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Computationally Efficient Covariance Steering for Systems Subject to Parametric Disturbances and Chance Constraints

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***Abstract*— This work investigates the finite-horizon optimal covariance steering problem for discrete-time linear systems subject to both additive and multiplicative uncertainties as well as state and input chance constraints. In particular, a tractable convex approximation of the optimal covariance steering problem is developed by tightening the chance constraints and by introducing a suitable change of variables. The solution of the convex approximation is shown to be a valid (albeit potentially suboptimal) solution to the original chance-constrained covariance steering problem.**

I. Introduction

Covariance control examines the problem of driving a stochastic system from a given initial distribution to a specified final distribution. The problem has been previously studied extensively for the infinite-horizon case [1]–[3], but has only recently been studied for the finite-horizon case (referred to as covariance steering). Specifically, in [4]–[7], the authors introduced the finite-horizon covariance steering problem. In [8], [9], the authors added expectation constraints, reference [10] introduced state chance constraints, and in [11] the authors considered hard input constraints. Additionally, covariance steering has been applied to robotic path planning in [12], [13], differential games in [14], and re-entry, descent, and landing tasks for space operations in [15]–[17]. However, most recent work on covariance steering focuses on linear systems subject only to additive noise, where the initial and final distributions as well as the noise characteristics are given as Gaussian distributions. A few notable exceptions are [18], which considered additive generic (non-Gaussian) noise, and [19], [20] which considered covariance control for nonlinear systems.

We examine the problem of steering a stochastic linear system from an initial distribution characterized by its first two moments to a given terminal mean and covariance in finite time, when the system is subject to parametric uncertainties (i.e. the disturbances enter both multiplicatively with the state and control as well as additively). Although the problem of steering the first two moments of a linear system subject to state and control chance constraints and mixed additive and multiplicative noise from initial to given final conditions is, in general, nonlinear, this work shows that the problem may be represented by a tightened convex problem formulation, similar to that used for a model predictive control scheme in [21], the solution of which ensures the satisfaction of the original problem's chance and terminal constraints. Therefore, the optimal solution of the convex reformulation is an admissible (albeit potentially suboptimal) solution of the original nonlinear program.

During the preparation of this manuscript, the authors became aware of a similar study [22]. The authors of [22] also studied the covariance steering problem for linear systems affected by multiplicative disturbances and solved the problem using semi-definite programming. However, the work of [22] assumes multiplicative disturbances affecting the state and control and the additive disturbances are all mutually independent, which simplifies the covariance propagation. Contrary, the current work does not assume that the disturbances affecting the system at a given time step are independent of each other. Additionally, the proposed approach in this paper includes chance constraints in the problem formulation and shows how they can be ensured using Boole's and Cantelli's inequalities and incorporated into the semi-definite program using a linear bounding technique, whereas the work of [22] does not address chance constraints.

This paper employs several standard notation practices. A random variable drawn from a normal distribution with mean μ and covariance matrix Σ is denoted by $x \sim \mathcal{N}(\mu, \Sigma)$. $\mathbb{E}[\cdot]$ denotes the expectation operator, and $\Pr(x)$ denotes the probability of event x. I_n denotes the $n \times n$ identity matrix, and $\mathtt{tr}(\cdot)$ denotes the trace operation. A symmetric positive (semi-)definite matrix is denoted by $M \succ 0$ ($M \succeq 0$).

II. Problem Formulation

Consider the system

$$x_{k+1} = A_k x_k + B_k u_k + d_k, \tag{1}$$

where $x_k \in \mathbb{R}^{n_x}$, $u_k \in \mathbb{R}^{n_u}$. Let the initial conditions be given as $\mathbb{E}[x_0] = \mu_I$ and $\mathbb{E}[(x_0 - \mathbb{E}[x_0])(x_0 - \mathbb{E}[x_0])^\top] = \Sigma_I$, where $\Sigma_I \succ 0$. Additionally, the system matrices are comprised of a constant, known component, and a time-varying stochastic component given by $A_k = \bar{A} + \sum_{j=1}^m \tilde{A}_j q_{j,k}$, $B_k = \bar{B} + \sum_{j=1}^m \tilde{B}_j q_{j,k}$, $d_k = \bar{d} + \sum_{j=1}^m \tilde{d}_j q_{j,k}$, where $q_{j,k} \sim \mathcal{N}(0,1)$, for all $k = 0, 1, \ldots,$ is a Gaussian white noise. Therefore,

$$\mathbb{E}[q_{j_1,k_1} q_{j_2,k_2}] = \begin{cases} 1 & \text{where } j_1 = j_2 \text{ and } k_1 = k_2, \\ 0 & \text{otherwise.} \end{cases} \tag{2a}$$

Furthermore, we assume

$$\mathbb{E}[x_k q_{j,k}] = \mathbb{E}[x_k]\mathbb{E}[q_{j,k}] = 0, \tag{2b}$$

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$$\mathbb{E}[u_k q_{j,k}] = \mathbb{E}[u_k]\mathbb{E}[q_{j,k}] = 0, \tag{2c}$$

for $k = 0, 1, \ldots, j = 1, \ldots, m$ which stem from causality considerations.

The state and control inputs in (1) are subject to the chance constraints

$$\Pr(x_k \in \mathcal{X}) \geq 1 - p_x, \quad \Pr(u_k \in \mathcal{U}) \geq 1 - p_u, \tag{3}$$

for all $k = 0, 1, \ldots, N-1$, where $\mathcal{X} \subseteq \mathbb{R}^{n_x}$ and $\mathcal{U} \subseteq \mathbb{R}^{n_u}$ are convex sets and $p_x, p_u \in (0, 0.5)$ are given maximal probabilities of constraint violation. We further assume that the sets $\mathcal{X}$ and $\mathcal{U}$ can be written as an intersection of a finite number of linear inequality constraints as follows

$$\mathcal{X} \triangleq \bigcap_{i_x=1}^{N_s} \left\{ x : \alpha_{x,i_x}^\top x \leq \beta_{x,i_x} \right\}, \tag{4}$$

$$\mathcal{U} \triangleq \bigcap_{i_u=1}^{N_c} \left\{ u : \alpha_{u,i_u}^\top u \leq \beta_{u,i_u} \right\}, \tag{5}$$

where $\alpha_{x,i_x} \in \mathbb{R}^{n_x}$ and $\alpha_{u,i_u} \in \mathbb{R}^{n_u}$ are constant vectors, and $\beta_{x,i_x} \geq 0$ and $\beta_{u,i_u} \geq 0$ are constant scalars.

