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Stability Conditions for Structured Multi-agent Linear Systems

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***Abstract*— In this work, multi-agent systems which consist of a finite set of agents featuring linear dynamics and influencing each other in a linear way are considered. On the assumption that the topology of the communication network that connects the various agents is known, while the gains of the communication channels, which can assume any real value, are unknown, the multi-agent system is modeled as a structured system whose dynamics is defined by a set of mutually independent real parameters. In this context, structural properties are defined as properties which hold for all the values that the parameters can take and, in particular, this work is focused on the study of the structural stability of the overall multi-agent system. In the case of interest, where all the agents are assumed to have asymptotically stable dynamics, it is shown that a necessary condition for the structural asymptotic stability of the multi-agent system is that the graph describing the relations between the state variables of the agents does not contain cycles of a special kind, defined herein as *simple outer cycles*. Namely, if the graph contains simple outer cycles, then the overall multi-agent system is not structurally asymptotically stable.**

I. Introduction

Multi-agent systems are complex dynamical structures which consist of a finite number of autonomous entities performing specific tasks and mutually interacting through a communication network where the information can be exchanged [1]. The multi-agent paradigm is employed in a wide variety of fields with different aims: e.g., to model and control multi-robot systems [2], [3] and swarms of mobile robots [4], [5]; to optimize the management of resources like physical equipment in decentralized production plants and supply chains [6]–[8], like power plants, transportation & distribution infrastructures, consumers & co-providers in power grids [9], [10], or even like sets of appliances in home automation [11]–[13]; to underpin AI-based decision support systems [14]–[16]; to model and simulate the behavior of cell, animal and human populations [17]–[20].

In this paper, we consider multi-agent systems in which the agents have a linear dynamics and interact with each other in a linear way. In general, both the dynamics of the single agents and the characteristics of the communication network (that is, its topology and the gains of the communication channels) may be only partially known with different levels of uncertainty. In the situation we consider, the topology of the communication network is assumed to be known, while the gain of the active communication channels are unknown and, in principle, they may take any real value. It is assumed that also the entries of the matrices that define the dynamics of the single agents are not necessarily known, although their qualitative properties, like, in particular, asymptotic stability, are known. Multi-agent system models with fixed and known subsystem dynamics, but unknown independent gain values for the interactions among the subsystems were previously considered, e.g., in [21], [22], with special emphasis on the analysis of their structural controllability properties.

These assumptions apply, for instance, to the case of a multi-agent system which consists in a swarm of mobile robots that can exchange information to coordinate their behavior. In practical situations, the multi-agent system can be designed by suitably choosing the individual agents and by specifying the pairs that are nominally linked by communication channels (e.g., letting each agent to take into account only the information broadcasted by neighbouring agents in a formation). On the other hand, the communication capability is influenced and may even be disrupted by environmental conditions, like noise and disturbances, which make the communication gains to be generally unknown.

In order to take into account the uncertainty, we model the multi-agent system as a structured system, that is a system whose dynamic matrix contains entries which are either known to be equal to zero, called fixed zeros, or are described by parameters that can vary in a subset of the set of real numbers, possibly depending on the parameter itself [23]. Then, since stability is a key property of dynamical systems, a natural problem concerns the possibility to assure the asymptotic stability of the multi-agent system, provided that the single agents are themselves stable, regardless of the values that the parameters which represent the gains of the communication network can take. In this regard, we will speak of structural asymptotic stability of the multi-agent system.

In the case where there are no cycles in the graph that describes the relations between the agents, asymptotic stability follows from [24, Theorem 5.1]. In a more complex, nonlinear framework, sufficient conditions for asymptotic stability of the equilibrium point were given in [25]. Further results about stability and stabilizability of multi-agent systems can be found in [26], where discrete-time systems are considered, and in [27], although in those cases the overall system's parameters are assumed to be known. Further results on stability and stabilizability of discrete-time multi-agent systems modelled as structured systems were derived in [28].

