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Adaptive output regulation of MIMO LTI systems with unmodeled input dynamics

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Abstract— **This paper addresses the problem of adaptive output regulation of a minimum-phase MIMO LTI system in the presence of un-modeled (fast) input dynamics. The adoption of a post-processing tunable internal model makes it possible to implement standard methods for the analysis of two-time-scale systems. The proposed adaptation law guarantees, under suitable hypotheses, convergence to zero of the regulation error as well as of the parameter estimation error.**

I. Introduction

The problem of controlling systems under conditions of significant lack of prior knowledge is a problem of high interest in feedback design. This is the case in the classical problem of output regulation, which is the problem of controlling a plant so as to have its output asymptotically tracking/rejecting exogenous commands/disturbances, where uncertainties may include unknown parameters, unmeasurable states, uncertain exogenous inputs, and unmodeled dynamics. Solutions to the problem in question, at different levels of generality and in different scenarios of uncertainty, have been known since a long time in control theory. A common feature of such solutions is the so-called internal model principle (IPM), which establishes the necessity of the presence, in the controller, of a suitable model of the exogenous inputs [1]. [1]

For a MIMO linear system, the IMP claims that asymptotic regulation is achieved in the presence of plant parameter variations "only if the controller utilizes feedback of the regulated variable, and incorporates in the feedback path a suitably reduplicated model of the dynamic structure of the exogenous signals". If such model is embedded in the feedback path, a problem of output regulation in the presence of structured/unstructured uncertainties is reduced to a problem of robust stabilization of a suitable augmented system [1][3]. Robust stabilization of such augmented system can be achieved – in the case of structured uncertainties – via (dynamic) high-gain output feedback if the system is minimum-phase (see, e.g. [4], for a recent review in which a notion of robust-minimum phase is exploited) and – in the case of unstructured uncertainties – via H_∞ and LMI methods (see, e.g., [5], [6], [7]).

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[1] A thorough analysis of the implications of the concept of internal model, not only in system science but also in various branches of natural sciences, can be found in the recent survey [2].

Extensions of this viewpoint to nonlinear systems have a long history in the literature, beginning with the series works of [8], [9], [10]. For nonlinear systems, in recent years, the issue of robustness has been thoroughly investigated and limits to the ambition of achieving robustness have been pointed out (see, in particular, [11], where it is claimed that no finite-dimensional robust regulator exists if unstructured perturbations are considered). This result has pushed a good part of the research toward the development of approximate but robust design methods (see e.g. [12]).

A kind of uncertainty that was not addressed in the original works [1][3] is the possible lack of knowledge of the parameters that characterize the so-called exosystem, the (finite-dimensional) autonomous system that models all possible exogenous exogenous inputs. Indeed, if such parameters are unknown, the design of an internal model becomes problematic. It was only with the works [13], in which a tunable internal model was proposed, and [14][15], in which the "modes" of the exosystem (assumed to be a bank of harmonic oscillators) were directly estimated, that the issue of robustness with respect exosystem uncertainties begun to be systematically addressed. Since then, a number of relevant contributions appeared (see, e.g., [16], [17], [18][19][20][11][21], among the most recent ones).

An issue that, to the best of our knowledge, is not explicitly addressed in the existing literature is the issue of robustness with respect to unmodeled (but fast) input dynamics. In dealing with such kind of unstructured uncertainty, one is intuitively tempted to believe that a design based on the nominal model (i.e. ignoring the input dynamics) would also work in the presence of unmodeled input dynamics, if the latter are sufficiently fast. Such intuition, though, has to be supported by rigorous arguments, and this is precisely the main goal of the present paper, where we deal with a rather general class of MIMO linear time invariant systems. The framework is essentially the same as of [4]. Specifically, it is shown that the choice of a post-processing adaptive internal model, as in [4], makes it possible to look at the augmented system as a two-time-scale system, with consequent use the pertinent methods of analysis. In this context, though, it is necessary to make sure that, in the so-called "reduced" subsystem, trajectories are uniformly attracted to a compact invariant set on which the regulated output vanishes. This is the case if the exogenous inputs have special properties, that are assumed to hold in section III. A byproduct of such assumptions is that the proposed design secures asymptotic convergence to zero of the regulation error as well as of the parameter estimation error.

