varepsilon}\sqrt{\tfrac{2\pi\alpha}{\varepsilon}}\textstyle\int_{t-\varepsilon}^{t}\cos\left(\tfrac{2\pi}{\varepsilon}\tau\right)\textstyle\int_{\tau}^{t}\sin(ky(s))\\ &\quad\times\tilde{\theta}(s-D)\dot{\tilde{\theta}}(s-D)\mathrm{d}s\mathrm{d}\tau\\ &=\tfrac{kH}{\varepsilon}\sqrt{\tfrac{2\pi\alpha}{\varepsilon}}\textstyle\int_{t-\varepsilon}^{t}\cos\left(\tfrac{2\pi}{\varepsilon}\tau\right)\textstyle\int_{\tau}^{t}\sin(ky(s))\tilde{\theta}(s-D)\\ &\quad\times\left[\sqrt{\tfrac{2\pi\alpha}{\varepsilon}}\cos\left(\tfrac{2\pi}{\varepsilon}s\right)\cos(ky(s-D))\right.\\ &\quad\left.-\sqrt{\tfrac{2\pi\alpha}{\varepsilon}}\sin\left(\tfrac{2\pi}{\varepsilon}s\right)\sin(ky(s-D))\right]\mathrm{d}s\mathrm{d}\tau\\ &=\tfrac{2\pi\alpha kH}{\varepsilon^2}\textstyle\int_{t-\varepsilon}^{t}\int_{\tau}^{t}\cos\left(\tfrac{2\pi}{\varepsilon}\tau\right)\cos\left(\tfrac{2\pi}{\varepsilon}s\right)\\ &\quad\times\sin(ky(s))\cos(ky(s-D))\tilde{\theta}(s-D)\mathrm{d}s\mathrm{d}\tau\\ &\quad-\tfrac{2\pi\alpha kH}{\varepsilon^2}\textstyle\int_{t-\varepsilon}^{t}\int_{\tau}^{t}\cos\left(\tfrac{2\pi}{\varepsilon}\tau\right)\sin\left(\tfrac{2\pi}{\varepsilon}s\right)\sin(ky(s))\\ &\quad\times\sin(ky(s-D))\tilde{\theta}(s-D)\mathrm{d}s\mathrm{d}\tau.\end{aligned} \tag{12}$$

By using

$$\begin{aligned}
&\int_{t-\varepsilon}^{t}\int_{\tau}^{t}\cos\left(\tfrac{2\pi}{\varepsilon}\tau\right)\cos\left(\tfrac{2\pi}{\varepsilon}s\right)\mathrm{d}s\mathrm{d}\tau\\
&=\tfrac{\varepsilon}{2\pi}\int_{t-\varepsilon}^{t}\cos\left(\tfrac{2\pi}{\varepsilon}\tau\right)\left[\sin\left(\tfrac{2\pi}{\varepsilon}t\right)-\sin\left(\tfrac{2\pi}{\varepsilon}\tau\right)\right]\mathrm{d}\tau\\
&=-\tfrac{\varepsilon}{2\pi}\int_{t-\varepsilon}^{t}\cos\left(\tfrac{2\pi}{\varepsilon}\tau\right)\sin\left(\tfrac{2\pi}{\varepsilon}\tau\right)\mathrm{d}\tau\\
&=-\tfrac{\varepsilon}{4\pi}\int_{t-\varepsilon}^{t}\sin\left(\tfrac{4\pi}{\varepsilon}\tau\right)\mathrm{d}\tau=0,
\end{aligned}$$

the first term on the right-hand side of (12) is calculated as follows:

$$\begin{aligned}
&\tfrac{2\pi\alpha kH}{\varepsilon^2}\int_{t-\varepsilon}^{t}\int_{\tau}^{t}\cos\left(\tfrac{2\pi}{\varepsilon}\tau\right)\cos\left(\tfrac{2\pi}{\varepsilon}s\right)\sin(ky(s))\\
&\quad\times\cos(ky(s-D))\tilde{\theta}(s-D)\mathrm{d}s\mathrm{d}\tau\\
&=\tfrac{2\pi\alpha kH}{\varepsilon^2}\int_{t-\varepsilon}^{t}\int_{\tau}^{t}\cos\left(\tfrac{2\pi}{\varepsilon}\tau\right)\cos\left(\tfrac{2\pi}{\varepsilon}s\right)\mathrm{d}s\mathrm{d}\tau\\
&\quad\times\sin(ky(t))\cos(ky(t-D))\tilde{\theta}(t-D)\\
&\quad-\tfrac{2\pi\alpha kH}{\varepsilon^2}\int_{t-\varepsilon}^{t}\int_{\tau}^{t}\cos\left(\tfrac{2\pi}{\varepsilon}\tau\right)\cos\left(\tfrac{2\pi}{\varepsilon}s\right)\\
&\quad\times\left[\sin(ky(t))\cos(ky(t-D))\tilde{\theta}(t-D)\right.\\
&\quad\left.-\sin(ky(s))\cos(ky(s-D))\tilde{\theta}(s-D)\right]\mathrm{d}s\mathrm{d}\tau\\
&=-\tfrac{2\pi\alpha kH}{\varepsilon^2}\int_{t-\varepsilon}^{t}\int_{\tau}^{t}\cos\left(\tfrac{2\pi}{\varepsilon}\tau\right)\cos\left(\tfrac{2\pi}{\varepsilon}s\right)\\
&\quad\times\int_{s}^{t}\tfrac{\mathrm{d}}{\mathrm{d}\nu}\left[\sin(ky(\nu))\cos(ky(\nu-D))\tilde{\theta}(\nu-D)\right]\mathrm{d}\nu\mathrm{d}s\mathrm{d}\tau,
\end{aligned}\tag{13}$$