With regard to the problem illustrated above, the contribution of this paper consists in proving that the presence of cycles of a special kind in the graph that describes the relations between the state variables of the agents prevents

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the multi-agent systems to be structurally asymptotically stable, i.e. to be asymptotically stable for all the values that the parameters which represent the gains of the communication network can take. This result provides a simple graph theoretical condition that is necessary for structural asymptotic stability and that, as such, should be satisfied in designing the communication network of a multi-agent system which is expected to be asymptotically stable when the communication gains may be arbitrarily altered.

The paper is organized as follows. In Section II, we introduce the class of multi-agent systems we consider and we describe the associated graph, whose vertices represent the state variables of the agents and whose edges represent the relationships between them. Cycles in the graph are called outer if they contain edges between vertices that represent state variables of different agents, while they are called simple if they do not have vertices in common with any other cycle, except auto-loops. In Section III, we introduce the notion of structural asymptotic stability and we prove the main result of the paper (Theorem 1). Namely, we show that the presence of a simple outer cycle in the associated graph prevents the multi-agent system to be structurally asymptotically stable. Section IV presents a couple of illustrative examples and Section V contains some concluding remarks.

II. Structured Multi-agent Systems

Let $S = \{S^i\}_{i \in \mathcal{I}}$, with $\mathcal{I} = \{1, \ldots, q\}$, denote a *set of agents* which interact with each other, where each agent, S^i, with $i \in \mathcal{I}$, is assumed to have a continuous-time linear dynamics described by equations of the form

$$S^i \equiv \begin{cases} \dot{x}^i(t) = A^{ii}x^i(t) + \displaystyle\sum_{j \in \mathcal{I}, j \neq i} A^{ij}x^j(t), \end{cases} \tag{1}$$

where $t \in \mathbb{R}^+$ denotes the continuous time variable; $x^i = [x_1^i \ \ldots \ x_{n_i}^i]^\top \in \mathbb{R}^{n_i}$ is the state of S^i and A^{ij} are real matrices of suitable dimensions for all $i, j \in \mathcal{I}$.

The term $A^{ii}x^i(t)$ in the state equation (1) describes the autonomous component of the state dynamics of S^i, which will be referred to as the *internal state dynamics* of S^i. The term $\sum_{j \in \mathcal{I}, j \neq i} A^{ij}x^j(t)$ describes the interaction between the agents. More precisely, it describes the effects on the dynamics of the state x^i of S^i that are due to the states x^j of the other agents S^j with $j \neq i$, or, from a different point of view, the way in which the information is transferred from the agents S^j, with $j \neq i$, to the agent S^i. The entries of A^{ij} can be seen as the gains that characterize each communication channel of the network that connects the agents. By removing the term $\sum_{j \in \mathcal{I}, j \neq i} A^{ij}x^j(t)$ from the state equation (1), one obtains the dynamics of the agent S^i in a *stand-alone configuration*, in which it is not influenced by any other agent.

With the notation previously introduced, the dynamics of the linear, multi-agent system Σ that groups $S^1, \ldots, S^q$ is described by equations of the form

$$\Sigma \equiv \begin{cases} \dot{x}(t) = Ax(t), \end{cases} \tag{2}$$

where

$$x(t) = \left[(x^1(t))^\top \, (x^2(t))^\top \, \ldots \, (x^q(t))^\top\right]^\top \in \mathbb{R}^n,$$

with $n = n_1 + \ldots + n_q$, and

$$A = \begin{bmatrix} A^{11} & A^{12} & \ldots & A^{1q} \\ A^{21} & A^{22} & \ldots & A^{2q} \\ \vdots & \vdots & \ddots & \vdots \\ A^{q1} & A^{q2} & \ldots & A^{qq} \end{bmatrix}.$$

Remark 1: Agents in Σ may have different dynamics of different dimensions. Multi-agent systems of the above kind can be used to model a variety of physical systems with interconnected components, provided that the individual components have a linear dynamics and that they mutually influence each other in a linear way. For instance, they can be used to describe the autonomous behavior of swarms of mobile robots, interconnected sets of production units in a plant, networks of sensors and measuring devices, sets of control units in a power distribution network, populations or groups of biological, social or economics entities.