II. PROBLEM SETUP

In this paper we consider a MIMO linear system with state $x \in \mathbb{R}^n$, input $v \in \mathbb{R}^m$ and regulated output $e \in \mathbb{R}^m$, modeled by equations of the form

$$\begin{array}{rcl} \dot{x} & = & Ax + Bv + Pw \\ e & = & Cx + Qw \,. \end{array} \qquad (1)$$

The exogenous input $w \in \mathbb{R}^d$ is generated by an exosystem

$$\dot{w} = S_\rho w \qquad (2)$$

in which ρ denotes a vector of uncertain parameters.

The input v of (1) is the output of an unmodeled dynamics

$$\begin{array}{rcl} v & = & C_0 x_0 \\ \mu \dot{x}_0 & = & A_0 x_0 + B_0 u \end{array} \qquad (3)$$

with state $x_0 \in \mathbb{R}^{n_0}$ and control $u \in \mathbb{R}^m$, in which $\mu \in \mathbb{R}^+$ is small.

Remark 1: The dynamics (3) represents dynamics that are neglected in the modeling process, such as the dynamics of the actuators or any dynamics affecting the plant's input. Typically, such dynamics are much faster compared with the plant's own dynamics, and stable. However, its parameters and even its order might be unknown.

We address the classical problem of (adaptive) output regulation, i.e. the design of a feedback controller – with input e and output u – yielding a closed-loop system in which trajectories are bounded and $\lim_{t\to\infty} e(t) = 0$, regardless of the actual values of the uncertain parameters in the exosystem.

III. BASIC ASSUMPTIONS

A. Assumptions about the plant

The plant (1) is assumed to be *invertible* and *minimum phase*. If this is the case, as observed in [4], there exists a change of coordinates by means of which the equations (1) can be transformed into equations of the form [2]

$$\begin{array}{rcl} \dot{z} & = & A_{00} z + A_{01} z_1 + A_{02} z_2 + P_0 w \\ \dot{z}_1 & = & A_{11} z_1 + A_{12} z_2 \\ \dot{z}_2 & = & A_{20} z + A_{21} z_1 + A_{22} z_2 + B_{22} v + P_2 w \\ e & = & H_1 z_1 + H_2 z_2 \end{array} \qquad (4)$$

in which $z_2 \in \mathbb{R}^m$, the matrix A_{00} is Hurwitz (a consequence of the assumption of minimum phase), the matrix A_{11} is also Hurwitz (a consequence of the special choice coordinates yielding (4)) and the matrix B_{22} is nonsingular. We assume, in addition, that the matrix B_{22} is positive definite. Borrowing a terminology from [22] such system could be called a *"hyper minimum-phase"* system. [3]

Remark 2: In this respect, it should be stress that in [4] we considered a broader class of plants, namely systems having $p < m$ regulated outputs and in which $m - p$ inputs could be used, via feedback from an auxiliary measured output y_r, to enforce the property of minimum phase. Extension of the approach presented in the current paper to this broader class of systems is possible, but it would imply a sensible increase of length. For such reason, we have limited the consideration to the class of systems defined in (1), and have chosen to put the emphasis on a rigorous proof of the intuition that a design based on the nominal model works also in the presence of unmodeled input dynamics, if the latter are sufficiently fast.

It is also assumed that the solution (Π, Ψ) of the so-called *regulator equations* of (1), namely the equations

$$\begin{array}{rcl} \Pi S_\rho & = & A\Pi + B\Psi + P \\ 0 & = & C\Pi + Q \,, \end{array} \qquad (5)$$

has the following property:

Assumption 1: Let Ψ_i denote the i-th row of Ψ. For some integer $1 \le i^* \le m$ the pair (S_ρ, Ψ_{i^*}) is observable.