We wish to steer (1) to a given final mean $\mu_F \in \mathcal{X}$ and covariance $\Sigma_F \succ 0$ at time N, such that

$$\mathbb{E}[x_N] = \mu_F, \quad \mathbb{E}[(x_N - \mathbb{E}[x_N])(x_N - \mathbb{E}[x_N])^\top] = \Sigma_F,$$

while minimizing the cost function

$$J(\mu_I, \Sigma_I; u_0, \ldots, u_{N-1}) = \mathbb{E}\left[\sum_{k=0}^{N-1} \ell(x_k, u_k)\right]. \tag{6}$$

In particular, we will examine the case where $\ell(\cdot,\cdot)$ has the quadratic form

$$\ell(x, u) = x^\top Q x + u^\top R u. \tag{7}$$

The problem may thus be summarized as follows. Given $\mu_I, \Sigma_I, \mu_F, \Sigma_F$, determine the control sequence $\mathbf{u} = \{u_0, \ldots, u_{N-1}\}$ that solves the following finite-time, optimal covariance steering problem

$$\min_{\mathbf{u}} \; J(\mu_I, \Sigma_I; \mathbf{u}) = \mathbb{E}\left[\sum_{k=0}^{N-1} x_k^\top Q x_k + u_k^\top R u_k\right], \tag{8a}$$

subject to

$$\mathbb{E}[x_0] = \mu_I, \tag{8b}$$

$$\mathbb{E}[(x_0 - \mathbb{E}[x_0])(x_0 - \mathbb{E}[x_0])^\top] = \Sigma_I, \tag{8c}$$

$$x_{k+1} = (\bar{A} + \sum_{j=1}^m \tilde{A}_j q_{j,k})x_k + (\bar{B} + \sum_{j=1}^m \tilde{B}_j q_{j,k})u_k + \bar{d} + \sum_{j=1}^m \tilde{d}_j q_{j,k}, \quad k = 0, \ldots, N-1 \tag{8d}$$

$$\Pr(x_k \in \mathcal{X}) \geq 1 - p_x, \quad k = 0, \ldots, N-1 \tag{8e}$$

$$\Pr(u_k \in \mathcal{U}) \geq 1 - p_u, \quad k = 0, \ldots, N-1 \tag{8f}$$

$$\mathbb{E}[x_N] = \mu_F, \tag{8g}$$

$$\mathbb{E}[(x_N - \mathbb{E}[x_N])(x_N - \mathbb{E}[x_N])^\top] = \Sigma_F. \tag{8h}$$

III. Covariance Steering Controller Design

We now turn our attention to formulating a computationally tractable approximation of Problem (8) using standard relaxations, the solution of which will provide a (suboptimal) feasible solution to the original problem. Specifically, we first formulate Problem (8) as a deterministic optimal control problem by evaluating the expectations in (8a)-(8h) and derive explicit expressions for the propagation of the system mean and covariance. Additionally, we use Boole's inequality and Cantelli's inequality to approximate the chance constraints (8e)-(8f) as deterministic inequality constraints. We then show that the deterministic problem can be cast as a convex programming problem by performing a change of variables, tightening the chance constraints, and relaxing the terminal covariance constraint (8h) to a linear matrix inequality (LMI) constraint.

A. Expectation and Uncertainty Propagation

We may write the nominal system as

$$\begin{aligned}\mathbb{E}\left[x_{k+1}\right] &= \mathbb{E}\left[A_k x_k + B_k u_k + d_k\right] \\ &= \mathbb{E}[(\bar{A} + \sum_{j=1}^m \tilde{A}_j q_{j,k})x_k + (\bar{B} + \sum_{j=1}^m \tilde{B}_j q_{j,k})u_k \\ &\quad + \bar{d} + \sum_{j=1}^m \tilde{d}_j q_{j,k}].\end{aligned} \tag{9}$$

Using the independence of the disturbances (2) and the fact that $\mathbb{E}[q_{j,k}] = 0$ for all $k = 0, 1, \ldots$ and $j = 1, \ldots, m$, the nominal system is given by

$$\bar{x}_{k+1} = \bar{A}\bar{x}_k + \bar{B}\bar{u}_k + \bar{d}, \tag{10}$$

where $\bar{x}_k = \mathbb{E}[x_k]$ and $\bar{u}_k = \mathbb{E}[u_k]$.

Next, let us define the deviation of the stochastic system from the nominal dynamics as $\tilde{x}_k = x_k - \bar{x}_k$. The error dynamics are given as

$$\begin{aligned}\tilde{x}_{k+1} &= \bar{A}\tilde{x}_k + \bar{B}\tilde{u}_k \\ &\quad + \sum_{j=1}^m \tilde{A}_j q_{j,k} x_k + \sum_{j=1}^m \tilde{B}_j q_{j,k} u_k + \sum_{j=1}^m \tilde{d}_j q_{j,k},\end{aligned} \tag{11}$$

where $\tilde{u}_k = u_k - \bar{u}_k$. Considering (2), the covariance dynamics are given by

