

Credit-Based Congestion Pricing: Equilibrium Properties and Optimal Scheme Design

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Abstract—Credit-based congestion pricing (CBCP) has emerged as a mechanism to alleviate the social inequity concerns of road congestion pricing — a promising strategy for traffic congestion mitigation — by providing low-income users with travel credits to offset some of their toll payments. While CBCP offers immense potential for addressing inequity issues that hamper the practical viability of congestion pricing, the deployment of CBCP in practice is nascent, and the potential efficacy and optimal design of CBCP schemes have yet to be formalized. In this work, we study the design of CBCP schemes to achieve particular societal objectives and investigate their influence on traffic patterns when routing heterogeneous users with different values of time (VoTs) on a multi-lane highway with an express lane (EL). To this end, we introduce a new non-atomic congestion game model of a *mixed-economy*, wherein eligible users receive travel credits while the remaining ineligible users pay *out-of-pocket* to use the EL. In this setting, we investigate the effect of CBCP schemes on traffic patterns by characterizing the properties (i.e., existence, comparative statics) of the corresponding Nash equilibria and, in the setting when eligible users have time-invariant VoTs, develop a convex program to compute these equilibria. We further present a bi-level optimization framework to design optimal CBCP schemes to achieve a central planner’s societal objectives. Finally, we conduct numerical experiments based on a case study of the San Mateo 101 Express Lanes Project, one of the first CBCP pilots. Our results demonstrate the potential of CBCP to enable low-income users to avail of the travel time savings provided by congestion pricing on ELs while having comparatively low impacts on the travel costs of other road users.

I. INTRODUCTION

With the ever-worsening traffic congestion in urban metropolises, congestion pricing has emerged as one of the most promising traffic management policies to reduce system inefficiencies caused by selfish travel behavior [1]. While network-wide deployments of congestion pricing, wherein tolls are placed on all or some cordoned portion of roads in the network, are less common at present, there has been a growing interest in introducing congestion fees on certain lanes on highways, known as *express lanes* (ELs), to provide users with a faster and more reliable travel option during peak traffic periods. In the Bay Area alone, there are more than 155 miles of ELs at the time of writing this manuscript [2], and in many cases, existing *high-occupancy vehicle* (HOV) lanes, which only grant access to vehicles with more than two or three passengers, have been converted to *high-occupancy toll* (HOT) lanes that enable single-occupant vehicles (SOVs) to pay for access.

Despite the proliferation of ELs to better manage highway traffic, congestion fees on ELs, as with network-wide

congestion pricing, have come under scrutiny due to social inequity concerns. In particular, ELs have been termed as elitist “Lexus lanes”, as they offer only those with the highest willingness to pay (i.e., the most wealthy) a higher quality of service through reduced travel times [3] while lower-income users bear the brunt of longer travel times on more congested general purpose lanes (GPLs). Thus, there has been a growing interest in designing equitable congestion pricing schemes, with a focus on *credit-based congestion pricing* (CBCP) [4], under which road users, particularly those with lower incomes, are given travel credits to use priced roads. Although numerous variations of CBCP have been explored, demonstrating their potential to provide positive equity benefits, their deployment in practice is nascent, with one of the first pilots launching in San Mateo County, California, in 2022. The *San Mateo 101 Express Lanes Project* recently launched the “Community Transportation Benefits Program” [5], which provides low-income residents with travel credits for using the EL. While CBCP programs, such as the one in San Mateo County, offer great potential to improve equity outcomes, a principled design of CBCP schemes is necessary to realize the benefits of its implementation, which is the focus of this work.

Contributions: We study CBCP schemes to route heterogeneous users with different values of time (VoTs) on a multi-lane highway with a tolled EL. In alignment with practically deployed CBCP schemes, such as in San Mateo County, we introduce a new model of a *mixed-economy* wherein eligible users receive travel credits while ineligible users pay *out-of-pocket* to use the EL (see Section III). Our mixed-economy model is unlike traditional *single-economy* settings in traffic routing, wherein either all users have quasi-linear costs [6] or budget constraints [7] as in markets with artificial currencies. Given the different optimization objectives of eligible and ineligible users, we initiate the study of mixed-economy traffic routing settings through an introduction of CBCP equilibria, an investigation of its properties to study the influence of CBCP schemes on traffic patterns, and a framework for designing optimal schemes.

In particular, we first establish the existence of CBCP equilibria and, in the setting when eligible users have time-invariant VoTs, develop a convex program to compute CBCP equilibria (see Section IV).

We then develop a bi-level optimization framework for designing optimal CBCP schemes to achieve specific societal objectives of a central planner in the setting when eligible users have time-invariant VoTs (see Section V). To solve the corresponding bi-level optimization problem, we present a *dense sampling* approach for computing an approximation to the optimal CBCP scheme that involves discretizing the set of feasible CBCP schemes and choosing the scheme that

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induces an equilibrium with the minimum societal cost.