In our setting, we assume that not all the entries of A^{ij} are known. More precisely, we assume that some of them are known to be equal to 0, while the others are described by mutually independent parameters that can assume any value in $\mathbb{R}$. In displaying the matrix A^{ij}, entries which are known to be equal to 0 are called *fixed zeros* and they are indicated by 0 or, e.g., for the entry a_{rs}, this feature is specified by writing $a_{rs} \equiv 0$. In the opposite situation, we will write $a_{rs} \not\equiv 0$. If $a_{rs} \equiv 0$ in A^{ij}, the component x_s^j of the state x^j of the agent S^j does not influence the dynamics of the component x_r^i of the state x^i of the agent S^i. In the opposite situation, i.e. if $a_{rs} \not\equiv 0$, its value (which may also be 0) is the gain with which such influence occurs.

Entries of A^{ij} which are not fixed zeros correspond to active communication channels between the agents and, from this point of view, our assumption means that the topology of the communication networks between the agents is known, but the gain of each communication channel is not and, in principle, it may be equal to any real value. This agrees with many real situations in which the topology of the communication network can be designed by establishing specific communication channels, but the actual gain on each active channel is subject to external unknown and unpredictable factors.

We make similar assumptions also for the entries of A^{ii}. More precisely, we assume that some of them are known to be fixed zeros, while the others may be known or be described by real parameters which are not necessarily mutually independent, but which satisfy known dependency relations (e.g., it may be known that two entries whose values are unknown are equal to each other) or which are constrained to assume their value in a specific subset $\Omega \subseteq \mathbb{R}$ that may vary with the entry itself. This makes it possible to include in our framework also situations in which it is known that some component of the internal dynamics of each agent satisfies physical or structural constraints or that the dynamics itself has specific qualitative properties, like, for instance, asymptotic stability.

By assuming that the entries of A are either described by parameters or by fixed zeros, the multi-agent system Σ is viewed as a structured system in the sense of [23]. Indeed, the fact that, in the systems considered in [23], all the parameters are generally assumed to be mutually independent does not represent, for our purposes, a significant difference. We can therefore qualify Σ as a *structured multi-agent system* and treat it as such.

Relevant information on the dynamics of Σ can be conveniently displayed by means of the associated directed graph $(G, \mathcal{E})$. The pair $(G, \mathcal{E})$ consists of the set of vertices G and of the set of edges $\mathcal{E}$, whose definitions are given below.

- The *set of vertices* G is given by

$$G = \bigcup_{i \in \mathcal{I}} G^i,$$

where, for any $i \in \mathcal{I}$, G^i is a set of vertices whose cardinality is equal to the dimension of the state space of the corresponding agent S^i: i.e.,

$$G^i = \{V_1^i, \ldots, V_{n_i}^i\}, \quad i \in \mathcal{I}.$$

Hence, the cardinality of G is equal to the dimension of the state space of Σ and it is the sum of the dimensions of the state spaces of the single agents which form Σ: i.e.

$$\operatorname{card} G = \sum_{i \in \mathcal{I}} \operatorname{card} G^i = \sum_{i \in \mathcal{I}} n_i = n.$$

- The *set of edges* $\mathcal{E}$ contains
  - an edge from the vertex V_s^i to the vertex V_r^i if and only if the entry a_{rs} of the matrix A^{ii} is not a fixed zero, i.e. $a_{rs} \not\equiv 0$,
  - an edge from the vertex V_s^j to the vertex V_r^i if and only if the entry a_{rs} of the matrix A^{ij}, with $j \neq i$, is not a fixed zero, i.e. $a_{rs} \not\equiv 0$.

The notation $e(V_s^j, V_r^i)$ will be used to indicate an edge from V_s^j to V_r^i in $\mathcal{E}$. Moreover, V_s^j and V_r^i are respectively called the *tail* and the *head* of the edge $e(V_s^j, V_r^i)$.

Definition 1: An edge $e(V_s^j, V_r^i)$ in $\mathcal{E}_k$ is said to be

- *inner* if its tail and its head belong to the same subset of vertices $G^i = G^j$, that is if $j = i$;
- *outer* if its tail and its head belong to two different subsets of vertices $G^i \neq G^j$, that is if $j \neq i$.