By using

$$\begin{aligned}
&\int_{t-\varepsilon}^{t}\int_{\tau}^{t}\cos\left(\tfrac{2\pi}{\varepsilon}\tau\right)\sin\left(\tfrac{2\pi}{\varepsilon}s\right)\mathrm{d}s\mathrm{d}\tau\\
&=-\tfrac{\varepsilon}{2\pi}\int_{t-\varepsilon}^{t}\cos\left(\tfrac{2\pi}{\varepsilon}\tau\right)\left[\cos\left(\tfrac{2\pi}{\varepsilon}t\right)-\cos\left(\tfrac{2\pi}{\varepsilon}\tau\right)\right]\mathrm{d}\tau\\
&=\tfrac{\varepsilon}{2\pi}\int_{t-\varepsilon}^{t}\cos\left(\tfrac{2\pi}{\varepsilon}\tau\right)\cos\left(\tfrac{2\pi}{\varepsilon}\tau\right)\mathrm{d}\tau\\
&=\tfrac{\varepsilon^2}{4\pi},
\end{aligned}$$

the second term on the right-hand side of (12) is calculated as follows:

$$\begin{aligned}
&-\tfrac{2\pi\alpha kH}{\varepsilon^2}\int_{t-\varepsilon}^{t}\int_{\tau}^{t}\cos\left(\tfrac{2\pi}{\varepsilon}\tau\right)\sin\left(\tfrac{2\pi}{\varepsilon}s\right)\sin(ky(s))\\
&\quad\times\sin(ky(s-D))\tilde{\theta}(s-D)\mathrm{d}s\mathrm{d}\tau\\
&=-\tfrac{\alpha kH}{2}\sin(ky(t))\sin(ky(t-D))\tilde{\theta}(t-D)\\
&\quad+\tfrac{2\pi\alpha kH}{\varepsilon^2}\int_{t-\varepsilon}^{t}\int_{\tau}^{t}\cos\left(\tfrac{2\pi}{\varepsilon}\tau\right)\sin\left(\tfrac{2\pi}{\varepsilon}s\right)\\
&\quad\times\int_{s}^{t}\tfrac{\mathrm{d}}{\mathrm{d}\nu}\left[\sin(ky(\nu))\sin(ky(\nu-D))\tilde{\theta}(\nu-D)\right]\mathrm{d}\nu\mathrm{d}s\mathrm{d}\tau\\
&=-\tfrac{\alpha kH}{2}\sin^2(ky(t))\tilde{\theta}(t-D)\\
&\quad+\tfrac{\alpha kH}{2}\sin(ky(t))\tilde{\theta}(t-D)\left(\sin(ky(t))-\sin(ky(t-D))\right)\\
&\quad+\tfrac{2\pi\alpha kH}{\varepsilon^2}\int_{t-\varepsilon}^{t}\int_{\tau}^{t}\cos\left(\tfrac{2\pi}{\varepsilon}\tau\right)\sin\left(\tfrac{2\pi}{\varepsilon}s\right)\\
&\quad\times\int_{s}^{t}\tfrac{\mathrm{d}}{\mathrm{d}\nu}\left[\sin(ky(\nu))\sin(ky(\nu-D))\tilde{\theta}(\nu-D)\right]\mathrm{d}\nu\mathrm{d}s\mathrm{d}\tau\\
&=-\tfrac{\alpha kH}{2}\sin^2(ky(t))\tilde{\theta}(t-D)\\
&\quad+\tfrac{\alpha k^2H^2}{2}\sin(ky(t))\tilde{\theta}(t-D)\int_{t-D}^{t}\cos(ky(s))\\
&\quad\times\tilde{\theta}(s-D)\dot{\tilde{\theta}}(s-D)\mathrm{d}s\\
&\quad+\tfrac{2\pi\alpha kH}{\varepsilon^2}\int_{t-\varepsilon}^{t}\int_{\tau}^{t}\cos\left(\tfrac{2\pi}{\varepsilon}\tau\right)\sin\left(\tfrac{2\pi}{\varepsilon}s\right)\\
&\quad\times\int_{s}^{t}\tfrac{\mathrm{d}}{\mathrm{d}\nu}\left[\sin(ky(\nu))\sin(ky(\nu-D))\tilde{\theta}(\nu-D)\right]\mathrm{d}\nu\mathrm{d}s\mathrm{d}\tau,
\end{aligned}\tag{14}$$

By (11), for the second term on the right-hand of (10), we have

$$\begin{aligned}
&-\tfrac{1}{\varepsilon}\sqrt{\tfrac{2\pi\alpha}{\varepsilon}}\int_{t-\varepsilon}^{t}\sin\left(\tfrac{2\pi}{\varepsilon}\tau\right)\sin(ky(\tau))\mathrm{d}\tau\\
&=-\tfrac{1}{\varepsilon}\sqrt{\tfrac{2\pi\alpha}{\varepsilon}}\int_{t-\varepsilon}^{t}\sin\left(\tfrac{2\pi}{\varepsilon}\tau\right)\left[\sin(ky(\tau))\pm\sin(ky(t))\right]\mathrm{d}\tau\\
&=\tfrac{k}{\varepsilon}\sqrt{\tfrac{2\pi\alpha}{\varepsilon}}\int_{t-\varepsilon}^{t}\sin\left(\tfrac{2\pi}{\varepsilon}\tau\right)\int_{\tau}^{t}\cos(ky(s))\dot{y}(s)\mathrm{d}s\mathrm{d}\tau\\
&=\tfrac{kH}{\varepsilon}\sqrt{\tfrac{2\pi\alpha}{\varepsilon}}\int_{t-\varepsilon}^{t}\sin\left(\tfrac{2\pi}{\varepsilon}\tau\right)\int_{\tau}^{t}\cos(ky(s))\\
&\quad\times\tilde{\theta}(s-D)\dot{\tilde{\theta}}(s-D)\mathrm{d}s\mathrm{d}\tau\\
&=\tfrac{2\pi\alpha kH}{\varepsilon^2}\int_{t-\varepsilon}^{t}\int_{\tau}^{t}\sin\left(\tfrac{2\pi}{\varepsilon}\tau\right)\cos\left(\tfrac{2\pi}{\varepsilon}s\right)\cos(ky(s))\\
&\quad\times\cos(ky(s-D))\tilde{\theta}(s-D)\mathrm{d}s\mathrm{d}\tau\\
&\quad-\tfrac{2\pi\alpha kH}{\varepsilon^2}\int_{t-\varepsilon}^{t}\int_{\tau}^{t}\sin\left(\tfrac{2\pi}{\varepsilon}\tau\right)\sin\left(\tfrac{2\pi}{\varepsilon}s\right)\cos(ky(s))\\
&\quad\times\sin(ky(s-D))\tilde{\theta}(s-D)\mathrm{d}s\mathrm{d}\tau.
\end{aligned}\tag{15}$$