Finally, we present numerical experiments based on a case study of the San Mateo 101 Express Lanes Project. Our results indicate that the optimal CBCP scheme can vary widely based on the central planner's objective, thereby highlighting that a principled approach using bi-level optimization is key to realizing the benefits of CBCP.

In the extended version of our paper [8], we present omitted proofs, our comparative statics analysis, policy implications of our study, implementation details of our experiments, and additional experimental results.

II. RELATED LITERATURE

There has been a growing interest in designing equitable congestion pricing schemes, with a focus on revenue redistribution, wherein the collected toll revenues are refunded as lump-sum transfers to users. As with the study of revenue refunding schemes [9], we also consider refunding a proportion of the collected revenues to users. However, compared to these works, we consider CBCP, wherein the credits function solely as a travel allowance for using an EL rather than additional money *in the pocket* of users.

CBCP schemes have been explored extensively with a focus on *tradable* credit schemes wherein additional credits may be purchased [10] or earned by exhibiting desirable travel behavior [11]. Moreover, these credits typically have monetary value beyond paying for access to priced facilities (e.g., for using public transit) [4]. However, the equity improvements from tradable CBCP typically result from lower-income users that reduce travel to sell their allocated credits or use them on driving alternatives [4], [11]. In contrast, we consider non-tradable CBCP wherein credits cannot be traded nor provide value for anything other than for paying EL tolls, as with the San Mateo 101 Express Lanes Project. Thus, as opposed to tradable CBCP, wherein low-income users generally use less convenient modes of travel to obtain monetary gains from selling excess credits, under non-tradable CBCP, eligible users will avail of a fast and reliable mode of travel, i.e., the EL.

Since travel credits provide no value to users beyond paying for EL tolls, our work is also closely related to artificial currency mechanisms [7]. However, in contrast to traditional artificial currency applications, we consider a mixed-economy wherein only a fraction of the users receive travel credits (artificial currencies).

From a methodological viewpoint, as in prior studies that characterize user equilibria in congestion games [6], [12], we also investigate the properties of equilibria induced by CBCP schemes. However, CBCP equilibria differ markedly from prior equilibrium notions in the congestion pricing literature (see Section III-C). Beyond studying equilibrium properties, we also develop a bi-level framework to optimize over CBCP schemes. While bi-level optimization [13] has been studied in many traffic routing contexts, e.g., second-best tolling [14], our bi-level framework involves optimizing over both tolls and budgets rather than only road tolls.

III. MODEL

In this section, we introduce the basic definitions of traffic flow (Section III-A), the operation of CBCP schemes and

corresponding user costs (Section III-B), and the notion of CBCP equilibria (Section III-C).

A. Preliminaries

We study the problem of designing CBCP schemes to route heterogeneous users with different VoTs in a multi-lane highway section. The highway consists of one EL that can be tolled while the remaining GPLs remain untolled. We focus on studying CBCP schemes for a multi-lane highway segment, as opposed to general road networks, in alignment with practically deployed congestion pricing schemes and, in particular, the San Mateo 101 Express Lanes Project. Without loss of generality, we model the freeway section as a two-edge Pigou network consisting of a source vertex s , a destination vertex d , and two directed edges $e \in \{1, 2\}$ between the source and destination vertices, where the first edge ($e = 1$) denotes the EL while the second edge ($e = 2$) corresponds to the GPLs. We note that modeling all GPLs as a single edge is without loss of generality, as we focus on equilibrium formation in this work; hence, these lanes are indistinguishable for users as none of these lanes are tolled. Furthermore, to model the travel times on each edge e , we consider a flow-dependent travel-time (latency) function $l_e : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, which maps x_e , the traffic flow rate on edge e , to the travel time $l_e(x_e)$. As in prior literature on traffic routing, we assume that the function l_e , for both edges e , is differentiable, convex, and monotonically increasing.

Users make trips over T periods (e.g., days) over which the CBCP scheme is run (see Section III-B) and belong to a finite set of discrete groups characterized by their (i) level of income and (ii) VoT. Let \mathcal{G} denote the set of all user groups, wherein users, based on their income, are subdivided into two categories, *eligible* and *ineligible*, depending on their eligibility to receive travel credits (i.e., a budget) to use the EL as determined by a central planner. We let \mathcal{G}_E and \mathcal{G}_I denote the sets of eligible and ineligible user groups, respectively. Further, a user in a group $g \in \mathcal{G}$ at each period $t \in [T]$ has a VoT $v_{t,g}$, which captures users' willingness to pay for travel time savings. The total travel demand of a user group g is given by d_g , which represents the flow of users in group g to be routed at each period. We assume that the travel demand for each user group stays fixed across time, as is consistent with weekday rush hour traffic wherein users commute to and from work. For the simplicity of exposition, we normalize d_g to one for all groups g and note that our results naturally extend to the general travel demands setting.