Letting

$$\mathcal{E}^i = \{e(V_s^i, V_r^i), \text{ for } s, r \in \{1, \ldots, n_i\}\}$$

denote the set of inner edges between the vertices of G^i, we have that $(G^i, \mathcal{E}^i)$ is a subgraph of $(G, \mathcal{E})$ that structurally describes the internal dynamic relations between the state variables of the agent S^i.

A *path* P of length $\bar{h} \geq 1$ in $(G, \mathcal{E})$ is an ordered set of $\bar{h}$ edges

$$P = \{e_1(V_{s_1}^{j_1}, V_{r_1}^{i_1}), \ldots, e_{\bar{h}}(V_{s_{\bar{h}}}^{j_{\bar{h}}}, V_{r_{\bar{h}}}^{i_{\bar{h}}})\}$$

in $\mathcal{E}$ such that the head $V_{r_h}^{i_h}$ of $e_h(V_{s_h}^{j_h}, V_{r_h}^{i_h})$ coincides with the tail $V_{s_{(h+1)}}^{j_{(h+1)}}$ of $e_{h+1}(V_{s_{(h+1)}}^{j_{(h+1)}}, V_{r_{(h+1)}}^{i_{(h+1)}})$ for $h = 1, \ldots, \bar{h} - 1$.

The tail of the first edge of the path is called the *tail of the path*. The head of the last edge of the path is called the *head of the path*. The edges that form a path and their tails and heads will be said, respectively, to be the edges and the vertices which belong to the path.

Definition 2: A path P is said to be

- *inner* if it consists of inner edges only (i.e., if it is a path of the subgraph $(G^i, \mathcal{E}^i)$ for some $i \in \mathcal{I}$);
- *outer* if it contains at least an outer edge.

Definition 3: An *inner cycle* (or, respectively, an *outer cycle*) is an inner path (respectively, an outer path)

$$P_c = \{e_1(V_{s_1}^{j_1}, V_{r_1}^{i_1}), \ldots, e_{\bar{h}}(V_{s_{\bar{h}}}^{j_{\bar{h}}}, V_{r_{\bar{h}}}^{i_{\bar{h}}})\}$$

of any length $\bar{h} \geq 1$ in $\mathcal{E}$ such that the head $V_{r_{\bar{h}}}^{i_{\bar{h}}}$ of the last edge coincides with the tail $V_{s_1}^{j_1}$ of first edge, i.e. such that $V_{r_{\bar{h}}}^{i_{\bar{h}}} = V_{s_1}^{j_1}$.

Definition 4: A cycle P_c in $(G, \mathcal{E})$ of length at least 2 is said to be *simple* if there are no other cycles, except possibly auto-loops, that have one or more than one vertex in common with it.

III. Main Results

Overall qualitative properties of the structured multi-agent system Σ depend not only on the dynamics of the single agents, but also on the topology and on the gains of the network that allows their mutual interactions. Since the topology of the network is characterized by the entries of A that are not fixed zeros and it is displayed by the graph $(G, \mathcal{E})$, our setting describes the situation in which the topology of the network is known, while its gains are unknown. In such situation, it is interesting to investigate how the network topology or, equivalently, the graph theoretic properties of $(G, \mathcal{E})$ influence specific qualitative properties of Σ, regardless of the values that the unknown parameters may take. In particular, assuming that each agent has an asymptotically stable dynamics, we aim to gain insight into how graph theoretic properties influence the possibility that Σ be asymptotically stable, regardless of the actual values that the unknown parameters may take.

If there are no outer cycles in the associated graph, it is easy to see that, possibly reordering the set of agents, the dynamic matrix A of the overall multi-agent system Σ can be given a block diagonal form, with the internal dynamics matrices A^{ii} on the main diagonal. Hence, in such situation, the asymptotic stability of each agent is clearly a necessary and sufficient condition for the asymptotic stability of Σ, regardless of the values that the unknown parameters may take. On the other hand, as will be shown in this section, the presence of simple outer cycles implies that there exist values of the parameters in the matrices A^{ij} for which Σ is not asymptotically stable.