The first term on the right-hand side of (15) is calculated as follows:

$$\begin{aligned}
&\tfrac{2\pi\alpha kH}{\varepsilon^2}\int_{t-\varepsilon}^{t}\int_{\tau}^{t}\sin\left(\tfrac{2\pi}{\varepsilon}\tau\right)\cos\left(\tfrac{2\pi}{\varepsilon}s\right)\cos(ky(s))\\
&\quad\times\cos(ky(s-D))\tilde{\theta}(s-D)\mathrm{d}s\mathrm{d}\tau\\
&=-\tfrac{\alpha kH}{2}\cos(ky(t))\cos(ky(t-D))\tilde{\theta}(t-D)\\
&\quad-\tfrac{2\pi\alpha kH}{\varepsilon^2}\int_{t-\varepsilon}^{t}\int_{\tau}^{t}\sin\left(\tfrac{2\pi}{\varepsilon}\tau\right)\cos\left(\tfrac{2\pi}{\varepsilon}s\right)\\
&\quad\times\int_{s}^{t}\tfrac{\mathrm{d}}{\mathrm{d}\nu}\left[\cos(ky(\nu))\cos(ky(\nu-D))\tilde{\theta}(\nu-D)\right]\mathrm{d}\nu\mathrm{d}s\mathrm{d}\tau\\
&=-\tfrac{\alpha kH}{2}\cos^2(ky(t))\tilde{\theta}(t-D)\\
&\quad+\tfrac{\alpha kH}{2}\cos(ky(t))\tilde{\theta}(t-D)\left[\cos(ky(t))-\cos(ky(t-D))\right]\\
&\quad-\tfrac{2\pi\alpha kH}{\varepsilon^2}\int_{t-\varepsilon}^{t}\int_{\tau}^{t}\sin\left(\tfrac{2\pi}{\varepsilon}\tau\right)\cos\left(\tfrac{2\pi}{\varepsilon}s\right)\\
&\quad\times\int_{s}^{t}\tfrac{\mathrm{d}}{\mathrm{d}\nu}\left[\cos(ky(\nu))\cos(ky(\nu-D))\tilde{\theta}(\nu-D)\right]\mathrm{d}\nu\mathrm{d}s\mathrm{d}\tau\\
&=-\tfrac{\alpha kH}{2}\cos^2(ky(t))\tilde{\theta}(t-D)\\
&\quad-\tfrac{\alpha k^2H^2}{2}\cos(ky(t))\tilde{\theta}(t-D)\int_{t-D}^{t}\sin(ky(s))\\
&\quad\times\tilde{\theta}(s-D)\dot{\tilde{\theta}}(s-D)\mathrm{d}s\\
&\quad-\tfrac{2\pi\alpha kH}{\varepsilon^2}\int_{t-\varepsilon}^{t}\int_{\tau}^{t}\sin\left(\tfrac{2\pi}{\varepsilon}\tau\right)\cos\left(\tfrac{2\pi}{\varepsilon}s\right)\\
&\quad\times\int_{s}^{t}\tfrac{\mathrm{d}}{\mathrm{d}\nu}\left[\cos(ky(\nu))\cos(ky(\nu-D))\tilde{\theta}(\nu-D)\right]\mathrm{d}\nu\mathrm{d}s\mathrm{d}\tau,
\end{aligned}\tag{16}$$

where we have used

$$\begin{aligned}
&\int_{t-\varepsilon}^{t}\int_{\tau}^{t}\sin\left(\tfrac{2\pi}{\varepsilon}\tau\right)\cos\left(\tfrac{2\pi}{\varepsilon}s\right)\mathrm{d}s\mathrm{d}\tau\\
&=\tfrac{\varepsilon}{2\pi}\int_{t-\varepsilon}^{t}\sin\left(\tfrac{2\pi}{\varepsilon}\tau\right)\left[\sin\left(\tfrac{2\pi}{\varepsilon}t\right)-\sin\left(\tfrac{2\pi}{\varepsilon}\tau\right)\right]\mathrm{d}\tau\\
&=-\tfrac{\varepsilon}{2\pi}\int_{t-\varepsilon}^{t}\sin\left(\tfrac{2\pi}{\varepsilon}\tau\right)\sin\left(\tfrac{2\pi}{\varepsilon}\tau\right)\mathrm{d}\tau\\
&=-\tfrac{\varepsilon^2}{4\pi}.
\end{aligned}$$

The second term on the right-hand side of (15) is calculated as follows:

$$\begin{aligned}
&-\tfrac{2\pi\alpha kH}{\varepsilon^2}\int_{t-\varepsilon}^{t}\int_{\tau}^{t}\sin\left(\tfrac{2\pi}{\varepsilon}\tau\right)\sin\left(\tfrac{2\pi}{\varepsilon}s\right)\cos(ky(s))\\
&\quad\times\sin(ky(s-D))\tilde{\theta}(s-D)\mathrm{d}s\mathrm{d}\tau\\
&=-\tfrac{2\pi\alpha kH}{\varepsilon^2}\int_{t-\varepsilon}^{t}\int_{\tau}^{t}\sin\left(\tfrac{2\pi}{\varepsilon}\tau\right)\sin\left(\tfrac{2\pi}{\varepsilon}s\right)\mathrm{d}s\mathrm{d}\tau\\
&\quad\times\cos(ky(t))\sin(ky(t-D))\tilde{\theta}(t-D)\\
&\quad+\tfrac{2\pi\alpha kH}{\varepsilon^2}\int_{t-\varepsilon}^{t}\int_{\tau}^{t}\sin\left(\tfrac{2\pi}{\varepsilon}\tau\right)\sin\left(\tfrac{2\pi}{\varepsilon}s\right)\\
&\quad\times\left[\cos(ky(t))\sin(ky(t-D))\tilde{\theta}(t-D)\right.\\
&\quad\left.-\cos(ky(s))\sin(ky(s-D))\tilde{\theta}(s-D)\right]\mathrm{d}s\mathrm{d}\tau\\
&=\tfrac{2\pi\alpha kH}{\varepsilon^2}\int_{t-\varepsilon}^{t}\int_{\tau}^{t}\sin\left(\tfrac{2\pi}{\varepsilon}\tau\right)\sin\left(\tfrac{2\pi}{\varepsilon}s\right)\\
&\quad\times\int_{s}^{t}\tfrac{\mathrm{d}}{\mathrm{d}\nu}\left[\cos(ky(\nu))\sin(ky(\nu-D))\tilde{\theta}(\nu-D)\right]\mathrm{d}\nu\mathrm{d}s\mathrm{d}\tau
\end{aligned}\tag{17}$$

where we have used

$$\begin{aligned}
&\int_{t-\varepsilon}^{t}\int_{\tau}^{t}\sin\left(\tfrac{2\pi}{\varepsilon}\tau\right)\sin\left(\tfrac{2\pi}{\varepsilon}s\right)\mathrm{d}s\mathrm{d}\tau\\
&=-\tfrac{\varepsilon}{2\pi}\int_{t-\varepsilon}^{t}\sin\left(\tfrac{2\pi}{\varepsilon}\tau\right)\left[\cos\left(\tfrac{2\pi}{\varepsilon}t\right)-\cos\left(\tfrac{2\pi}{\varepsilon}\tau\right)\right]\mathrm{d}\tau\\
&=\tfrac{\varepsilon}{2\pi}\int_{t-\varepsilon}^{t}\sin\left(\tfrac{2\pi}{\varepsilon}\tau\right)\cos\left(\tfrac{2\pi}{\varepsilon}\tau\right)\mathrm{d}\tau\\
&=0.
\end{aligned}$$