A flow pattern $\mathbf{y} = \{y_{e,t}^g : e \in \{1, 2\}, t \in [T], g \in \mathcal{G}\}$ specifies for each user group g the amount of flow $y_{e,t}^g \geq 0$ routed on edge e at period t . The resulting flows must satisfy the user demand at each period t , i.e., $\sum_{e=1}^2 y_{e,t}^g = 1$, for all $g \in \mathcal{G}, t \in [T]$. Furthermore, we represent the edge flows corresponding to the flow pattern \mathbf{y} by the vector $\mathbf{x} = \{x_{e,t} : e \in \{1, 2\}, t \in [T]\}$, where $x_{e,t} = \sum_{g \in \mathcal{G}} y_{e,t}^g$, for all $e \in \{1, 2\}, t \in [T]$.

B. CBCP Schemes and User Optimization

A CBCP scheme is characterized by a tuple $(\boldsymbol{\tau}, B)$, where $\boldsymbol{\tau} \in \mathbb{R}_{\geq 0}^T$ is the vector of tolls on the EL over T periods, and B is the travel credit (budget) given to eligible users to use the EL over the T periods. Both eligible and ineligible users pay the toll when using the EL; however, ineligible users pay

out-of-pocket while eligible users pay using their available budget. For simplicity, we assume that eligible users never spend out-of-pocket to use the EL. Such an assumption is consistent with real-world traffic networks, wherein lower-income users, i.e., those in the eligible group, are less likely to spend out-of-pocket to use tolled roads and thus generally bear the burden of longer travel times [15]. However, this assumption can readily be relaxed to the setting where eligible users also spend out-of-pocket, and we defer a thorough treatment of this setting to future research. Further, the finiteness of the time horizon T is crucial to successfully deploying a CBCP scheme to improve EL access for eligible users as the ratio $\frac{B}{T}$ represents the per-period budget for these users to use the EL. Finally, as in the traffic routing literature [4], [9], we focus on SOVs in this work and defer the consideration of HOVs (e.g., carpools) to future work.

We now present the individual optimization problems for both ineligible and eligible users who seek to minimize their total travel cost given a CBCP scheme (τ, B) .

Ineligible Users: Since ineligible users spend out-of-pocket to use the EL, their cumulative travel cost is assumed to be a linear function of their travel time and tolls, a commonly used modeling assumption [6]. Given a CBCP scheme (τ, B) and a vector of edge flows \mathbf{x} , the individual optimization of an ineligible user in a group $g \in \mathcal{G}_I$ is

$$\begin{aligned} \mu^{g*}(\mathbf{x}, \tau, B) = \min_{\mathbf{z}^g \in \mathbb{R}_{\geq 0}^{2 \times T}} \sum_{t \in [T]} \sum_{e \in [2]} (v_{t,g} l_e(x_{e,t}) + \mathbb{I}_{e=1} \tau_t) z_{e,t}^g, \quad (1a) \\ \text{s.t.} \quad z_{1,t}^g + z_{2,t}^g = 1, \quad \forall t \in [T] \quad (1b) \end{aligned}$$

where (1a) is the travel cost objective of the ineligible users and (1b) are user allocation constraints at each period $t \in [T]$. Observe that Problem (1a)-(1b) is a linear program where $\mathbf{z}^g = \{z_{e,t}^g : e \in \{1, 2\}, t \in [T]\}$ corresponds to the actions of an infinitesimal user and thus does not influence the edge flow \mathbf{x} . Here, we denote the decision variables for any user in group $g \in \mathcal{G}_I$ as \mathbf{z}^g to distinguish it from the cumulative flow $\mathbf{y}^g = \{y_{e,t}^g : e \in \{1, 2\}, t \in [T]\}$ of all users in $g \in \mathcal{G}_I$ and note that the decision variables $z_{e,t}^g \in [0, 1]$ can be interpreted as the fraction of flow or probability that a user in group g uses edge e at period t . Further, for succinctness, we denote the travel cost for users in a group $g \in \mathcal{G}_I$ as $\mu_{\mathbf{z}^g}(\mathbf{x}, \tau, B) = \sum_{t=1}^T \sum_{e=1}^2 (v_{t,g} l_e(x_{e,t}) + \mathbb{I}_{e=1} \tau_t) z_{e,t}^g$, the travel cost on edge e at period t for as $\mu_{e,t}^g(\mathbf{x}, \tau, B) = v_{t,g} l_e(x_{e,t}) + \mathbb{I}_{e=1} \tau_t$, and let $\mu^{g*}(\mathbf{x}, \tau, B)$ denote the minimum travel cost.