To proceed, let us introduce the following definition.

Definition 5: Assume that each agent in the structured multi-agent system Σ of the form (2) is asymptotically stable. Then, Σ is said to be *structurally asymptotically stable* if, for any choice of the parameters in the matrices A^{ij}, with $i \neq j$, the resulting multi-agent system $\bar{\Sigma}$, with the dynamics described by $\bar{A}$, is asymptotically stable.

The main result of this work is then given by the following theorem.

Theorem 1: Let Σ be a structured multi-agent system of the form (2) in which all the agents are asymptotically stable and let $(G, \mathcal{E})$ denote the associated graph. If there exists a simple outer cycle P_c in $(G, \mathcal{E})$, then Σ is not structurally asymptotically stable.

Proof: Let P_c be described by

$$P_c = \{e_1(V_{s_1}^{j_1}, V_{r_1}^{i_1}), \dots, e_{\bar{h}}(V_{s_{\bar{h}}}^{j_{\bar{h}}}, V_{r_{\bar{h}}}^{i_{\bar{h}}})\}$$

and assume, without loss of generality, that the last edge $e_{\bar{h}}(V_{s_{\bar{h}}}^{j_{\bar{h}}}, V_{r_{\bar{h}}}^{i_{\bar{h}}})$ of P_c is an outer edge. Rename the vertices of $(G, \mathcal{E})$ (without taking into account to which subset G^i they belong to), in such a way that $V_{s_1}^{j_1} = V_{r_{\bar{h}}}^{i_{\bar{h}}}$ becomes V_1 (namely, the tail of the first edge, which coincides with the head of the last edge, in above representation of P_c, becomes V_1); the tails of the other edges in P_c become $V_2, \dots, V_{\bar{h}}$, according to the order in which they are encountered by following the path along P_c starting from V_1. Then, rename $V_{\bar{h}+1}, \dots, V_{\bar{h}+k}$ in an arbitrary way all the vertices that are the head of a path in $(G, \mathcal{E})$ whose tail is one of the vertices $V_1, \dots, V_{\bar{h}}$. Finally, rename $V_{\bar{h}+k+1}, \dots, V_n$, in an arbitrary way, the remaining vertices of $(G, \mathcal{E})$.

Note that renaming in this way all the vertices corresponds to a change of basis in the state space of Σ. In particular, after applying the procedure described above, the dynamics matrix A takes the form

$$A = \begin{bmatrix} A_{11} & 0 & A_{13} \\ A_{21} & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix} \tag{3}$$

where

$$A_{11} = \begin{bmatrix} a_{11} & 0 & \dots & \dots & 0 & a_{1\bar{h}} \\ a_{21} & a_{22} & 0 & \dots & \dots & 0 \\ 0 & a_{32} & a_{33} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & a_{\bar{h}\,\bar{h}-1} & a_{\bar{h}\,\bar{h}} \end{bmatrix} \tag{4}$$

A_{22} is a square matrix of dimension k, A_{33} is a square matrix of dimension $n - (\bar{h} + k)$. The entries represented by 0 in A_{11} and the entries of the null blocks in A are fixed zeros due to the fact that P_c is a simple cycle. Note that the elements $a_{i\,i-1}$ and $a_{1\,\bar{h}}$, for $i = 1, \dots, \bar{h}$, in A_{11} correspond to the edges of P_c and, therefore, they are not fixed zeros. Moreover, since $a_{1\,\bar{h}}$ corresponds to an outer edge, it is an independent parameter that can assume any value in $\mathbb{R}$.