Let

$$G(t)=\frac{1}{\varepsilon}\int_{t-\varepsilon}^{t}(\tau-t+\varepsilon)\dot{\tilde{\theta}}(\tau)\mathrm{d}\tau,\ t\geq 2D+\varepsilon. \tag{18}$$

Then we can present

$$\tfrac{1}{\varepsilon}\int_{t-\varepsilon}^{t}\dot{\tilde{\theta}}(\tau)\mathrm{d}\tau=\tfrac{\mathrm{d}}{\mathrm{d}t}\left[\tilde{\theta}(t)-G(t)\right], \tag{19}$$

When $t \geq 2D+\varepsilon$, we denote

$$\begin{aligned}
Y_1(t) = & \tfrac{\alpha k^2 H^2}{2}\tilde{\theta}(t-D)\int_{t-D}^{t}\sin\left(ky(t)-ky(s)\right) \\
& \times \tilde{\theta}(s-D)\dot{\tilde{\theta}}(s-D)\mathrm{d}s, \\
Y_2(t) = & -\tfrac{\pi\alpha kH}{\varepsilon^2}\int_{t-\varepsilon}^{t}\int_{\tau}^{t}\int_{s}^{t}[\sin\left(ky(\nu)+ky(\nu-D)\right. \\
& \left.+\tfrac{2\pi}{\varepsilon}(\tau+s)\right)+\sin\left(ky(\nu)-ky(\nu-D)\right. \\
& \left.+\tfrac{2\pi}{\varepsilon}(\tau-s)\right)]\dot{\tilde{\theta}}(\nu-D)\mathrm{d}\nu\mathrm{d}s\mathrm{d}\tau, \\
Y_3(t) = & -\tfrac{\pi\alpha k^2 H^2}{\varepsilon^2}\int_{t-\varepsilon}^{t}\int_{\tau}^{t}\int_{s}^{t}[\cos\left(ky(\nu)-ky(\nu-D)\right. \\
& \left.+\tfrac{2\pi}{\varepsilon}(\tau-s)\right)+\cos\left(ky(\nu)+ky(\nu-D)\right. \\
& \left.+\tfrac{2\pi}{\varepsilon}(\tau+s)\right)]\tilde{\theta}^2(\nu-D)\dot{\tilde{\theta}}(\nu-D)\mathrm{d}\nu, \\
Y_4(t) = & \tfrac{\pi\alpha k^2 H^2}{\varepsilon^2}\int_{t-\varepsilon}^{t}\int_{\tau}^{t}\int_{s}^{t}[\cos\left(ky(\nu)-ky(\nu-D)\right. \\
& \left.+\tfrac{2\pi}{\varepsilon}(\tau-s)\right)-\cos\left(ky(\nu)+ky(\nu-D)\right. \\
& \left.+\tfrac{2\pi}{\varepsilon}(\tau+s)\right)]\tilde{\theta}(\nu-D)\tilde{\theta}(\nu-2D) \\
& \times \dot{\tilde{\theta}}(\nu-2D)\mathrm{d}\nu\mathrm{d}s\mathrm{d}\tau.
\end{aligned} \tag{20}$$

Substituting (13) and (14) into (12), (16) and (17) into (15), and further substituting (12) and (15) into (10), employing (19) and (20), we finally arrive at the time-delay system

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\left[\tilde{\theta}(t)-G(t)\right] = -\tfrac{\alpha kH}{2}\tilde{\theta}(t-D)+\textstyle\sum_{i=1}^{4}Y_i,\ t\geq 2D+\varepsilon.$$

By denoting

$$\begin{aligned}
& z(t) = \tilde{\theta}(t)-G(t),\ t\geq D+\varepsilon, \\
& G(t) = 0,\ t\in[D+\varepsilon,2D+\varepsilon),
\end{aligned} \tag{21}$$

we further have

$$\begin{aligned}
& \dot{z}(t) = -\tfrac{\alpha kH}{2}z(t-D)+w(t),\ t\geq 2D+\varepsilon, \\
& w(t) = -\tfrac{\alpha kH}{2}G(t-D)+\textstyle\sum_{i=1}^{4}Y_i(t).
\end{aligned} \tag{22}$$

Note that if $\tilde{\theta}(t)$ (and thus $z(t)$) is of the order of $\mathrm{O}(1)$ and let the tuning parameter $\alpha$ be of the order of $\mathrm{O}(\varepsilon)$, then the terms $G(t)$ and $Y_1$ defined by (18) and (20) are of the order of $\mathrm{O}(\varepsilon)$, $Y_i (i=2,3,4)$ defined by (20) are of the order of $\mathrm{O}\left(\varepsilon^2\right)$. Therefore, $w(t)$ defined by (22) is of the order of $\mathrm{O}(\varepsilon)$. Similar to our previous work [19], we will analyze (22) as linear delayed system w.r.t. $z(t)$ with delayed disturbance-like $\mathrm{O}(\varepsilon)$ term $w(t)$ that depends on the solutions of (8). By utilizing solution representation formula in Lemma 1, we get the bound on $z(t)$ which will lead to the bound on $\tilde{\theta}(t)$ by (21).