$$\begin{aligned}\Sigma_{x_{k+1}} &= \bar{A}\Sigma_{x_k}\bar{A}^\top + \bar{A}\Sigma_{x_k u_k}\bar{B}^\top + \bar{B}\Sigma_{x_k u_k}^\top \bar{A}^\top + \bar{B}\Sigma_{u_k}\bar{B}^\top \\ &+ \sum_{j=1}^m (\tilde{A}_j \Sigma_{x_k} \tilde{A}_j^\top + \tilde{A}_j \Sigma_{x_k u_k} \tilde{B}_j^\top + \tilde{B}_j \Sigma_{x_k u_k}^\top \tilde{A}_j^\top + \tilde{B}_j \Sigma_{u_k} \tilde{B}_j^\top) \\ &+ \sum_{j=1}^m (\tilde{A}_j \bar{x}_k + \tilde{B}_j \bar{u}_k + \tilde{d}_j)(\tilde{A}_j \bar{x}_k + \tilde{B}_j \bar{u}_k + \tilde{d}_j)^\top\end{aligned} \tag{12}$$

where $\Sigma_{u_k} = \mathbb{E}[(u_k - \mathbb{E}[u_k])(u_k - \mathbb{E}[u_k])^\top]$, and $\Sigma_{x_k u_k} = \mathbb{E}[(x_k - \mathbb{E}[x_k])(u_k - \mathbb{E}[u_k])^\top]$ are properties of the particular control policy under consideration.

B. Control Policy

Rather than optimizing over control actions, we optimize over control policies. In order to steer the mean and covariance of the system dynamics, we propose the following affine state-feedback control policy

$$u_k = L_k \tilde{x}_k + c_k, \tag{13}$$

where $L_k \in \mathbb{R}^{n_u \times n_x}$ and $c_k \in \mathbb{R}^{n_u}$ are new optimization variables, for $k = 0, \ldots, N-1$. Thus, $\bar{u}_k = c_k$, $\Sigma_{u_k} = L_k \Sigma_{x_k} L_k^\top$, $\Sigma_{x_k u_k} = \Sigma_{x_k} L_k^\top$, for $k = 0, \ldots, N-1$.

C. Deterministic Reformulation

We are now ready to formulate a deterministic approximation of Problem (8). First, the expectation of the stage cost function (7) can be written in terms of the mean and covariance as $\mathbb{E}[\ell(x_k, u_k)] = \bar{x}_k^\top Q \bar{x}_k + \mathtt{tr}(Q\Sigma_{x_k}) + \bar{u}_k^\top R \bar{u}_k + \mathtt{tr}(R\Sigma_{u_k})$. Next, using Boole's inequality [23] and (4)-(5), conservative approximations of the chance constraints (3) are given by the inequality constraints

$$\Pr\left(\alpha_{x,i_x}^\top x_k \le \beta_{x,i_x}\right) \ge 1 - p_{x,i_x}, \ i_x = 1, \ldots, N_s, \tag{14a}$$
$$\Pr\left(\alpha_{u,i_u}^\top u_k \le \beta_{u,i_u}\right) \ge 1 - p_{u,i_u}, \ i_u = 1, \ldots, N_c, \tag{14b}$$

where $p_{x,i_x}, p_{u,i_u} \ge 0$ are such that $\sum_{i_x=1}^{N_s} p_{x,i_x} \le p_x$, $\sum_{i_u=1}^{N_c} p_{u,i_u} \le p_u$, for all $k = 0, 1, \ldots, N-1$. We introduce the following lemma to convert (14a) and (14b) to deterministic inequalities.

Lemma 1 ([24]): The state chance constraints (14a) are ensured by the tightened deterministic constraints given in terms of the mean and covariance as

$$\alpha_{x,i_x}^\top \bar{x}_k + \sqrt{\alpha_{x,i_x}^\top \Sigma_{x_k} \alpha_{x,i_x}} \sqrt{\frac{1 - p_{x,i_x}}{p_{x,i_x}}} - \beta_{x,i_x} \le 0, \tag{15}$$

where $i_x = 1, \ldots, N_s, \quad k = 0, \ldots, N-1$. Likewise, the input constraints (14b) are ensured by the tightened deterministic constraints given in terms of the mean control and control covariance as

$$\alpha_{u,i_u}^\top \bar{u}_k + \sqrt{\alpha_{u,i_u}^\top \Sigma_{u_k} \alpha_{u,i_u}} \sqrt{\frac{1 - p_{u,i_u}}{p_{u,i_u}}} - \beta_{u,i_u} \le 0, \tag{16}$$

where $i_u = 1, \ldots, N_c, \quad k = 0, \ldots, N-1$. Furthermore, the satisfaction of the original chance constraints (3) is ensured by the satisfaction of (15) and (16).

Proof: Due to space limitations, the proof is omitted. However, the proof may be found in [24], [25]. ■

Thus, a deterministic version of Problem (8) is given by

$$\min_{\mathbf{c}, \mathbf{L}} \quad \bar{J}(\mu_I, \Sigma_I; \mathbf{c}, \mathbf{L}) = \sum_{k=0}^{N-1} \bar{x}_k^\top Q_k \bar{x}_k + \mathtt{tr}(Q_k \Sigma_{x_k}) + \bar{u}_k^\top R_k \bar{u}_k + \mathtt{tr}(R_k \Sigma_{u_k}) \tag{17a}$$