B. Assumptions about the exosystem

The matrix $S_\rho \in \mathbb{R}^{d\times d}$ is a matrix in companion form with characteristic polynomial

$$\psi(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_d)$$

with $\lambda_i \neq \lambda_j \neq 0$ for $i \neq j$ and $\mathrm{Re}[\lambda_i] = 0$ for $i = 1, \ldots, d$.[4] This being the case, it is clear that d is even and – after a suitable coordinates transformation $\tilde{w} = Tw$ – the exosystem can be expressed as a bank of $d/2$ harmonic oscillators of the form

$$\dot{\tilde{w}}_i = \begin{pmatrix} 0 & \rho_i \\ -\rho_i & 0 \end{pmatrix} \tilde{w}_i \qquad \tilde{w}_i \in \mathbb{R}^2 \qquad (6)$$

in which $\rho_i \neq \rho_j$ for $i \neq j$. The ρ_i's, the components of the vector ρ of uncertain parameters, are real numbers assumed to vary between fixed upper and lower bounds.

In this paper, we make two additional assumption on the exosystem. The first assumption is that the initial conditions, taken in a compact invariant set, are such that *all modes of the system are excited*. In the special coordinates yielding a model consisting of a bank of $d/2$ harmonic oscillators of the form (6) the assumption in question is that each $\tilde{w}_i(0)$ is taken in an "annular" set of the form

$$B_a^b = \{\tilde{w}_i \in \mathbb{R}^2 : 0 < a \le \|\tilde{w}_i\| \le b\} \,.$$

In the original coordinates of (2) the assumption reads as follows:

Assumption 2: The initial conditions of (2) are taken in the set

$$\mathcal{W} = T^{-1}(B_a^b \times \cdots \times B_a^b) \,.$$

Clearly, if $w(t)$ is expressed in the form

$$w(t) = \sum_{j=1}^{d} \bar{v}_j e^{\lambda_j t} \bar{u}_j^{\mathrm{T}} w_0 \,, \qquad w_0 = w(0) \qquad (7)$$

Assumption 2 is that $\bar{u}_j^{\mathrm{T}} w_0 \neq 0$ for all $j = 1, \ldots, d$.

[2] Note that in (1) the exogenous input w does not affect the dynamics of z_1 nor the regulated output e. This is a consequence of the special choice of coordinates z, z_1, z_2.

[3] For convenience, in the present paper we have considered the case of a system having the same number of inputs and outputs. If the system has more inputs then outputs, and is right-invertible, the results of the paper can be extended without much effort.

[4] Note that we exclude the case of an exosystem with an eigenvalue at 0. Such assumption is related to Assumption 2.

The second assumption is that all ρ_i's are rationally related.

Assumption 3: There exist a real number Ω and integers $N_1, N_2, \dots$ such that $\rho_i N_i = \Omega$ for all i.

If this is the case, the exogenous input $w(t)$ is a periodic function of t.

C. Assumptions about the unmodeled dynamics

As indicated in Remark 1, the unmodeled dynamics (3) has to be stable and sufficiently fast. Thus, we assume that the matrix A_0 is Hurwitz and $\mu > 0$ is sufficiently small number. Moreover, we assume $C_0 A_0^{-1} B_0 = -I$ so as to make $v = -C_0 A_0^{-1} B_0 u = u$ for the case with the unmodeled dynamics excluded, i.e. when $\mu = 0$.