Given any large $D$ and small enough $\alpha$, we will find $k$ from the inequality

$$\tfrac{\alpha kHD}{2}-\tfrac{1}{\mathrm{e}}<0, \tag{23}$$

which guarantees the exponential stability with a decay rate $\delta_i = \frac{\alpha kH}{2} < \frac{1}{\mathrm{e}D}$ of the averaged system

$$\dot{z}(t) = -\frac{\alpha kH}{2}z(t-D). \tag{24}$$

*Theorem 1:* Given $D>0$, consider the quadratic map (4) subject to **A1** and **A2** under the delayed measurements (5), and the ES system (7) with $|\tilde{\theta}(0)|\leq\sigma_0$. Given tuning parameters $p>1$, $\mu$, $k$ and $\sigma>\sigma_0>0$ and choosing $\alpha=\mu\varepsilon^p$, let small enough $\varepsilon^*>0$ satisfies

$$\begin{aligned}
\Phi_1 = & \tfrac{\mu\varepsilon^{*p}kH_{\mathrm{M}}D}{2}-\tfrac{1}{\mathrm{e}}\leq 0, \\
\Phi_2 = & \mathrm{e}^{\frac{\mu\varepsilon^{*p}kH_{\mathrm{M}}D}{2}}\left[\sigma_0+\varepsilon^{*\frac{p-1}{2}}\left(D+\tfrac{3\varepsilon^*}{2}\right)\sqrt{2\pi\mu}\right. \\
& \left.+W(\mu,\varepsilon^*)\right]+\varepsilon^{*\frac{p+1}{2}}\sqrt{\tfrac{\pi\mu}{2}}-\sigma<0,
\end{aligned} \tag{25}$$

where

$$\begin{aligned}
W(\mu,\varepsilon) = & \varepsilon^{\frac{p+1}{2}}\left(\tfrac{1}{2}+\tfrac{2\pi}{3}+\tfrac{4\pi kH_{\mathrm{M}}\sigma^2}{3}\right)\sqrt{2\pi\mu} \\
& +\varepsilon^{\frac{p-1}{2}}kH_{\mathrm{M}}\sigma^2 D\sqrt{2\pi\mu}.
\end{aligned} \tag{26}$$

Then for all $\varepsilon\in(0,\varepsilon^*]$ subject to (9), the solution of system (8) satisfies

$$\begin{aligned}
& \left|\tilde{\theta}(t)\right| \leq \left|\tilde{\theta}(D)\right|+(D+\varepsilon)\sqrt{2\pi\mu}\varepsilon^{\frac{p-1}{2}}<\sigma,\ t\in[D,2D+\varepsilon], \\
& \left|\tilde{\theta}(t)\right| < \left(1+\tfrac{\mu\varepsilon^p kH_{\mathrm{M}}D}{2}\right)\left[\left|\tilde{\theta}(D)\right|+\varepsilon^{\frac{p-1}{2}}\left(D+\tfrac{3\varepsilon}{2}\right)\sqrt{2\pi\mu}\right] \\
& \quad +\tfrac{\mu\varepsilon^p kH_{\mathrm{M}}D}{2}W(\mu,\varepsilon)+\varepsilon^{\frac{p+1}{2}}\sqrt{\tfrac{\pi\mu}{2}} \\
& \quad <\sigma,\ t\in[2D+\varepsilon,3D+\varepsilon]. \\
& \left|\tilde{\theta}(t)\right| < \mathrm{e}^{-\frac{\mu\varepsilon^p kH_{\mathrm{m}}}{2}(t-3D-\varepsilon)}\mathrm{e}^{\frac{\mu\varepsilon^p kH_{\mathrm{M}}D}{2}}\left[\left|\tilde{\theta}(D)\right|\right. \\
& \quad \left.+\varepsilon^{\frac{p-1}{2}}\left(D+\tfrac{3\varepsilon}{2}\right)\sqrt{2\pi\mu}\right]+\mathrm{e}^{\frac{\mu\varepsilon^p kH_{\mathrm{M}}D}{2}}W(\mu,\varepsilon)+\varepsilon^{\frac{p+1}{2}}\sqrt{\tfrac{\pi\mu}{2}} \\
& \quad <\sigma,\ t\geq 3D+\varepsilon,
\end{aligned} \tag{27}$$

Moreover, for all $\varepsilon\in(0,\varepsilon^*]$ subject to (9) and each constant initial function with $|\tilde{\theta}(0)|\leq\sigma_0$, the interval

$$\left\{\tilde{\theta}(t)\in\mathbf{R}:\left|\tilde{\theta}(t)\right|<\mathrm{e}^{\frac{\mu\varepsilon^p kH_{\mathrm{M}}D}{2}}W(\mu,\varepsilon)+\varepsilon^{\frac{p+1}{2}}\sqrt{\tfrac{\pi\mu}{2}}\right\} \tag{28}$$

is exponential attractive with a decay rate $\delta = \frac{\mu\varepsilon^p kH_{\mathrm{m}}}{2}$.

*Remark 1:* Theorem 1 guarantees for any delay $D$ semi-global convergence for small enough $\varepsilon^*$, $\mu$ and $\alpha=\mu\varepsilon^p$. Given any $D>0$ and $\sigma_0>0$ and choosing $k$ to satisfy (23), the ES algorithm converges for small enough $\varepsilon^*$ and $\mu$.

*Remark 2:* In [18], we proposed a robust time-delay approach to the classical ES without delay. Due to the fact that the unknown output function enters the control scheme in an affine way, the uncertainty of convergence rate and control effort exists. Comparatively to that, in this work we study bounded ES with large constant measurement delay, in which the uncertainty is confined to the argument of a sine function, resulting in guaranteed bounds on update rate and control effort. Moreover, the existing delay makes theoretical research more challenging than that in [18].

When $D=0$ in (5), from (20) we find that

$$\begin{aligned}
& Y_1(t) = 0, \\
& Y_2(t) = -\tfrac{\pi\alpha kH}{\varepsilon^2}\int_{t-\varepsilon}^{t}\int_{\tau}^{t}\int_{s}^{t}\left[\sin\left(2ky(\nu)+\tfrac{2\pi}{\varepsilon}(\tau+s)\right)\right. \\
& \quad \left.+\sin\left(\tfrac{2\pi}{\varepsilon}(\tau-s)\right)\right]\dot{\tilde{\theta}}(\nu)\mathrm{d}\nu\mathrm{d}s\mathrm{d}\tau, \\
& Y_3(t)+Y_4(t) = -\tfrac{2\pi\alpha k^2H^2}{\varepsilon^2} \\
& \quad \times\int_{t-\varepsilon}^{t}\int_{\tau}^{t}\int_{s}^{t}[\cos\left(2ky(\nu)+\tfrac{2\pi}{\varepsilon}(\tau+s)\right)]\tilde{\theta}^2(\nu)\dot{\tilde{\theta}}(\nu)\mathrm{d}\nu.
\end{aligned}$$

Then, by using Theorem 1, we have the following corollary.