Optimal Solution of Problem (1a)-(1b): By the separability of the travel cost function and the constraints across periods, the optimal solution of Problem (1a)-(1b) corresponds to users choosing routes with the minimum travel cost at each period, which is akin to the well-studied model of heterogeneous users in non-atomic congestion games [6].

Eligible Users: On the other hand, since eligible users only utilize travel credit to use the EL, their cumulative travel cost only consists of the travel time component of the cost of the ineligible users. Thus, given a CBCP scheme (τ, B) and a vector of edge flows \mathbf{x} , the individual optimization of an eligible user in a group $g \in \mathcal{G}_E$ is

$$\begin{aligned} \mu^{g*}(\mathbf{x}, \tau, B) = \min_{\mathbf{z}^g \in \mathbb{R}_{\geq 0}^{2 \times T}} \sum_{t \in [T]} \sum_{e \in [2]} v_{t,g} l_e(x_{e,t}) z_{e,t}^g, \quad (2a) \\ \text{s.t.} \quad z_{1,t}^g + z_{2,t}^g = 1, \quad \forall t \in [T] \quad (2b) \\ \sum_{t \in [T]} z_{1,t}^g \tau_t \leq B, \quad (2c) \end{aligned}$$

where (2b) are allocation constraints, and (2c) is the budget constraint that ensures no user spends more credits than the provided allowance. Problem (2a)-(2c) is a linear program and each user's travel cost in a group $g \in \mathcal{G}_E$ is denoted as $\mu_{\mathbf{z}^g}(\mathbf{x}, \tau, B) = \sum_{t=1}^T \sum_{e=1}^2 v_{t,g} l_e(x_{e,t}) z_{e,t}^g$. Unlike ineligible users, the travel decisions of eligible users are coupled across periods due to the budget Constraint (2c).

C. CBCP Equilibria

We evaluate the efficacy of a CBCP scheme in achieving particular societal scale goals of a central planner based on the induced Nash equilibria (see Section V). To this end, we present the Nash equilibrium notion, which we term a CBCP equilibrium, studied in this work. In particular, given a CBCP scheme (τ, B) , a flow pattern \mathbf{y} is a CBCP equilibrium if no user can reduce their travel cost through a unilateral deviation, as is formalized by the following definition.

Definition 1 (CBCP Equilibrium). *For a CBCP scheme (τ, B) , the flow \mathbf{y} , with corresponding edge flows \mathbf{x} , is a CBCP (τ, B) -equilibrium if for each ineligible group $g \in \mathcal{G}_I$ with $y_{e,t}^g > 0$, $\mu_{e,t}^g(\mathbf{x}, \tau, B) \leq \mu_{e',t}^g(\mathbf{x}, \tau, B)$, for all $e' \in \{1, 2\}, t \in [T]$, and for each eligible group $g \in \mathcal{G}_E$ with $y_{e,t}^g > 0$, it holds that $z_{e,t}^{g*} > 0$ for some optimal solution \mathbf{z}^* to Problem (2a)-(2c), i.e., $\mu_{\mathbf{z}^*}^g(\mathbf{x}, \tau, B) \leq \mu_{\mathbf{z}}^g(\mathbf{x}, \tau, B)$, for all $\mathbf{z} \geq \mathbf{0}$ satisfying Constraints (2b)-(2c).*

A few comments about this equilibrium notion are in order. First, if all users are ineligible, CBCP equilibria reduce to standard non-atomic Nash equilibria with heterogeneous users [6]. Next, since Definition 1 accounts for the preferences of eligible users whose travel decisions are coupled across the periods through their budget constraints, it differs from prior works on the single-economy setting wherein all users have quasi-linear costs [6]. Finally, the cumulative flow \mathbf{y} of all users represents the sum of their individual probabilities (or fraction of flow on each edge) and we note that without loss of generality it suffices to focus on CBCP equilibrium flows \mathbf{y} such that for eligible (ineligible) user group $g \in \mathcal{G}_E$ ($g \in \mathcal{G}_I$), $y_{e,t}^g = z_{e,t}^{g*}$ for some optimal solution \mathbf{z}^* to Problem (2a)-(2c) (Problem (1a)-(1b)), as all users in a given group incur the same travel cost at any CBCP equilibrium. For a further discussion on CBCP equilibria, see the extended version of our paper [8].

IV. PROPERTIES OF CBCP EQUILIBRIA

We initiate our study of CBCP schemes in a mixed-economy setting by studying the properties of CBCP equilibria. In particular, we establish the existence of CBCP equilibria (Section IV-A) and present a convex program to compute CBCP equilibria in the setting when eligible users' VoTs are time-invariant (Section IV-B).

A. Equilibrium Existence and Edge Flow Uniqueness

We show that CBCP equilibria exist in the general setting when all users have time-varying VoTs.