IV. The internal model and the control

Let the system be augmented with a *post-processing* internal model

$$\dot\eta \;=\; F\eta + G[\hat\Gamma(t)\eta + z_2]$$

in which

$$\begin{array}{rcll} F &=& I_m \otimes F_0 & F_0 \in \mathbb{R}^{d\times d} \\ G &=& I_m \otimes G_0 & G_0 \in \mathbb{R}^{d\times 1} \\ \hat\Gamma &=& I_m \otimes \hat\Gamma_0 & \hat\Gamma_0 \in \mathbb{R}^{1\times d} \\ \eta &=& \mathrm{col}(\eta_1,\dots,\eta_m) & \eta_i \in \mathbb{R}^{d\times 1}\,, \end{array}$$

where "$\otimes$" denotes the Kronecker product of matrices. The matrix F_0 is a Hurwitz matrix in companion form and $G_0 = \begin{pmatrix}0 & \cdots & 0 & 1\end{pmatrix}^{\mathrm{T}}$. Choose

$$u = -k[\hat\Gamma(t)\eta + z_2]\,.$$

The resulting system is modeled by equations of the form

$$\begin{array}{rcl} \dot z &=& A_{00}z + A_{01}z_1 + A_{02}z_2 + P_0 w \\ \dot\eta &=& (F + G\hat\Gamma(t))\eta + Gz_2 \\ \dot z_1 &=& A_{11}z_1 + A_{12}z_2 \\ \dot z_2 &=& A_{20}z + A_{21}z_1 + A_{22}z_2 + B_{22}C_0x_0 + P_2w \\ \mu\dot x_0 &=& A_0x_0 - kB_0(\hat\Gamma(t)\eta + z_2) \\ e &=& H_1z_1 + H_2z_2\,. \end{array} \qquad (8)$$

Let $\Gamma_{0,\rho}$ be such that $F_0 + G_0\Gamma_{0,\rho} = S_\rho$, set

$$\Gamma_\rho = I_m \otimes \Gamma_{0,\rho}$$

and define an estimation error as

$$\tilde\Gamma = \hat\Gamma - \Gamma_\rho = I_m \otimes \tilde\Gamma_0 \qquad \text{where} \qquad \tilde\Gamma_0 = \hat\Gamma_0 - \Gamma_{0,\rho}\,.$$

Set $\mathbf{x} = \mathrm{col}(z, \eta, z_1, z_2)$ and

$$\mathbf{A}_\rho = \begin{pmatrix} A_{00} & 0 & A_{01} & A_{02} \\ 0 & (F + G\Gamma_\rho) & 0 & G \\ 0 & 0 & A_{11} & A_{12} \\ A_{20} & 0 & A_{21} & A_{22} \end{pmatrix} \quad \mathbf{B}_\rho = \begin{pmatrix} 0 \\ 0 \\ 0 \\ B_{22} \end{pmatrix}$$

$$\mathbf{P} = \begin{pmatrix} P_0 \\ 0 \\ 0 \\ P_2 \end{pmatrix} \quad \mathbf{Q} = \begin{pmatrix} 0 \\ G \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{C}_\rho = \begin{pmatrix} 0 & \Gamma_\rho & 0 & I \end{pmatrix}.$$

Then, the resulting closed-loop system can be expressed in the form

$$\begin{array}{rcl} \dot{\mathbf{x}} &=& \mathbf{A}_\rho\mathbf{x} + \mathbf{B}_\rho C_0x_0 + \mathbf{P}w + \mathbf{Q}(\tilde\Gamma\eta) \\ \mu\dot x_0 &=& A_0x_0 - kB_0\mathbf{C}_\rho\mathbf{x} - kB_0(\tilde\Gamma\eta)\,. \end{array}$$

To these equations we need to add the adaptation law for $\hat\Gamma$, which determines the dynamics of $\tilde\Gamma$.

As a preliminary step in the choice of the adaptation law, it is important to examine first the influence of the parameters μ and k. To this end, we introduce appropriate additional notations, defining[5]

$$\begin{array}{rcl} \mathbf{A} &=& \mathbf{A}_\rho - k\mathbf{B}_\rho\mathbf{C}_\rho \\ \mathbf{B} &=& \mathbf{Q} - k\mathbf{B}_\rho \\ \mathbf{C} &=& \begin{pmatrix}0 & 0 & 0 & I\end{pmatrix} \\ \mathbf{D} &=& \begin{pmatrix}0 & I & 0 & 0\end{pmatrix}. \end{array} \qquad (9)$$

Lemma 1: There is a number k^* such that, if $k > k^*$, the matrix $\mathbf{A}$ defined in (9) is Hurwitz.