*Corollary 1:* Let **A1-A2** be satisfied. Consider the system (8) with $|\tilde{\theta}(0)|\leq\sigma_0$. Given tuning parameters $\alpha$, $k$ and $\sigma>\sigma_0>0$, let there exists $\varepsilon^*>0$ that satisfy $\Phi_2<0$ in (25) with $D=0$ and

$$W(\alpha,\varepsilon) = \left(\tfrac{1}{2}+\tfrac{2\pi}{3}+\tfrac{2\pi kH_{\mathrm{M}}\sigma^2}{3}\right)\sqrt{2\pi\alpha\varepsilon}. \tag{29}$$

Then for all $\varepsilon\in(0,\varepsilon^*]$, the solution of system (8) with $|\tilde{\theta}(0)|\leq\sigma_0$ will exponentially converge to the interval

$$\left\{\tilde{\theta}(t)\in\mathbf{R}:\left|\tilde{\theta}(t)\right|<W(\alpha,\varepsilon)+\sqrt{\tfrac{\pi\alpha\varepsilon}{2}}\right\}$$

with a decay rate $\delta=\frac{\alpha kH_{\mathrm{m}}}{2}$, where $W(\alpha,\varepsilon)$ is given by (29).

## IV. EXAMPLES

Consider the single-input map (4) with $\theta^*=0$ and

$$H=2. \tag{30}$$

If $H>0$ is unknown and satisfies **A2**, we consider

$$1.0 \leq H \leq 3.0. \tag{31}$$

Case 1: When $D=0$, we select the tuning parameters as [15]

$$\alpha=0.0001,\ k=11. \tag{32}$$

The results that follow from Corollary 1 and Theorem 3 (Corollary 2 for unknown $H$) in [15] are shown in Table I. By comparing the data, we find that our results allow much larger upper bound $\varepsilon^*$ (lower bound frequency bound $\omega^*$) and much smaller UB than those in [15]. Moreover, our results allow larger uncertainties in $H$ than that in [15].

TABLE I
COMPARISON OF $\varepsilon^*$ AND UB IN SCALAR SYSTEMS WITH $D=0$

| BES | $\sigma_0$ | $\sigma$ | $\delta$ | $\varepsilon^*$ | UB |
|---|---|---|---|---|---|
| Corollary 1 with (30) | 1 | 2 | $1.1\cdot10^{-3}$ | 0.0754 | 0.017 |
| [15] with (30) | 1 | 2 | $1.0\cdot10^{-3}$ | 0.013 | 1.52 |
| Corollary 1 with (31) | 1 | 2 | $0.55\cdot10^{-3}$ | 0.02 | 0.011 |
| [15] with (31) | 1 | 2 | - | - | - |

Case 2: When $D=0.5$, we select the tuning parameters

$$\mu=0.001,\ k=5,\ p=1.5,\ \alpha=0.001\varepsilon^{1.5}.$$

The results of Theorem 1 are shown in Table II. It follows that our method performs well in the presence of large delay.

TABLE II
VALUES OF $\delta$, $\varepsilon^*$ AND UB IN SCALAR SYSTEMS WITH $D=0.5$

| BES | $\sigma_0$ | $\sigma$ | $\delta$ | $\varepsilon^*$ | UB |
|---|---|---|---|---|---|
| Theorem 1 with (30) | 0.5 | 1 | $0.19\cdot10^{-3}$ | 0.1148 | 0.0186 |
| Theorem 1 with (31) | 0.5 | 1 | $0.42\cdot10^{-4}$ | 0.0659 | 0.0091 |

## V. CONCLUSION

This paper developed a time-delay approach to gradient-based bounded ES with a large measurement delay. By employing the solution representation formula, explicit conditions in terms of inequalities were established to guarantee the practical stability of the ES control systems. The resulting time-delay method provides a quantitative analysis of the control parameters and the ultimate bound of seeking error. Compared with the L-K method utilized in [15], the established method not only greatly simplifies the stability conditions and improves the results, but also allows large time delay. Future work includes the extension to multi-variable bounded ES in the presence of large delays.

## APPENDIX: PROOF OF THEOREM 1

The proof is divided into three parts. (A) First, we give a group of upper bounds under the assumption that $\tilde{\theta}(t)$ is bounded for $t \geqslant D$; (B) Second, we employ the solution representation formula for delay differential equations on $z$-system (22) for the practical stability (and thus $\theta$-system (8)); (C) Third, we show the availability of the assumption that $\tilde{\theta}(t)$ is bounded for $t \geqslant D$ by contradiction.

*Proof of part A.* Assume that

$$\left|\tilde{\theta}(t)\right|<\sigma,\ t\geq D. \tag{33}$$

When $t\in[0,D]$, we note that $\tilde{\theta}(t)$ is a constant satisfying

$$\left|\tilde{\theta}(t)\right|=\left|\tilde{\theta}(0)\right|=\left|\tilde{\theta}(D)\right|\leq\sigma_0<\sigma,\ t\in[0,D]. \tag{34}$$

Via (7), we have

$$\left|\dot{\tilde{\theta}}(t)\right|\leq\sqrt{\tfrac{2\pi\alpha}{\varepsilon}},\ t\geq 0, \tag{35}$$

by which and $\alpha=\mu\varepsilon^p$, we further have

$$\begin{aligned}\left|\tilde{\theta}(t)\right|&=\left|\tilde{\theta}(D)+\textstyle\int_D^t\dot{\tilde{\theta}}(s)\mathrm{d}s\right|\\ &\leq\left|\tilde{\theta}(D)\right|+(D+\varepsilon)\sqrt{2\pi\mu}\varepsilon^{\frac{p-1}{2}},\ t\in[D,2D+\varepsilon].\end{aligned} \tag{36}$$

This implies the first inequality in (27) since $\Phi_2<0$ in (25) implies that $\sigma_0+(D+\varepsilon^*)\sqrt{2\pi\mu}\varepsilon^{*\frac{p-1}{2}}<\sigma$.

When $t\geq 2D+\varepsilon$, we have from (18), (20) and (33)-(35) that

$$\begin{aligned}|G(t)|&\leq\tfrac{1}{\varepsilon}\textstyle\int_{t-\varepsilon}^t\left|(\tau-t+\varepsilon)\dot{\tilde{\theta}}(\tau)\right|\mathrm{d}\tau\\ &\leq\tfrac{1}{\varepsilon}\sqrt{\tfrac{2\pi\alpha}{\varepsilon}}\textstyle\int_{t-\varepsilon}^t(\tau-t+\varepsilon)\mathrm{d}\tau\\ &=\sqrt{\tfrac{\pi\alpha\varepsilon}{2}},\end{aligned} \tag{37}$$

$$|Y_1(t)|<\tfrac{\alpha k^2H^2}{2}\sigma^2\sqrt{\tfrac{2\pi\alpha}{\varepsilon}}\textstyle\int_{t-D}^t 1\mathrm{d}s=k^2H^2\sigma^2D\sqrt{\tfrac{\pi\alpha^3}{2\varepsilon}}, \tag{38}$$