subject to

$$\bar{x}_0 = \mu_I, \quad \Sigma_{x_0} = \Sigma_I, \quad \bar{x}_N = \mu_F, \quad \Sigma_{x_N} = \Sigma_F, \tag{17b}$$
$$\bar{x}_{k+1} = \bar{A}\bar{x}_k + \bar{B}\bar{u}_k + \bar{d}, \tag{17c}$$
$$\Sigma_{x_{k+1}} = \bar{A}\Sigma_{x_k}\bar{A}^\top + \bar{A}\Sigma_{x_k u_k}\bar{B}^\top + \bar{B}\Sigma_{x_k u_k}^\top \bar{A}^\top + \bar{B}\Sigma_{u_k}\bar{B}^\top + \sum_{j=1}^{m} (\tilde{A}_j \Sigma_{x_k} \tilde{A}_j^\top + \tilde{A}_j \Sigma_{x_k u_k} \tilde{B}_j^\top + \tilde{B}_j \Sigma_{x_k u_k}^\top \tilde{A}_j^\top + \tilde{B}_j \Sigma_{u_k} \tilde{B}_j^\top) + \sum_{j=1}^{m} (\tilde{A}_j \bar{x}_k + \tilde{B}_j \bar{u}_k + \tilde{d}_j)(\tilde{A}_j \bar{x}_k + \tilde{B}_j \bar{u}_k + \tilde{d}_j)^\top, \tag{17d}$$
$$\bar{u}_k = c_k, \quad \Sigma_{u_k} = L_k \Sigma_{x_k} L_k^\top, \quad \Sigma_{x_k u_k} = \Sigma_{x_k} L_k^\top, \tag{17e}$$
$$\alpha_{x,i_x}^\top \bar{x}_k + \sqrt{\alpha_{x,i_x}^\top \Sigma_{x_k} \alpha_{x,i_x}} \sqrt{\frac{1 - p_{x,i_x}}{p_{x,i_x}}} - \beta_{x,i_x} \le 0, \tag{17f}$$
$$\alpha_{u,i_u}^\top \bar{u}_k + \sqrt{\alpha_{u,i_u}^\top \Sigma_{u_k} \alpha_{u,i_u}} \sqrt{\frac{1 - p_{u,i_u}}{p_{u,i_u}}} - \beta_{u,i_u} \le 0, \tag{17g}$$

for $i_x = 1, \ldots, N_s$, $i_u = 1, \ldots, N_c$, and $k = 0, \ldots, N-1$, where $\mathbf{c} = \{c_0, \ldots, c_{N-1}\}$ and $\mathbf{L} = \{L_0, \ldots, L_{N-1}\}$. We denote the optimal solution of Problem (17) as $(\mathbf{c}^*, \mathbf{L}^*)$ which generates the stochastic control sequence given by $u_k^* = L_k^* \tilde{x}_k^* + c_k^*$, where, from (11) and (8d), follows that $\tilde{x}_{k+1}^* = (\bar{A} + \bar{B}L_k^*)\tilde{x}_k^* + \sum_{j=1}^m \tilde{A}_j q_{j,k} x_k^* + \sum_{j=1}^m \tilde{B}_j q_{j,k} u_k^* + \sum_{j=1}^m \tilde{d}_j q_{j,k}$, $x_{k+1}^* = (\bar{A} + \sum_{j=1}^m \tilde{A}_j q_{j,k}) x_k^* + (\bar{B} + \sum_{j=1}^m \tilde{B}_j q_{j,k}) u_k^* + \bar{d} + \sum_{j=1}^m \tilde{d}_j q_{j,k}$, $\bar{x}_{k+1}^* = \bar{A}\bar{x}_k^* + \bar{B}\bar{u}_k^* + \bar{d}$, and $\bar{u}_k^* = c_k^*$ for all $k = 0, \ldots, N-1$, and where $\tilde{x}_0^* = x_0 - \mu_I$, $x_0^* = x_0$, and $\bar{x}_0^* = \mu_I$.

Theorem 1: Let $\mathbf{c}^\# = \{c_0^\#, \ldots, c_{N-1}^\#\}$ and $\mathbf{L}^\# = \{L_0^\#, \ldots, L_{N-1}^\#\}$ be a feasible solution of Problem (17). Problem (17) is a conservative approximation of Problem (8), such that, any feasible solution of Problem (17) will generate a control sequence given by $\mathbf{u} = \{u_k^\#\}_{k=0}^{N-1} = \{L_k^\# \tilde{x}_k^\# + c_k^\#\}_{k=0}^{N-1}$, which is a feasible solution of Problem (8). Moreover, by letting $\{u_k^*\}_{k=0}^{N-1} = \{L_k^* \tilde{x}_k^* + c_k^*\}_{k=0}^{N-1}$, the optimal solution of Problem (17), in particular, is always a feasible solution of Problem (8) satisfying the constraints (8b)-(8h).

Proof: The mean and covariance propagation equations (17c) and (17d) are exact representations for the first two moments of system (8d), given the control policy (13). Therefore, constraints (17b) are exact representations of (8b), (8c), (8g), and (8h), respectively. For the chance constraints, by Lemma 1, (17f) and (17g) are tightened versions of (8e) and (8f), respectively, such that the satisfaction of (17f) and (17g) implies satisfaction of (8e) and (8f), respectively. Thus, since $\{u_k^\#\}_{k=0}^{N-1}$ and $\{u_k^*\}_{k=0}^{N-1}$ necessarily satisfy constraints (17b)-(17g), they satisfy constraints (8b)-(8h). ■

IV. Convex Reformulation

Although Problem (17) is deterministic, it still depends on nonconvex constraints. First, the propagation of the state covariance (17d) is nonconvex due to the multiplicity of $\bar{x}_k$ and $\bar{u}_k$. Second, the input and state-input covariance constraints (17e) are nonconvex owing to the multiplicity of L_k and Σ_{x_k}. Third, the state and control constraints (17f)-(17g) are nonconvex owing to their nonlinear dependence on Σ_{x_k} and Σ_{u_k}.

We show that the first two issues can be overcome by relaxing the covariance propagation to linear matrix inequality (LMI) constraints, which requires us to relax the terminal covariance constraint (17b) to an inequality constraint, such that we merely ensure the terminal covariance satisfies an upper bound. The third issue is overcome by further tightening the chance constraints such that they can be written as linear inequality constraints.

Remark 1: Relaxing the terminal covariance constraint to an inequality is not an issue in most applications as, in general, the goal is to bound the covariance rather than to drive it to a specific value, as most applications are concerned with designing a controller to reduce the system's uncertainty rather than increase it.