Proof: The proof uses arguments identical to arguments used in the proof of Proposition 4 in [4] and is not repeated here. ∎

Lemma 2: Pick $k > k^*$. There is a number $\mu^* > 0$ such that, if $0 < \mu < \mu^*$, the linear matrix equation

$$\begin{pmatrix} \Pi \\ \mu\Pi_0 \end{pmatrix} S_\rho = \begin{pmatrix} \mathbf{A}_\rho & \mathbf{B}_\rho C_0 \\ -kB_0\mathbf{C}_\rho & A_0 \end{pmatrix}\begin{pmatrix} \Pi \\ \Pi_0 \end{pmatrix} + \begin{pmatrix} \mathbf{P} \\ 0 \end{pmatrix} \qquad (10)$$

has a (unique) solution (Π, Π_0). Moreover, if Π is partitioned, accordingly with the partition of $\mathbf{x}$, as

$$\Pi = \begin{pmatrix} \Pi_z \\ \Sigma \\ \Pi_1 \\ \Pi_2 \end{pmatrix},$$

then $\Pi_1 = 0$ and $\Pi_2 = 0$.

Proof: It is known[6] in [23] that the equation in question has a unique solution if the matrix

$$A(\lambda,\mu) = \begin{pmatrix} I & 0 \\ 0 & \mu I \end{pmatrix}\lambda - \begin{pmatrix} \mathbf{A}_\rho & \mathbf{B}_\rho C_0 \\ -kB_0\mathbf{C}_\rho & A_0 \end{pmatrix}$$

is nonsingular for each λ which is an eigenvalue of S_ρ.

Indeed, the matrix $A(\lambda,\mu)$ is nonsingular if so is the matrix

$$\begin{array}{rcl} \bar A(\lambda,\mu) &=& A(\lambda,\mu)\begin{pmatrix} I & 0 \\ kA_0^{-1}B_0\mathbf{C}_\rho & I \end{pmatrix} \\ &=& \begin{pmatrix} \lambda I - (\mathbf{A}_\rho - k\mathbf{B}_\rho\mathbf{C}_\rho) & -\mathbf{B}_\rho C_0 \\ \mu\lambda kA_0^{-1}B_0\mathbf{C}_\rho & \mu\lambda I - A_0 \end{pmatrix}. \end{array}$$

We know from Lemma 1 that, since $k > k^*$, the matrix $(\mathbf{A}_\rho - k\mathbf{B}_\rho\mathbf{C}_\rho)$ is Hurwitz. Moreover, A_0 is a Hurwitz matrix by assumption. Let $d(\lambda,\mu)$ denote the determinant of the matrix $\bar A(\lambda,\mu)$, let $\lambda \in \sigma(S_\rho)$[7] and observe that

$$d(\lambda,0) = \det[\lambda I - (\mathbf{A}_\rho - k\mathbf{B}_\rho\mathbf{C}_\rho)]\det[-A_0] \neq 0$$

because A_0 is nonsingular and no eigenvalue of S_ρ can be an eigenvalue of $(\mathbf{A}_\rho - k\mathbf{B}_\rho\mathbf{C}_\rho)$. By continuity, $d(\lambda,\mu) \neq 0$ also for small μ.

[5]The partitions in $\mathbf{C}$ and $\mathbf{D}$ are consistent with the partitions of $\mathbf{x}$.

[6]See Theorem A.1 in [23].

[7]The notation $\sigma(S_\rho)$ denotes spectrum of the matrix S_ρ.