$$\begin{aligned}|Y_2(t)|&\leq\tfrac{\pi\alpha kH}{\varepsilon^2}2\sqrt{\tfrac{2\pi\alpha}{\varepsilon}}\textstyle\int_{t-\varepsilon}^t\int_\tau^t\int_s^t 1\mathrm{d}\nu\mathrm{d}s\mathrm{d}\tau\\ &=\tfrac{\pi\alpha kH}{\varepsilon^2}2\sqrt{\tfrac{2\pi\alpha}{\varepsilon}}\tfrac{\varepsilon^3}{6}=\tfrac{\pi\alpha kH}{3}\sqrt{2\pi\alpha\varepsilon},\end{aligned} \tag{39}$$

$$\begin{aligned}|Y_3(t)|&<\tfrac{\pi\alpha k^2H^2}{\varepsilon^2}2\sigma^2\sqrt{\tfrac{2\pi\alpha}{\varepsilon}}\textstyle\int_{t-\varepsilon}^t\int_\tau^t\int_s^t 1\mathrm{d}\nu\mathrm{d}s\mathrm{d}\tau\\ &=\tfrac{\pi\alpha k^2H^2}{\varepsilon^2}2\sigma^2\sqrt{\tfrac{2\pi\alpha}{\varepsilon}}\tfrac{\varepsilon^3}{6}=\tfrac{\pi\alpha k^2H^2\sigma^2}{3}\sqrt{2\pi\alpha\varepsilon}\end{aligned} \tag{40}$$

and

$$\begin{aligned}|Y_4(t)|&<\tfrac{\pi\alpha k^2H^2}{\varepsilon^2}2\sigma^2\sqrt{\tfrac{2\pi\alpha}{\varepsilon}}\textstyle\int_{t-\varepsilon}^t\int_\tau^t\int_s^t 1\mathrm{d}\nu\mathrm{d}s\mathrm{d}\tau\\ &=\tfrac{\pi\alpha k^2H^2}{\varepsilon^2}2\sigma^2\sqrt{\tfrac{2\pi\alpha}{\varepsilon}}\tfrac{\varepsilon^3}{6}=\tfrac{\pi\alpha k^2H^2\sigma^2}{3}\sqrt{2\pi\alpha\varepsilon}.\end{aligned} \tag{41}$$

By using (37)-(41) and $G(t)=0,\ t\in[D+\varepsilon,2D+\varepsilon)$ in (21), we find from the second equation in (22) that

$$\begin{aligned}|w(t)|&\leq\left|\tfrac{\alpha kH}{2}G(t-D)\right|+\textstyle\sum_{i=1}^4|Y_i(t)|\\ &<\tfrac{\alpha kH}{4}\sqrt{2\pi\alpha\varepsilon}+k^2H^2\sigma^2D\sqrt{\tfrac{\pi\alpha^3}{2\varepsilon}}\\ &\quad+\tfrac{\pi\alpha kH}{3}\sqrt{2\pi\alpha\varepsilon}+\tfrac{2\pi\alpha k^2H^2\sigma^2}{3}\sqrt{2\pi\alpha\varepsilon}\\ &=\left(\tfrac{\alpha kH}{4}+\tfrac{\pi\alpha kH}{3}+\tfrac{2\pi\alpha k^2H^2\sigma^2}{3}\right)\sqrt{2\pi\alpha\varepsilon}\\ &\quad+k^2H^2\sigma^2D\sqrt{\tfrac{\pi\alpha^3}{2\varepsilon}}\\ &\leq\tfrac{\alpha kH}{2}W(\alpha,\varepsilon),\ t\geq 2D+\varepsilon\end{aligned} \tag{42}$$

with $W(\mu,\varepsilon)$ given by (26). In addition, via (21), (36), (37) and $\alpha=\mu\varepsilon^p$, we find

$$\begin{aligned}|z(t)| &\le \left|\tilde{\theta}(t)\right|+|G(t)|\\ &\leqslant \left|\tilde{\theta}(D)\right|+(D+\varepsilon)\sqrt{2\pi\mu}\varepsilon^{\frac{p-1}{2}}+\sqrt{\tfrac{\pi\alpha\varepsilon}{2}}\\ &= \left|\tilde{\theta}(D)\right|+\varepsilon^{\frac{p-1}{2}}\left(D+\tfrac{3\varepsilon}{2}\right)\sqrt{2\pi\mu},\ t\in[D+\varepsilon,2D+\varepsilon].\end{aligned} \tag{43}$$

*Proof of part B.* Define $X(t)$ as the solution of the following homogeneous equation

$$\dot{z}(t)=-\frac{\alpha kH}{2}z(t-D),\ z(t)=0,t<0,\ z(0)=1. \tag{44}$$

By using Lemma 1, under the condition $\Phi_1\le 0$ in (25), there hold

$$0<X(t)\le\begin{cases}1, & 0\le t\le D,\\ \mathrm{e}^{-\frac{\alpha kH}{2}(t-D)}, & t\ge D.\end{cases} \tag{45}$$

By using (3) in Lemma 1 for (22) we further have

$$\begin{aligned}z(t)=&X(t-2D-\varepsilon)z(2D+\varepsilon)\\ &-\tfrac{\alpha kH}{2}\int_{D+\varepsilon}^{2D+\varepsilon}X(t-s-D)\varphi(s)\mathrm{d}s\\ &+\int_{2D+\varepsilon}^{t}X(t-s)w(s)\mathrm{d}s,\end{aligned} \tag{46}$$

where $\varphi(s)=z(s)$ if $D+\varepsilon\le s\le 2D+\varepsilon$. Then when $t\in[2D+\varepsilon,3D+\varepsilon]$, via (42)-(43) and (45)-(46), we get