Theorem 1 (Existence of CBCP Equilibria). *For any CBCP scheme (τ, B) , where $\tau \geq \mathbf{0}$ and $B \geq 0$, there exists a CBCP (τ, B) -equilibrium.*

Theorem 1 establishes that introducing eligible user budgets does not preclude the existence of an equilibrium and augments the literature on investigating equilibrium existence under congestion pricing alternatives [16], [17]. Further, Theorem 1 extends equilibrium existence results in traffic routing focusing on single-economies [6] to mixed-economies. To prove Theorem 1, we develop the following variational inequality characterization of CBCP equilibria.

Lemma 1 (Variational Inequality Characterization of CBCP Equilibria). *For a CBCP scheme (τ, B) , a flow $\mathbf{y}^* = (y_{e,t}^{g*})$, with edge flows \mathbf{x}^* , is a CBCP (τ, B) -equilibrium if and only if it solves the following variational inequality problem:*

$$\sum_{t \in [T]} \sum_{e \in [2]} \left[\sum_{g \in \mathcal{G}_I} (v_{t,g} l_e(x_{e,t}^*) + \mathbb{I}_{e=1} \tau_t) (y_{e,t}^g - y_{e,t}^{g*}) \right. \quad (3)$$

$$\left. + \sum_{g \in \mathcal{G}_E} v_{t,g} l_e(x_{e,t}^*) (y_{e,t}^g - y_{e,t}^{g*}) \right] \geq 0, \forall \text{ feasible } \mathbf{y} \in \Omega,$$

where the set Ω is described by $\mathbf{y} \geq \mathbf{0}$, $y_{1,t}^g + y_{2,t}^g = 1$ for all $t \in [T]$ and $g \in \mathcal{G}$, and all eligible users satisfy their budget Constraint (2c).

The proof of Lemma 1 involves arguments similar to that in [18], as the variational Inequality (3) is reminiscent of the variational inequalities used to study heterogeneous user equilibria in classical traffic routing settings [6], [17], [18]. However, in contrast to these approaches that consider a single-economy setting, Equation (3) has two separate terms to capture the differing travel costs for the eligible and ineligible users in a mixed-economy setting.

We now complete the proof of Theorem 1 by showing that the variational Inequality (3) admits a feasible solution.

Lemma 2 (Feasibility of Variational Inequality). *There exists a solution \mathbf{y}^* to the variational Inequality (3).*

Proof. First, note that the variational Inequality (3) can be expressed in standard form $F(\mathbf{y}^*)^T(\mathbf{y} - \mathbf{y}^*)$ for all feasible \mathbf{y} , where $F = (F_{e,t}^g)_{e \in \{1,2\}, t \in [T], g \in \mathcal{G}}$, $F_{e,t}^g(\mathbf{y}^*) = v_{t,g} t_e(x_{e,t}^*)$ for $g \in \mathcal{G}_E$ and $F_{e,t}^g(\mathbf{y}^*) = v_{t,g} t_e(x_{e,t}^*) + \mathbb{I}_{e=1} \tau_t$ for $g \in \mathcal{G}_I$. Then, following standard variational inequality theory [19], [20], a feasible solution \mathbf{y}^* to Equation (3) exists as the feasible set Ω , defined in Lemma 1, is compact and the travel time functions are continuous. \square

Lemmas 1 and 2 jointly imply Theorem 1.

B. Convex Program to Compute Equilibria

While Theorem 1 established the existence of CBCP equilibria, determining a feasible solution to the variational Inequality (3) may, in general, be challenging. Given the difficulty in solving variational inequalities, we now present a convex program to compute CBCP equilibria in the setting when eligible users have time-invariant VoTs, i.e., for all $g \in \mathcal{G}_E$, $v_{t,g} = v_{t',g} > 0$ for all $t, t' \in [T]$, while ineligible users can, in general, have time-varying VoTs. While eligible users' VoTs can, in practice, vary over time, we defer the question of computing CBCP equilibria when eligible users' VoTs vary with time to future research and

note that the time-invariance of eligible users' VoTs has important practical significance. First, since the EL is likely to be tolled during morning and evening rush hour periods on weekdays, the VoTs of users commuting to and from work are unlikely to differ much between one period and the next, e.g., between subsequent days. Furthermore, the individual optimization Problem (2a)-(2c) for the eligible users can involve quite sophisticated decision-making as eligible users' travel decisions are coupled across periods. Since users may not have complete information on their VoT over the T periods, eligible users may prefer to minimize their total travel time rather than the more complex Objective (2a).