Rewrite (10) as

$$\begin{pmatrix}\Pi_z\\ \Sigma\\ \Pi_1\\ \Pi_2\\ \mu\Pi_0\end{pmatrix} S_\rho = \begin{pmatrix} A_{00} & 0 & A_{01} & A_{02} & 0\\ 0 & (F+G\Gamma_\rho) & 0 & G & 0\\ 0 & 0 & A_{11} & A_{12} & 0\\ A_{20} & 0 & A_{21} & A_{22} & B_{22}C_0\\ 0 & -kB_0\Gamma_\rho & 0 & -kB_0 & A_0\end{pmatrix}\begin{pmatrix}\Pi_z\\ \Sigma\\ \Pi_1\\ \Pi_2\\ \Pi_0\end{pmatrix} + \begin{pmatrix}P_0\\ 0\\ 0\\ P_2\\ 0\end{pmatrix}. \tag{11}$$

It is known[8] that, because of the special structure chosen for F, G, Γ_ρ, the equation on the second block-row, namely

$$\Sigma S_\rho = (F + G\Gamma_\rho)\Sigma + G\Pi_2$$

implies $\Pi_2 = 0$. This being the case, the resulting equation on the third block-row, namely

$$\Pi_1 S_\rho = A_{11}\Pi_1 + A_{12}\Pi_2 = A_{11}\Pi_1$$

yields $\Pi_1 = 0$ (because the spectra of S_ρ and A_{11} are disjoint). ■

Rescaling state variables as $\tilde{z} = z - \Pi_z w$, $\tilde{\eta} = \Sigma w$, $\tilde{x}_0 = x_0 - \Pi_0 w$ (no rescaling is needed for z_1, z_2 because $\Pi_1 = 0$ and $\Pi_2 = 0$) and setting $\tilde{\mathbf{x}} = \mathrm{col}(\tilde{z}, \tilde{\eta}, z_1, z_2)$, yields

$$\begin{aligned} \dot{\tilde{\mathbf{x}}} &= \mathbf{A}_\rho\tilde{\mathbf{x}} + \mathbf{B}_\rho C_0\tilde{x}_0 + \mathbf{Q}(\tilde{\Gamma}\eta)\\ \mu\dot{\tilde{x}}_0 &= A_0\tilde{x}_0 - kB_0\mathbf{C}_\rho\tilde{\mathbf{x}} - kB_0(\tilde{\Gamma}\eta)\end{aligned}$$

Note that we have not rescaled the variable η in the product $(\tilde{\Gamma}\eta)$.

We analyze this system choosing the standard change of variables used in dealing with two-time-scale systems (see Example 5.14 or proof of Theorem 11.4 of [24]). Change $\tilde{x}_0$ into

$$y = \tilde{x}_0 - kA_0^{-1}B_0[\mathbf{C}_\rho\tilde{\mathbf{x}} + (\tilde{\Gamma}\eta)].$$

to obtain equations of the form

$$\dot{\tilde{\mathbf{x}}} = \mathbf{A}\tilde{\mathbf{x}} + \mathbf{B}(\tilde{\Gamma}\eta) + \mathbf{B}_\rho C_0 y \tag{12}$$

in which $\mathbf{A}, \mathbf{B}$ are defined as in (9), and

$$\begin{aligned}&\mu\dot{y} = A_0 y\\ &\quad - \mu kA_0^{-1}B_0\Big[\mathbf{C}_\rho[\mathbf{A}\tilde{\mathbf{x}} + \mathbf{B}(\tilde{\Gamma}\eta) + \mathbf{B}_\rho C_0 y] + \tfrac{d}{dt}(\tilde{\Gamma}\eta)\Big].\end{aligned} \tag{13}$$

We proceed now with the choice of the adaptation law, which is suggested by the following result, whose proof is identical to the proof of a similar result in [4].