$$\begin{aligned}|z(t)|\le&|X(t-2D-\varepsilon)|\,|z(2D+\varepsilon)|\\ &+\tfrac{\alpha kH}{2}\int_{D+\varepsilon}^{2D+\varepsilon}|X(t-s-D)|\,|\varphi(s)|\,\mathrm{d}s\\ &+\int_{2D+\varepsilon}^{t}|X(t-s)|\,|w(s)|\,\mathrm{d}s\\ <&\left[\left|\tilde{\theta}(D)\right|+\varepsilon^{\frac{p-1}{2}}\left(D+\tfrac{3\varepsilon}{2}\right)\sqrt{2\pi\mu}\right]|X(t-2D-\varepsilon)|\\ &+\tfrac{\alpha kH}{2}\left[\left|\tilde{\theta}(D)\right|+\varepsilon^{\frac{p-1}{2}}\left(D+\tfrac{3\varepsilon}{2}\right)\sqrt{2\pi\mu}\right]\\ &\times\int_{2D+\varepsilon}^{3D+\varepsilon}|X(t-s)|\,\mathrm{d}s+\tfrac{\alpha kH}{2}W(\alpha,\varepsilon)\int_{2D+\varepsilon}^{3D+\varepsilon}|X(t-s)|\,\mathrm{d}s\\ <&\left(1+\tfrac{\alpha kH_{\mathrm{M}}D}{2}\right)\left[\left|\tilde{\theta}(D)\right|+\varepsilon^{\frac{p-1}{2}}\left(D+\tfrac{3\varepsilon}{2}\right)\sqrt{2\pi\mu}\right]\\ &+\tfrac{\alpha kH_{\mathrm{M}}D}{2}W(\alpha,\varepsilon),\end{aligned} \tag{47}$$

by which, (21) and (37), we further have

$$\begin{aligned}\left|\tilde{\theta}(t)\right|<&\left(1+\tfrac{\mu\varepsilon^p kH_{\mathrm{M}}D}{2}\right)\left[\left|\tilde{\theta}(D)\right|+\varepsilon^{\frac{p-1}{2}}\left(D+\tfrac{3\varepsilon}{2}\right)\right.\\ &\left.\times\sqrt{2\pi\mu}\right]+\tfrac{\mu\varepsilon^p kH_{\mathrm{M}}D}{2}W(\mu,\varepsilon)+\varepsilon^{\frac{p+1}{2}}\sqrt{\tfrac{\pi\mu}{2}},\end{aligned} \tag{48}$$

which implies the second inequality in (27) due to $\Phi_2<0$ in (25) since $\mathrm{e}^{\frac{\alpha kH_{\mathrm{M}}D}{2}}\ge 1+\frac{\alpha kH_{\mathrm{M}}D}{2}$.

When $t\ge 3D+\varepsilon$, via (42)-(43) and (45)-(46), we further have

$$\begin{aligned}|z(t)|<&\mathrm{e}^{-\frac{\alpha kH}{2}(t-3D-\varepsilon)}\left[\left|\tilde{\theta}(D)\right|+\varepsilon^{\frac{p-1}{2}}\left(D+\tfrac{3\varepsilon}{2}\right)\sqrt{2\pi\mu}\right]\\ &+\tfrac{\alpha kH}{2}\left[\left|\tilde{\theta}(D)\right|+\varepsilon^{\frac{p-1}{2}}\left(D+\tfrac{3\varepsilon}{2}\right)\sqrt{2\pi\mu}\right]\\ &\times\int_{2D+\varepsilon}^{3D+\varepsilon}\mathrm{e}^{-\frac{\alpha kH}{2}(t-s-D)}\mathrm{d}s+\tfrac{\alpha KH}{2}W(\alpha,\varepsilon)\int_{2D+\varepsilon}^{t}|X(t-s)|\,\mathrm{d}s\\ \le&\mathrm{e}^{-\frac{\alpha kH}{2}(t-3D-\varepsilon)}\left[\left|\tilde{\theta}(D)\right|+\varepsilon^{\frac{p-1}{2}}\left(D+\tfrac{3\varepsilon}{2}\right)\sqrt{2\pi\mu}\right]\\ &+\mathrm{e}^{-\frac{\alpha kH}{2}(t-3D-\varepsilon)}\left(\mathrm{e}^{\frac{\alpha kHD}{2}}-1\right)\left[\left|\tilde{\theta}(D)\right|+\varepsilon^{\frac{p-1}{2}}\right.\\ &\left.\times\left(D+\tfrac{3\varepsilon}{2}\right)\sqrt{2\pi\mu}\right]+W(\alpha,\varepsilon)\left[\mathrm{e}^{\frac{\alpha kHD}{2}}-\mathrm{e}^{-\frac{\alpha kH}{2}(t-3D-\varepsilon)}\right]\\ \le&\mathrm{e}^{-\frac{\alpha kH_{\mathrm{m}}}{2}(t-3D-\varepsilon)}\mathrm{e}^{\frac{\alpha kH_{\mathrm{M}}D}{2}}\left[\left|\tilde{\theta}(D)\right|+\varepsilon^{\frac{p-1}{2}}\left(D+\tfrac{3\varepsilon}{2}\right)\sqrt{2\pi\mu}\right]\\ &+\mathrm{e}^{\frac{\alpha kH_{\mathrm{M}}D}{2}}W(\alpha,\varepsilon),\end{aligned} \tag{49}$$

where we have used $\mathrm{e}^{\frac{\alpha kHD}{2}}\ge 1+\frac{\alpha kHD}{2}$ and

$$\begin{aligned}\int_{2D+\varepsilon}^{t}|X(t-s)|\mathrm{d}s&=\int_{2D+\varepsilon}^{t-D}|X(t-s)|\mathrm{d}s+\int_{t-D}^{t}|X(t-s)|\mathrm{d}s\\ &\leqslant\int_{2D+\varepsilon}^{t-D}\mathrm{e}^{-\frac{\alpha kH}{2}(t-s-D)}\mathrm{d}s+D\\ &=\tfrac{2}{\alpha KH}\left[1-\mathrm{e}^{-\frac{\alpha kH}{2}(t-3D-\varepsilon)}\right]+D.\end{aligned}$$

By (21), (37) and (49), we have

$$\begin{aligned}\left|\tilde{\theta}(t)\right|<&\mathrm{e}^{-\frac{\mu\varepsilon^p kH_{\mathrm{m}}}{2}(t-3D-\varepsilon)}\mathrm{e}^{\frac{\mu\varepsilon^p kH_{\mathrm{M}}D}{2}}\left[\left|\tilde{\theta}(D)\right|\right.\\ &\left.+\varepsilon^{\frac{p-1}{2}}\left(D+\tfrac{3\varepsilon}{2}\right)\sqrt{2\pi\mu}\right]+\mathrm{e}^{\frac{\mu\varepsilon^p kH_{\mathrm{M}}D}{2}}W(\alpha,\varepsilon)+\varepsilon^{\frac{p+1}{2}}\sqrt{\tfrac{\pi\mu}{2}},\end{aligned}$$

which implies the third inequality in (27) due to $\Phi_2<0$ in (25).

*Proof of part C.* By contradiction-based arguments in [15] (see Appendix A), it can be proved that (25) guarantees (33). The proof is finished.

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