1) Covariance Propagation Relaxation: The propagation of the state uncertainty is a nonlinear constraint owing to the multiplicities of $\bar{x}_k\bar{x}_k^\top$, $\bar{x}_k\bar{u}_k^\top$, and $\bar{u}_k\bar{u}_k^\top$. However, this problem can be overcome by relaxing the equality constraint to a LMI. We introduce a new optimization variable $\bar{\Sigma}_{jk}$ such that

$$\bar{\Sigma}_{jk} \succeq (\tilde{A}_j\bar{x}_k + \tilde{B}_j\bar{u}_k + \tilde{d}_j)(\tilde{A}_j\bar{x}_k + \tilde{B}_j\bar{u}_k + \tilde{d}_j)^\top, \quad (18)$$

which can be written as a positive semidefinite constraint using the Schur complement

$$\begin{bmatrix} \bar{\Sigma}_{jk} & \tilde{A}_j\bar{x}_k + \tilde{B}_j\bar{u}_k + \tilde{d}_j \\ (\tilde{A}_j\bar{x}_k + \tilde{B}_j\bar{u}_k + \tilde{d}_j)^\top & I \end{bmatrix} \succeq 0, \quad (19)$$

for all $j = 1, \ldots, m$ and $k = 0, \ldots, N-1$.

Similarly, to handle the nonconvexity of constraints (17e), we utilize the following change of variables previously used by [26], [27]. Let $\bar{\Sigma}_{ux_k}$ be a new optimization variable such that

$$\bar{\Sigma}_{ux_k} = L_k\Sigma_{x_k} = \Sigma_{x_k u_k}^\top, \quad (20)$$

for all $k = 0, \ldots, N-1$. For $\Sigma_{x_k} \succ 0$, the original control policy can then be recovered as $L_k = \bar{\Sigma}_{ux_k}\Sigma_{x_k}^{-1}$. Using (20), Σ_{u_k} may be written as $\Sigma_{u_k} = \bar{\Sigma}_{ux_k}\Sigma_{x_k}^{-1}\bar{\Sigma}_{ux_k}^\top$. We then introduce a new optimization variable $\bar{\Sigma}_{u_k}$ and relax the expression for Σ_{u_k} to an inequality given by

$$\bar{\Sigma}_{u_k} \succeq \bar{\Sigma}_{ux_k}\Sigma_{x_k}^{-1}\bar{\Sigma}_{ux_k}^\top = \Sigma_{u_k}, \quad (21)$$

which, using the Schur complement, is given by the positive semidefinite constraint $\begin{bmatrix} \bar{\Sigma}_{u_k} & \bar{\Sigma}_{ux_k} \\ \bar{\Sigma}_{ux_k}^\top & \Sigma_{x_k} \end{bmatrix} \succeq 0$, for all $k = 0, \ldots, N-1$. We may then write the relaxed covariance dynamics as

$$\begin{aligned} \bar{\Sigma}_{x_{k+1}} &= \bar{A}\bar{\Sigma}_{x_k}\bar{A}^\top + \bar{A}\bar{\Sigma}_{ux_k}^\top\bar{B}^\top + \bar{B}\Sigma_{ux_k}\bar{A}^\top + \bar{B}\bar{\Sigma}_{u_k}\bar{B}^\top \\ &+ \sum_{j=1}^{m}(\tilde{A}_j\bar{\Sigma}_{x_k}\tilde{A}_j^\top + \tilde{A}_j\Sigma_{ux_k}^\top\tilde{B}_j^\top + \tilde{B}_j\Sigma_{ux_k}\tilde{A}_j^\top \\ &+ \tilde{B}_j\bar{\Sigma}_{u_k}\tilde{B}_j^\top + \bar{\Sigma}_{jk}), \end{aligned} \quad (22)$$

for $k = 0, \ldots, N-1$, where $\bar{\Sigma}_{x_0} = \Sigma_{x_0}$.

Relaxing (17d) and the third inclusion in (17e) with (18), (20), (21), and (22) provides a bound on the state and input covariances.

Lemma 2: If the inequalities

$$\alpha_{x,i_x}^\top\bar{x}_k + \sqrt{\alpha_{x,i_x}^\top\bar{\Sigma}_{x_k}\alpha_{x,i_x}}\sqrt{\frac{1-p_{x,i_x}}{p_{x,i_x}}} - \beta_{x,i_x} \leq 0, \quad (23a)$$

$$\alpha_{u,i_u}^\top\bar{u}_k + \sqrt{\alpha_{u,i_u}^\top\bar{\Sigma}_{u_k}\alpha_{u,i_u}}\sqrt{\frac{1-p_{u,i_u}}{p_{u,i_u}}} - \beta_{u,i_u} \leq 0, \quad (23b)$$

$$\bar{\Sigma}_{x_N} \preceq \Sigma_F, \quad (23c)$$

hold, then the inequalities

$$\alpha_{x,i_x}^\top\bar{x}_k + \sqrt{\alpha_{x,i_x}^\top\Sigma_{x_k}\alpha_{x,i_x}}\sqrt{\frac{1-p_{x,i_x}}{p_{x,i_x}}} - \beta_{x,i_x} \leq 0, \quad (24a)$$

$$\alpha_{u,i_u}^\top\bar{u}_k + \sqrt{\alpha_{u,i_u}^\top\Sigma_{u_k}\alpha_{u,i_u}}\sqrt{\frac{1-p_{u,i_u}}{p_{u,i_u}}} - \beta_{u,i_u} \leq 0, \quad (24b)$$

$$\Sigma_{x_N} \preceq \Sigma_F, \quad (24c)$$

also hold for $i_x = 1, \ldots, N_s$, $i_u = 1, \ldots, N_c$, and $k = 0, \ldots, N-1$.