We compute CBCP (τ, B) -equilibria when eligible users have time-invariant VoTs with the following convex program

$$\mathbf{y} \in \mathbb{R}_{\geq 0}^{2 \times T \times |\mathcal{G}|} \quad \min \left[\sum_{t \in [T]} \left[\sum_{e \in [2]} \int_0^{x_{e,t}} l_e(\omega) d\omega + \sum_{g \in \mathcal{G}_I} \frac{y_{1,t}^g \tau_t}{v_{t,g}} \right] \right], \quad (4a)$$

$$\text{s.t.} \quad y_{1,t}^g + y_{2,t}^g = 1, \quad \forall t \in [T], g \in \mathcal{G} \quad (4b)$$

$$\sum_{t \in [T]} \tau_t y_{1,t}^g \leq B, \quad \forall g \in \mathcal{G}_E, \quad (4c)$$

$$\sum_{g \in \mathcal{G}} y_{e,t}^g = x_{e,t}, \quad \forall e \in E, t \in [T], \quad (4d)$$

where (4b) are allocation constraints, (4c) are eligible user budget constraints, and (4d) are edge flow constraints. Problem (4a)-(4d) is akin to the convex program to compute heterogeneous user equilibria given road tolls [6]. However, as opposed to the convex program in [6] that considers a single-economy setting, Problem (4a)-(4d), which applies to a mixed-economy setting, only has a toll component in the Objective (4a) for ineligible users and instead has a budget Constraint (4c) for eligible users. We now show that any solution of Problem (4a)-(4d) is a CBCP (τ, B) -equilibrium.

Theorem 2 (Convex Program for CBCP Equilibrium Computation). *Consider a CBCP scheme (τ, B) and the setting when the VoTs of all eligible users do not vary with time. Then, the optimal solution \mathbf{y}^* of the convex Program (4a)-(4d) is a CBCP (τ, B) -equilibrium.*

From Theorem 2, note that Problem (4a)-(4d) provides an efficient method to compute CBCP equilibria as it can be solved using computationally tractable approaches, e.g., Frank-Wolfe [21], used for traffic assignment problems.

V. OPTIMAL CBCP SCHEME DESIGN

While an analysis of the properties of CBCP equilibria, as in the previous section, aids in understanding the influence of CBCP schemes on traffic patterns, a central planner is typically interested in deploying an optimal policy to achieve particular societal goals. To this end, in this section, we present a bi-level optimization framework to design optimal CBCP schemes (Section V-A) and develop an algorithmic approach based on dense sampling to compute an approximation to the optimal scheme (Section V-B).

A. Bi-Level Optimization Framework

We now present a bi-level optimization framework for optimal CBCP design to achieve particular societal objectives of a central planner. We focus on the setting when eligible

users have time-invariant VoTs, in which case CBCP equilibria can be computed using Problem (4a)-(4d).

To present the bi-level optimization problem, we model the central planner's societal objective through a cost function $f : \mathbb{R}^{2 \times T \times |\mathcal{G}|} \rightarrow \mathbb{R}$, where $f(\mathbf{y})$ denotes the societal cost of the flow $\mathbf{y} \geq \mathbf{0}$ that lies in a feasible set Ω defined by Constraints (4b)-(4d). Further, we denote $\mathcal{F}_U \subseteq \mathbb{R}_{\geq 0}^{T+1}$ as the set of feasible CBCP schemes (τ, B) . Then, the goal of the central planner is to find a feasible CBCP scheme $(\tau^*, B^*) \in \mathcal{F}_U$ such that the resulting equilibria $\mathbf{y}(\tau^*, B^*)$ has the lowest societal cost among all feasible CBCP schemes, i.e., $f(\mathbf{y}(\tau^*, B^*)) \leq f(\mathbf{y}(\tau, B))$ for all $(\tau, B) \in \mathcal{F}_U$, where $\mathbf{y}(\tau, B)$ is an equilibrium flow given by the solution of Problem (4a)-(4d) for the scheme (τ, B) . In particular, the objective of the central planner can be captured through the following bi-level optimization problem

$$\min_{\substack{\mathbf{y} \in \Omega, \\ (\tau, B) \in \mathcal{F}_U}} f(\mathbf{y}(\tau, B)), \quad (5a)$$

$$\text{s.t. } \mathbf{y}(\tau, B) = \text{Solution of Problem (4a)-(4d)}, \quad (5b)$$

where Constraint (5b) represents the lower-level problem of computing the equilibrium flow given a scheme (τ, B) .

B. Algorithmic Approach for Bi-level Problem

Since solving bi-level programs is, in general, challenging [22], we use *dense sampling* to compute an approximate solution to Problem (5a)-(5b). While our approach is applicable for a broad range of feasibility sets, for simplicity, we suppose that the set \mathcal{F}_U is given by interval constraints, i.e., $\tau_t \in [\underline{\tau}, \bar{\tau}]$ at each period t for some $\underline{\tau}, \bar{\tau} \geq 0$ and $B \in [\underline{B}, \bar{B}]$ for some $\underline{B}, \bar{B} \geq 0$.