From (3) and (4), we have

$$\det(sI - A) = \det(sI - A_{11}) \det(sI - A_{22}) \det(sI - A_{33}),$$

with

$$\det(sI - A_{11}) = \left(\prod_{i=1}^{\bar{h}} (s - a_{ii}) + (-1)^{2\bar{h}+1} a_{1\bar{h}} \prod_{i=2}^{\bar{h}} a_{ii-1} \right).$$

Note that the constant term

$$(-1)^{2\bar{h}+1} a_{1\bar{h}} \prod_{i=2}^{\bar{h}} a_{ii-1},$$

can be made negative by a suitable choice of $a_{1\bar{h}}$, no matter which values the nonzero parameters a_{ii-1} assume. Hence, $\det(sI - A_{11})$ is not a Hurwitz polynomial for all the possible values that the parameters of A may take. Consequently, the multi-agent system Σ is not structurally asymptotically stable. ■

Theorem 1 states that a necessary condition for the structural asymptotic stability of a structured multi-agent system Σ of the form (2) is the absence of simple outer cycles in the associated graph $(G, \mathcal{E})$. This result is useful in the design of the communication network that *connects* a set of agents, each having an asymptotically stable linear dynamics, since it indicates how to choose a network topology avoiding connections which would definitely prevent the asymptotic stability of the resulting multi-agent system for some values of the gains of the communication channels.

Remark 2: The condition of Theorem 1 can be checked by finding all the cycles in $(G, \mathcal{E})$ by means of, e.g., the $\mathrm{MATLAB}^{\mathrm{TM}}$ function $allcycles$ and, then, by looking for those which have some vertex in common, with the exclusion of auto-loops.

IV. Illustrative Examples

In this section, we illustrate the content of Theorem 1 by a straightforward example, the first one. Then, with the second example, we show that the hypothesis that the outer cycle is simple cannot be dropped.

To avoid unnecessarily heavy notation, in these examples, the vertices of the graphs are denoted with the same symbols of the corresponding state variables in the mathematical models.

A. Example 1

The simplest example in which we can see an application of Theorem 1 consists of a structured multi-agent system Σ, of the form (2), with two agents, S^1 and S^2, of dimension 1, defined by the following equations

$$S^1 \equiv \left\{ \dot{x}^1(t) = a^{11} x^1(t) + a^{12} x^2(t), \right.$$

$$S^2 \equiv \left\{ \dot{x}^2(t) = a^{22} x^2(t) + a^{21} x^1(t). \right.$$

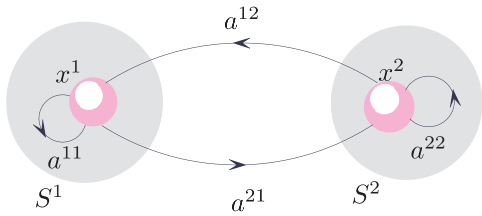

Fig. 1. Example 1 - The directed graph of the multi-agent system Σ.

To guarantee asymptotic stability of each agent, the entries a^{ii}, with $i=1,2$, are assumed to be negative. Moreover, the entries $a^{i,j}$, with $i \neq j$, are assumed not to be fixed zeros.

The dynamics of Σ is described by the matrix

$$A = \begin{bmatrix} a^{11} & a^{12} \\ a^{21} & a^{22} \end{bmatrix},$$

where the unknown, off-diagonal entries correspond to the outer edges of the associated graph $(G, \mathcal{E})$ shown in Fig. 1. Those entries represent the gains of the communications channels between the agents S^1 and S^2.

Since the graph $(G, \mathcal{E})$ contains the simple outer cycle

$$P_c = \{e(x^1, x^2), e(x^2, x^1)\},$$

according to Theorem 1, Σ is not structurally asymptotically stable. In fact, we have

$$\begin{aligned} &\det(sI - A) = \\ &\quad (s - a^{11})(s - a^{22}) - a^{12}a^{21} = \\ &\quad s^2 - (a^{11} + a^{22})\, s + (a^{11}a^{22} - a^{12}a^{21}) \end{aligned}$$

and, no matter which values a^{11} and a^{22} actually assume, there exist nonzero values of a^{12} and a^{21} which prevent $\det(sI - A)$ from being a Hurwitz polynomial, by making its constant term negative.

B. Example 2

In this example, we will show that the presence of outer cycles in the directed graph $(G, \mathcal{E})$ does not prevent the structural asymptotic stability of the corresponding multi-agent system Σ, provided that none of the outer cycles is simple.