Proposition 1: Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be defined as in (9). There exists a positive definite symmetric matrix $\mathbf{P}$ such that, if k is sufficiently large,

$$\begin{aligned}\mathbf{PA} + \mathbf{A}^{\mathrm{T}}\mathbf{P} &< \mathbf{0}\\ \mathbf{PB} &= -\mathbf{C}^{\mathrm{T}}.\end{aligned}$$

Proof: See proof of Proposition 4 in [4]. ■

In view of this, one is tempted to use the adaptation law already used in [4], which is a law of the form

$$\dot{\tilde{\Gamma}}_0^{\mathrm{T}} = \dot{\tilde{\Gamma}}_0^{\mathrm{T}} = \begin{pmatrix}\eta_1 & \cdots & \eta_m\end{pmatrix} z_2 = \begin{pmatrix}\eta_1 & \cdots & \eta_m\end{pmatrix}\mathbf{C}\tilde{\mathbf{x}}. \tag{14}$$

In fact, if such law is adopted, along the trajectories of

$$\begin{aligned}\dot{\tilde{\mathbf{x}}} &= \mathbf{A}\tilde{\mathbf{x}} + \mathbf{B}(\tilde{\Gamma}\eta)\\ \dot{\tilde{\Gamma}}_0^{\mathrm{T}} &= \begin{pmatrix}\eta_1 & \cdots & \eta_m\end{pmatrix}\mathbf{C}\tilde{\mathbf{x}}\end{aligned}$$

the positive definite function $V(\tilde{\mathbf{x}}, \tilde{\Gamma}_0^{\mathrm{T}}) = \tilde{\mathbf{x}}^{\mathrm{T}}\mathbf{P}\tilde{\mathbf{x}} + \tilde{\Gamma}_0\tilde{\Gamma}_0^{\mathrm{T}}$ satisfies

$$\dot{V} = \tilde{\mathbf{x}}^{\mathrm{T}}(\mathbf{PA} + \mathbf{A}^{\mathrm{T}}\mathbf{P})\tilde{\mathbf{x}} \le 0.$$

Having chosen the adaptation law in this way, let's augment the dynamics of (12) and (14) with that of the exosystem, so as to obtain an *autonomous* system of the form

$$\begin{pmatrix}\dot{w}\\ \dot{\tilde{\mathbf{x}}}\\ \dot{\tilde{\Gamma}}_0^{\mathrm{T}}\end{pmatrix} = \begin{pmatrix} S_\rho w\\ \mathbf{A}\tilde{\mathbf{x}} + \mathbf{B}(I_m\otimes\tilde{\Gamma}_0)\eta\\ \begin{pmatrix}\eta_1 & \cdots & \eta_m\end{pmatrix}\mathbf{C}\tilde{\mathbf{x}}\end{pmatrix} + \begin{pmatrix}0\\ \mathbf{B}_\rho C_0\\ 0\end{pmatrix} y \tag{15}$$

in which $\eta = \mathbf{D}\tilde{\mathbf{x}} + \Sigma w$ with $\mathbf{D}$ defined as in (9) and Σ defined as in Lemma 2

V. The asymptotic properties of the reduced system

To proceed with the analysis of the two-time-scale system consisting of (15) and (13), it is convenient to begin with the discussion of the asymptotic properties of the "reduced" system, which – since $\mu = 0$ in (13) implies $y = 0$ – is

$$\begin{pmatrix}\dot{w}\\ \dot{\tilde{\mathbf{x}}}\\ \dot{\tilde{\Gamma}}_0^{\mathrm{T}}\end{pmatrix} = \begin{pmatrix} S_\rho w\\ \mathbf{A}\tilde{\mathbf{x}} + \mathbf{B}(I_m\otimes\tilde{\Gamma}_0)\eta\\ \begin{pmatrix}\eta_1 & \cdots & \eta_m\end{pmatrix}\mathbf{C}\tilde{\mathbf{x}}\end{pmatrix} \quad \text{where } \eta = \mathbf{D}\tilde{\mathbf{x}} + \Sigma w. \tag{16}$$