Dense Sampling: To solve the bi-level Problem (5a)-(5b), we discretize the feasible set \mathcal{F}_U given by interval constraints as a grid with a step size of s in each component (in general, the step size can vary across each component). That is, the EL toll at any period t lies in the set $\mathcal{A}_s = \{\underline{\tau}, \underline{\tau} + s, \dots, \bar{\tau}\}$ and the eligible user budget lies in the set $\mathcal{B}_s = \{\underline{B}, \underline{B} + s, \dots, \bar{B}\}$. Further, we let \mathcal{C}_s be the set of all toll and budget combinations (τ, B) in this discretized grid. Then, to compute a good solution to Problem (5a)-(5b) with a low societal cost, we evaluate the optimal solution of the convex Program (4a)-(4d) for each CBCP scheme (τ, B) in the set \mathcal{C}_s and return the CBCP scheme with an equilibrium flow with the least societal cost. That is, we return a CBCP scheme $(\tau_s^*, B_s^*) \in \mathcal{C}_s$ with a corresponding equilibrium flow $\mathbf{y}(\tau_s^*, B_s^*)$, such that $f(\mathbf{y}(\tau_s^*, B_s^*)) \leq f(\mathbf{y}(\tau, B))$ for all $(\tau, B) \in \mathcal{C}_s$ with equilibrium flows $\mathbf{y}(\tau, B)$.

We note that while dense sampling involves solving Problem (4a)-(4d) in a discretized grid over a $T + 1$ dimensional space, in practical settings tolls tend to remain static over time, i.e., $\tau_t = \tau_{t'}$ for all $t \neq t'$. Thus, the dense sampling approach can be reduced from $T + 1$ to two dimensions, thereby providing a computationally tractable method to compute an optimal CBCP scheme in \mathcal{C}_s in practical settings. For a more detailed discussion on the computational tractability and practical viability of dense sampling, see the extended version of our paper [8].

VI. NUMERICAL EXPERIMENTS

We now investigate the influence of CBCP schemes on traffic patterns and study their optimal design through a

case study of the San Mateo 101 Express Lanes Project. We present the implementation details and our calibration method of the model parameters of a four-lane highway in San Mateo (with one EL and three GPLs) in the extended version of our paper [8]. Here, we present sensitivity results on EL usage and travel times with changes in tolls and budgets (Section VI-A) and apply dense sampling to solve Problem (5a)-(5b) (Section VI-B).

A. Express Lane Usage and User Travel Times

In this section, we present the variation in the travel time and proportion of users on the EL as the EL tolls and eligible user budgets are varied. We focus on the setting when eligible users have time-invariant VoTs, and the tolls are fixed across five periods over which the CBCP scheme is run. Further, we discretize the tolls to lie between \$0 to \$20, with \$1 increments, and budgets to lie between \$0 to \$90, with \$5 increments, and compute the solution to the convex Program (4a)-(4d) at each of the discretized toll and budget combinations. The resulting distributions of equilibrium lane choices and travel times are presented in Figure 1.

Express Lane Usage: As seen in Figure 1e, users are split evenly across lanes for \$0 tolls, with one-quarter of all users on the EL and the remaining three-quarters on the three GPLs. This observation aligns with equilibria in congestion games without tolls, wherein all users traveling between the same origin and destination incur the same travel time. Further, the proportion of eligible users using the EL ranges from 0% when the budget is \$0 to 100% when the budget exceeds the total cost of tolls over the five periods (i.e., for a toll τ and budget B where $5\tau \leq B$), as reflected by the yellow portion in Figure 1a. On the other hand, the share of ineligible users on the EL is at a maximum of 29% at the smallest non-zero toll of \$1 and \$0 budget and decreases with either increasing toll or budget (see Figure 1c). From Figure 1e, we also observe that the overall share of users on the EL monotonically decreases (increases) with toll (budget) values. Further, the proportion of all users using the EL smoothly varies with the change in the tolls and budgets.

User Travel Times: From Figures 1b and 1d, we observe that the travel times on the EL and GPLs decrease and increase, respectively, with the overall share of users on the EL. Further, the travel time savings on the EL increases monotonically with tolls, with a maximum of about 14.8 minutes (a 43% difference) with a \$20 toll and \$0 budget.

We also note that the overall EL usage and travel time savings depicted in Figure 1 are comparable to and of the same order of magnitude as the data obtained from Caltrans' PeMS database [23] for US 101 ELs in September 2022.