Let Σ be the multi-agent system of the form (2), which consists of the two agents, S^1 and S^2, described by the equations

$$S^1 \equiv \begin{cases} \dot{x}_1^1(t) \;=\; -x_1^1(t) + a\, x_1^2(t), \\ \dot{x}_2^1(t) \;=\; x_1^1(t) - x_2^1(t), \\ \dot{x}_3^1(t) \;=\; x_1^1(t) - x_3^1(t), \\ \dot{x}_4^1(t) \;=\; x_2^1(t) - x_3^1(t) - x_4^1(t), \end{cases} \tag{5}$$

$$S^2 \equiv \left\{ \; \dot{x}_1^2(t) \;=\; -x_1^2(t) + b\, x_4^1(t). \right. \tag{6}$$

Consistently with our general assumptions, the entries of the dynamic matrix A of the overall multi-agent system Σ corresponding to outer edges of the associated graph $(G, \mathcal{E})$ (which represent the gains of the communication channels between S^1 and S^2) are unknown, except for those that are fixed zeros, and are indicated by independent real parameters – namely, a, b in (5), (6). The entries of the matrix A corresponding to inner edges, which define, respectively, the internal dynamics of S^1 and of S^2 are supposed to be known and to assume the numerical values indicated in (5), (6). Note that both the internal dynamics are asymptotically stable.

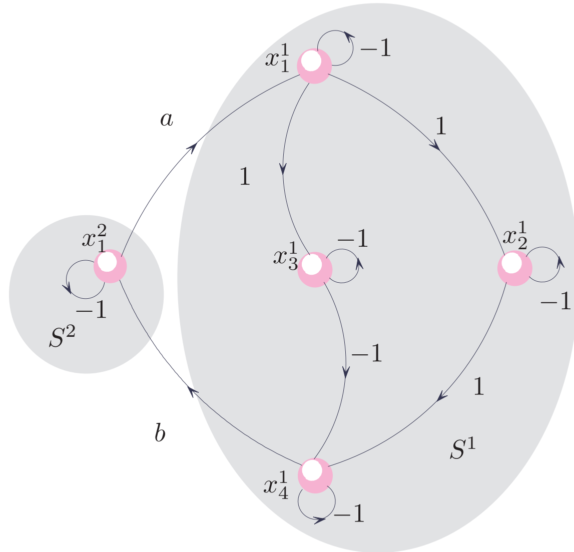

Fig. 2. Example 2 - The directed graph of the multi-agent system Σ.

The associated graph $(G, \mathcal{E})$ is shown in Fig. 2. It contains only two cycles of length at least equal to 2: namely,

$$P_{c1} = \{e(x_1^1, x_3^1), e(x_3^1, x_4^1), e(x_4^1, x_1^2), e(x_1^2, x_1^1)\}$$

and

$$P_{c2} = \{e(x_1^1, x_2^1), e(x_2^1, x_4^1), e(x_4^1, x_1^2), e(x_1^2, x_1^1)\}.$$

Note that none of these two cycles is simple. In fact, they have the vertices x_1^1, x_4^1, and x_1^2 in common.

The matrix A which defines the dynamics of Σ is given by

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & a \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & b & -1 \end{bmatrix}.$$

Therefore,

$$\det(sI - A) = (s+1)^5,$$

regardless of the values that a and b actually take.

Hence, Σ is structurally asymptotically stable.

V. Conclusion

A necessary condition for the structural asymptotic stability of structured multi-agent systems has been given in Theorem 1. It is worthwhile noting that there is a relevant gap between the condition that there are no outer cycles in the graph associated to a given system (which, as already mentioned, is sufficient for structural asymptotic stability provided that all the agents have an asymptotically stable dynamics) and the necessary condition expressed in Theorem 1, which calls into question the existence of outer cycles. In particular, Example 2 shows that the absence of outer cycles is not necessary, while the absence of simple outer cycles is easily seen not to be sufficient. In general, it does not seem possible to state necessary and sufficient conditions for structural asymptotic stability of structured systems akin to those considered in this work.

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