Proof: Due to space limitations, we omit the proof, which can be found in [25]. ■

2) Chance Constraint Tightening: Finally, the state and input chance constraints (17f)-(17g) include a nonlinear dependence on the state and input covariance, Σ_{x_k} and Σ_{u_k}, respectively. To resolve this, we use a linearization procedure similar to [24]. An upper bound on the square root of the state covariance can be derived using a tangent line approximation evaluated at $\lambda_{x,i_x,k}$ given by

$$\begin{aligned} \sqrt{\alpha_{x,i_x}^\top\Sigma_{x_k}\alpha_{x,i_x}} \leq &\frac{1}{2\sqrt{\lambda_{x,i_x,k}}}(\alpha_{x,i_x}^\top\Sigma_{x_k}\alpha_{x,i_x} - \lambda_{x,i_x,k}) \\ &+ \sqrt{\lambda_{x,i_x,k}}, \end{aligned} \quad (25)$$

for $i_x = 1, \ldots, N_s$ and $k = 0, \ldots, N-1$. A similar bound is given for the input covariance as

$$\begin{aligned} \sqrt{\alpha_{u,i_u}^\top\Sigma_{u_k}\alpha_{u,i_u}} \leq &\frac{1}{2\sqrt{\lambda_{u,i_u,k}}}(\alpha_{u,i_u}^\top\Sigma_{u_k}\alpha_{u,i_u} - \lambda_{u,i_u,k}) \\ &+ \sqrt{\lambda_{u,i_u,k}}, \end{aligned} \quad (26)$$

for $i_u = 1, \ldots, N_c$ and $k = 0, \ldots, N-1$. This leads to a conservative tightening of the chance constraints; therefore, the original constraints are still guaranteed to be satisfied by a solution of the convex reformulation at the expense of reducing the size of the feasible solution set, as stated in the following lemma.

Lemma 3: The satisfaction of the inequalities given by

$$\begin{aligned} \alpha_{x,i_x}^\top\bar{x}_k + (\frac{\sqrt{\lambda_{x,i_x,k}}}{2} + \frac{1}{2\sqrt{\lambda_{x,i_x,k}}}\alpha_{x,i_x}^\top\bar{\Sigma}_{x_k}\alpha_{x,i_x})\sqrt{\frac{1-p_{x,i_x}}{p_{x,i_x}}} \\ - \beta_{x,i_x} \leq 0, \end{aligned} \quad (27a)$$

$$\begin{aligned} \alpha_{u,i_u}^\top\bar{u}_k + (\frac{\sqrt{\lambda_{u,i_u,k}}}{2} + \frac{1}{2\sqrt{\lambda_{u,i_u,k}}}\alpha_{u,i_u}^\top\bar{\Sigma}_{u_k}\alpha_{u,i_u})\sqrt{\frac{1-p_{u,i_u}}{p_{u,i_u}}} \\ - \beta_{u,i_u} \leq 0, \end{aligned} \quad (27b)$$

for $i_x = 1, \ldots, N_s$, $i_u = 1, \ldots, N_c$, and $k = 0, \ldots, N-1$, imply satisfaction of (17f) and (17g).

Proof: We refer to [25] for the proof of Lemma 3. ■

Remark 2: The approximations given by (25) and (26) are exact (i.e., satisfied as an equality) when $\lambda_{x,i_x,k} = \alpha_{x,i_x}^\top\Sigma_{x_k}\alpha_{x,i_x}$ or $\lambda_{u,i_u,k} = \alpha_{u,i_u}^\top\Sigma_{u_k}\alpha_{u,i_u}$, respectively. Therefore, if estimates of Σ_{x_k} and Σ_{u_k} are available, they should be used for the selection of the linearization points.

3) Convex Covariance Steering Problem: Using the above relaxations, a convex formulation of Problem (17) is given by

$$\min_{\mathbf{c},\bar{\boldsymbol{\Sigma}}_{\mathbf{ux}},\bar{\boldsymbol{\Sigma}}_{\mathbf{u}},\bar{\boldsymbol{\Sigma}}} \bar{x}_k^\top Q_k \bar{x}_k + \mathtt{tr}(Q_k \bar{\Sigma}_{x_k}) + \bar{u}_k^\top R_k \bar{u}_k + \mathtt{tr}(R_k \bar{\Sigma}_{u_k}) \tag{28a}$$

subject to

$$\bar{x}_0 = \mu_I, \quad \bar{\Sigma}_{x_0} = \Sigma_I, \quad \bar{x}_N = \mu_F, \quad \bar{\Sigma}_{x_N} \preceq \Sigma_F, \tag{28b}$$

$$\bar{x}_{k+1} = \bar{A}\bar{x}_k + \bar{B}\bar{u}_k + \bar{d} \tag{28c}$$

$$\begin{aligned}\bar{\Sigma}_{x_{k+1}} &= \bar{A}\bar{\Sigma}_{x_k}\bar{A}^\top + \bar{A}\bar{\Sigma}_{ux_k}^\top \bar{B}^\top + \bar{B}\bar{\Sigma}_{ux_k}\bar{A}^\top \\ &\quad + \bar{B}\bar{\Sigma}_{u_k}\bar{B}^\top + \sum_{j=1}^{m}(\tilde{A}_j \bar{\Sigma}_{x_k}\tilde{A}_j^\top + \tilde{A}_j \bar{\Sigma}_{ux_k}^\top \tilde{B}_j^\top \\ &\quad + \tilde{B}_j \bar{\Sigma}_{ux_k}\tilde{A}_j^\top + \tilde{B}_j \bar{\Sigma}_{u_k}\tilde{B}_j^\top + \bar{\Sigma}_{jk}),\end{aligned} \tag{28d}$$

$$\bar{u}_k = c_k \quad \bar{\Sigma}_{u_k} \succeq \bar{\Sigma}_{ux_k}\bar{\Sigma}_{x_k}^{-1}\bar{\Sigma}_{ux_k}^\top \tag{28e}$$

$$\bar{\Sigma}_{jk} \succeq (\tilde{A}_j \bar{x}_k + \tilde{B}_j \bar{u}_k + \tilde{d}_j)(\tilde{A}_j \bar{x}_k + \tilde{B}_j \bar{u}_k + \tilde{d}_j)^\top \tag{28f}$$

$$\begin{aligned}&\Big(\frac{\sqrt{\lambda_{x,i_x,k}}}{2} + \frac{1}{2\sqrt{\lambda_{x,i_x,k}}}\alpha_{x,i_x}^\top \bar{\Sigma}_{x_k}\alpha_{x,i_x}\Big)\sqrt{\frac{1-p_{x,i_x}}{p_{x,i_x}}} \\ &\quad + \alpha_{x,i_x}^\top \bar{x}_k - \beta_{x,i_x} \le 0,\end{aligned} \tag{28g}$$