B. Optimal CBCP Schemes

We now design optimal CBCP schemes for a well-studied societal objective (i.e., the Pareto weighted combination of different cost (or welfare) measures) [24], given by

$$f_{\lambda}(\mathbf{y}(\tau, B)) = \lambda_E \sum_{g \in \mathcal{G}_E} v_g \sum_{t \in [T]} \sum_{e \in [2]} l_e(x_{e,t}) y_{e,t}^g - \lambda_R \sum_{g \in \mathcal{G}_I} \sum_{t \in [T]} \tau_t y_{1,t}^g + \lambda_I \sum_{g \in \mathcal{G}_I} \sum_{t \in [T]} \sum_{e \in [2]} (v_{t,g} l_e(x_{e,t}) + \mathbb{I}_{e=1} \tau_t) y_{e,t}^g,$$

which is parameterized by a Pareto weight vector $\lambda = (\lambda_E, \lambda_I, \lambda_R)$ applied to the i) eligible user travel costs, ii) ineligible user travel costs, and iii) negative toll revenue,

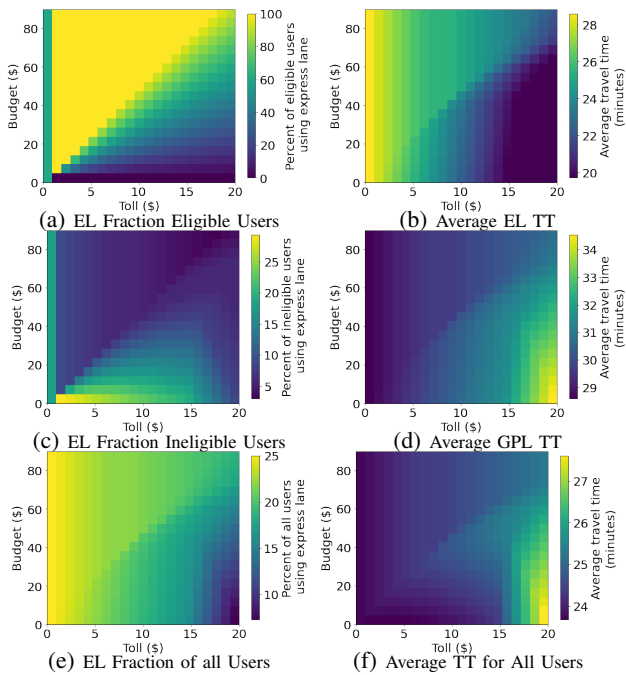


Fig. 1: Percentage of users on the EL (left) and average travel times (right), abbreviated TT, corresponding to the optimal solution of the convex Program (4a)-(4d) for different toll and budget combinations.

TABLE I: Optimal CBCP schemes for various Pareto weights with the associated travel times (TTs) and fraction of users on the EL

Weights ($\lambda_E, \lambda_I, \lambda_R$)	Optimal CBCP		% using express lane			Average TT	
	τ	B	Overall	Eligible	Ineligible	Express	GPL
(1, 0, 0)	19	90	19	95	3	22.1	30.3
(0, 1, 0)	0	0	25	60	18	28.2	28.2
(0, 0, 1)	15	0	16	0	19	19.4	31.4
(1, 1, 1)	13	0	17	0	21	20.3	30.9
(5, 1, 1)	11	0	18	0	22	21.5	30.5
(10, 1, 1)	10	0	19	0	23	22.1	30.3
(11, 1, 1)	10	15	19	30	17	22.3	30.2
(12, 1, 1)	11	45	19	82	7	22.6	30.1
(15, 1, 1)	13	55	19	85	6	22.2	30.3

respectively. For our experiments, we solve Problem (5a)-(5b) using dense sampling for the Pareto weights in Table I.

Table I presents the optimal CBCP schemes for each Pareto weight λ and lists the proportion of users on the EL and corresponding average travel times under the optimal scheme. From Table I, we observe that the optimal CBCP scheme can vary widely based on the central planner's objective, thus demonstrating that a principled approach using bi-level optimization is key to realizing the benefits of CBCP schemes. For instance, if the central planner solely optimizes for eligible users' travel costs, i.e., $\lambda = (1, 0, 0)$, then the optimal CBCP scheme involves providing high budgets and setting high tolls (to push most ineligible users out of the express lane), while the optimal revenue-maximizing CBCP scheme, i.e., $\lambda = (0, 0, 1)$, involves providing no budgets and setting a lower toll of \$15 (to incentivize enough eligible users to use the EL).

VII. CONCLUSION AND FUTURE WORK

In this paper, we studied CBCP schemes, akin to those implemented in the San Mateo 101 Express Lanes Project, to route heterogeneous users in a multi-lane highway.

There are several directions for future research. First, it would be worthwhile to investigate whether equilibria

can be computed efficiently in the general setting when eligible users' VoTs are time-varying. Further, several model extensions, e.g., considering time-varying travel demand, are of interest to further mirror the real-world operation of ELs. Moreover, including HOVs and incorporating mode choices would further improve understanding of the role of modal shift incentives in optimal CBCP. Lastly, investigating more general budget allocation structures is a promising direction.

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