$$\begin{aligned}&\Big(\frac{\sqrt{\lambda_{u,i_u,k}}}{2} + \frac{1}{2\sqrt{\lambda_{u,i_u,k}}}\alpha_{u,i_u}^\top \bar{\Sigma}_{u_k}\alpha_{u,i_u}\Big)\sqrt{\frac{1-p_{u,i_u}}{p_{u,i_u}}} \\ &\quad + \alpha_{u,i_u}^\top \bar{u}_k - \beta_{u,i_u} \le 0,\end{aligned} \tag{28h}$$

for $i_x = 1, \ldots, N_s$, $i_u = 1, \ldots, N_u$, and $k = 0, \ldots, N-1$, where $\bar{\boldsymbol{\Sigma}}_{\mathbf{ux}} = \{\bar{\Sigma}_{ux_0}, \ldots, \bar{\Sigma}_{ux_{N-1}}\}$, $\bar{\boldsymbol{\Sigma}}_{\mathbf{u}} = \{\bar{\Sigma}_{u_0}, \ldots, \bar{\Sigma}_{u_{N-1}}\}$, and $\bar{\boldsymbol{\Sigma}} = \{\bar{\Sigma}_{j,k}\}_{j=1,k=0}^{m,N-1}$.

Let us denote the optimal solution of Problem (28) as $(\mathbf{c}^\star, \bar{\boldsymbol{\Sigma}}_{\mathbf{ux}}^\star, \bar{\boldsymbol{\Sigma}}_{\mathbf{u}}^\star, \bar{\boldsymbol{\Sigma}}^\star)$, which gives the optimal control policy $(\mathbf{c}^\star, \mathbf{L}^\star)$, where $L_k^\star = \bar{\Sigma}_{ux_k}^\star \bar{\Sigma}_{x_k}^{\star-1}$, $\bar{\Sigma}_{x_{k+1}}^\star = \bar{A}\bar{\Sigma}_{x_k}^\star \bar{A}^\top + \bar{A}\bar{\Sigma}_{ux_k}^{\star\top}\bar{B}^\top + \bar{B}\bar{\Sigma}_{ux_k}^\star \bar{A}^\top + \bar{B}\bar{\Sigma}_{u_k}^\star \bar{B}^\top + \sum_{j=1}^m(\tilde{A}_j\bar{\Sigma}_{x_k}^\star\tilde{A}_j^\top + \tilde{A}_j\bar{\Sigma}_{ux_k}^{\star\top}\tilde{B}_j^\top + \tilde{B}_j\bar{\Sigma}_{ux_k}^\star\tilde{A}_j^\top + \tilde{B}_j\bar{\Sigma}_{u_k}^\star\tilde{B}_j^\top + \bar{\Sigma}_{jk}^\star)$, and $\bar{\Sigma}_{x_0}^\star = \Sigma_I$.

Theorem 2: The optimal solution of Problem (28), $(\mathbf{c}^\star, \bar{\boldsymbol{\Sigma}}_{\mathbf{ux}}^\star, \bar{\boldsymbol{\Sigma}}_{\mathbf{u}}^\star, \bar{\boldsymbol{\Sigma}}^\star)$, yields the optimal control policy $\{c_k^\star, L_k^\star\}_{k=0}^{N-1}$ which is a feasible solution of Problem (17) when constraint the terminal covariance constraint in (17b) is relaxed to (24c). Furthermore, let $\mathbf{u}^\star$ be the control sequence given by $\mathbf{u}^\star = \{u_k^\star = L_k^\star \tilde{x}_k^\star + c_k^\star\}_{k=0}^{N-1}$, where $\tilde{x}_{k+1}^\star = (\bar{A} + \bar{B}L_k^\star)\tilde{x}_k^\star + \sum_{j=1}^m \tilde{A}_j q_{j,k} x_k^\star + \sum_{j=1}^m \tilde{B}_j q_{j,k} u_k^\star + \sum_{j=1}^m \tilde{d}_j q_{j,k}$, $x_{k+1}^\star = (\bar{A} + \sum_{j=1}^m \tilde{A}_j q_{j,k})x_k^\star + (\bar{B} + \sum_{j=1}^m \tilde{B}_j q_{j,k})u_k^\star + \bar{d} + \sum_{j=1}^m \tilde{d}_j q_{j,k}$, $\bar{x}_{k+1}^\star = \bar{A}\bar{x}_k^\star + \bar{B}\bar{u}_k^\star + \bar{d}$, and $\bar{u}_k^\star = c_k^\star$ for all $k = 0, \ldots, N-1$, and where $\tilde{x}_0^\star = x_0 - \mu_I$, $x_0^\star = x_0$, and $\bar{x}_0^\star = \mu_I$. The control sequence $\mathbf{u}^\star$ is a feasible solution of Problem (8) when (8h) is relaxed to (24c).

Proof: The constraints given by the first three inclusions in (28b), (28c), and the first inclusion in (28e) are identical to the corresponding constraints in Problem (17). Per Lemma 2, the satisfaction of the last inclusion in (28b) implies (24c) will be satisfied. By Lemmas 2 and 3 satisfaction of (28g) and (28h) implies satisfaction of (17f) and (17g). Thus, $\{c_k^\star, L_k^\star\}_{k=0}^{N-1}$ is a feasible solution of Problem (17) when the terminal covariance constraint given in (17b) is relaxed to (24c). Feasibility of Problem (8) with the relaxation of (8h) to (24c) is then given by Theorem 1. ■

V. Numerical Results

The proposed approach is demonstrated through a vehicle control example. The kinematic bicycle model is commonly used to model the motion of a vehicle with respect to a given reference path. Although the kinematic bicycle model is nonlinear, a linear approximation may be obtained by assuming a constant velocity and assuming the steering angle and the heading error with respect to the reference path are small, which is an approximation technique commonly used in the literature [28].

The linear kinematic bicycle model [28] is given by

$$\begin{aligned}\begin{bmatrix}\varphi_{k+1}\\ e_{\psi_{k+1}}\\ e_{y_{k+1}}\end{bmatrix} &= \begin{bmatrix}1 & 0 & 0\\ \frac{\bar{\nu}_x}{r_f+r_r}\Delta t & 1 & 0\\ \frac{r_r}{r_f+r_r}\bar{\nu}_x\Delta t & \bar{\nu}_x\Delta t & 1\end{bmatrix}\begin{bmatrix}\varphi_k\\ e_{\psi_k}\\ e_{y_k}\end{bmatrix} + \begin{bmatrix}0\\ \dot{\psi}_k\Delta t\\ 0\end{bmatrix} \\ &+ \begin{bmatrix}\Delta t\\ \frac{r_r}{r_f+r_r}\Delta t\\ 0\end{bmatrix}(\dot{\varphi}_k\theta_{u_k}) + \begin{bmatrix}0 & 0 & 0\\ \frac{\Delta t}{r_f+r_r} & 0 & 0\\ \frac{r_r}{r_f+r_r}\Delta t & \Delta t & 0\end{bmatrix}\begin{bmatrix}\varphi_k\\ e_{\psi_k}\\ e_{y_k}\end{bmatrix}\tilde{\nu}_{x_k},\end{aligned} \tag{29}$$

where $\dot{\psi}_k = (\tilde{\dot{\psi}}_k - \bar{\dot{\psi}}_{\text{ref}})$, e_ψ is the heading error with respect to the reference heading ψ_{ref}, ν_x is the velocity parameter, r_f and r_r are the length from the center of mass to the front and rear wheels respectively, φ is the steering angle, e_y is the lateral error with respect to the reference path, $\dot{\varphi}_k$ is the control input and where $\bar{\nu}_x$ and $\bar{\dot{\psi}}_{\text{ref}}$ are the nominal parameters and $\tilde{\nu}_x, \tilde{\dot{\varphi}}_k, \theta_{u_k}$ are the random disturbances. We set $\bar{\nu}_x = 10$, $r_f = 1.2$, $r_r = 3.6$, $\Delta t = 0.1$, and $\bar{\dot{\psi}}_{\text{ref}} = 0.1$. The initial and terminal conditions are given as $\mu_I = [0,0,0]^\top$, $\Sigma_I = \mathtt{diag}(0.003, 0.03, 0.03)$, $\mu_F = [0.1, 0, 0]^\top$, and $\Sigma_F = \mathtt{diag}(0.05, 0.05, 0.05)$. The state chance constraints are given by $\alpha_{x,1} = [0, 0, 1.0]^\top$, $\beta_{x,1} = 1.5$, $p_{x,1} = 0.2$, $\alpha_{x,2} = [0, 0, -1.0]^\top$, $\beta_{x,2} = 1.5$, $p_{x,2} = 0.2$ and no control chance constraints are included so that $N_s = 2$ and $N_c = 0$. $\lambda_{x,i_x,k} = \alpha_{x,i_x}^\top \Sigma_{x,k}^{\text{nom}} \alpha_{x,i_x}$ for $i_x = 1, 2$ and $k = 0, \ldots, N-1$, and where $\Sigma_{x,k}^{\text{nom}}$ is computed as an N-step linear interpolation between Σ_I and Σ_F. The trajectory is planned over 5 sec, so that $N = 50$.

The proposed approach is compared for three different noise distributions with an optimistic approach that does not consider the multiplicative uncertainties arising from $\tilde{A}_j$ and $\tilde{B}_j$ and with a conservative approach in which the $\tilde{d}_j$ vectors are set as an over-approximation of the noise arising from input uncertainty (over-approximating the state transition matrix uncertainty was also investigated; however, we found this led to infeasibility in non-trivial cases). Fig. 1 shows the trajectories resulting from the three approaches transformed into a Cartesian coordinate frame, and it may be seen all three approaches track the reference trajectory through the curve. However, as highlighted in Fig. 1, the naïve approach fails to meet the terminal constraints and the conservative approach exceeds the specified terminal covariance, while the proposed approach successfully meets the required terminal mean and covariance.

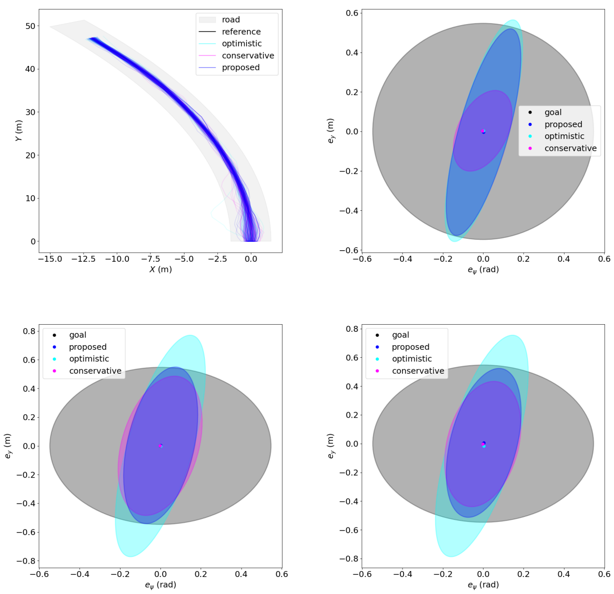

Fig. 1: Covariance steering results for a path-following vehicle application with parametric noise sampled from Gaussian, Uniform, and Gamma distributions, respectively.

VI. Conclusion

This work has presented a general problem formulation for stochastic linear systems subjected to both additive and multiplicative noise, while subject to state and control chance constraints as well as terminal constraints on the first and second moments. Although the problem is, in general, stochastic and nonconvex, a tightened, deterministic, convex problem formulation is derived, the optimal solution of which is guaranteed to be a valid (albeit potentially sub-optimal) solution of the original nonconvex problem. Finally, the results are demonstrated using Monte Carlo simulations on an autonomous vehicle path following problem.

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