Note that this system, setting $p = \mathrm{col}(\tilde{\mathbf{x}}, \tilde{\Gamma}_0^{\mathrm{T}})$, can be regarded as a system of the form

$$\begin{aligned}\dot{w} &= S_\rho w\\ \dot{p} &= f(w, p).\end{aligned} \tag{17}$$

Since $w(t)$ evolves on the compact invariant set $\mathcal{W}$, it is natural to consider system (17) as a system evolving on the (closed) "cylinder" $\mathcal{C} = \{(w, p) : w \in \mathcal{W}, p \in \mathbb{R}^{n_p}\}$, where $n_p = n + md + m$. Note that the set

$$\mathcal{A} = \{(w, p) \in \mathcal{C} : p = 0\} \tag{18}$$

is an invariant set of (17).

Proposition 2: The invariant set $\mathcal{A}$ is uniformly stable (in the sense of Lyapunov).

Proof: Consider the Lyapunov function

$$U(w, p) = W(w) + V(p)$$

with $V(p) = \tilde{\mathbf{x}}^{\mathrm{T}}\mathbf{P}\tilde{\mathbf{x}} + \tilde{\Gamma}_0\tilde{\Gamma}_0^{\mathrm{T}}$ and $W(w) = w^{\mathrm{T}}Qw$, with $Q = Q^{\mathrm{T}} > 0$ satisfying $QS_\rho + S_\rho^{\mathrm{T}}Q = 0$ (which is admissible because all eigenvalues of S_ρ have zero real part and multiplicity one in the minimal polynomial). Then, $\dot{U} = \tilde{\mathbf{x}}^{\mathrm{T}}(\mathbf{PA} + \mathbf{A}^{\mathrm{T}}\mathbf{P})\tilde{\mathbf{x}} \le 0$. Thus, the function $U(w, p)$ is non-increasing along the trajectories of (17). Observe that $W(w(t))$ is constant along trajectories, because $\dot{W}(w(t)) = 0$. Hence, for $t \ge 0$, we have

$$V(p(t)) \le V(p(0)). \tag{19}$$

[8] See Lemma 4.3 in [23].

Since $V(p)$ is positive definite, there exist positive numbers a_1, a_2 such that $a_1\|p\|^2 \leq V(p) \leq a_2\|p\|^2$. Using such inequalities in (19) it is seen that

$$\|p(t)\| \leq (a_1^{-1}a_2)^{\frac{1}{2}}\|p(0)\| \qquad \forall t \geq 0$$

which proves the uniform stability of the invariant set $\mathcal{A}$. ■

Proposition 3: All trajectories of (17) are bounded in positive time and the $\tilde{\mathbf{x}}(t)$ component of $p(t)$ satisfies

$$\lim_{t\to\infty} \tilde{\mathbf{x}}(t) = 0\,. \tag{20}$$

Proof: With the proof of the previous proposition in mind and in particular the fact that $\dot{U}(t) \leq -\lambda\|\tilde{\mathbf{x}}(t)\|^2$ for some $\lambda > 0$, use standard results due to Barbashin-Krasovskii-LaSalle to conclude that all trajectories are bounded and (20) holds. ■

Remark 3: Note that since $e(t) = H_1 z_1(t) + H_2 z_2(t)$ and z_1, z_2 are components of $\tilde{\mathbf{x}}$ (because such variables are not rescaled), if $y(t)$ were zero the limit (20) would imply

$$\lim_{t\to\infty} e(t) = 0. \tag{21}$$

In the following two Propositions we take advantage of Assumptions 1 and 2 to analyze the asymptotic properties of system (17).

Proposition 4: [9] Suppose Assumptions 1 and 2 hold. Then

$$\lim_{t\to\infty} \tilde{\Gamma}_0(t) = 0\,, \tag{22}$$

i.e. the invariant set $\mathcal{A}$ is globally attractive.

Remark 4: Note that, under such assumptions, if $y(t)$ were zero the chosen adaptation law would imply asymptotic convergence of the estimate $\hat{\Gamma}(t)$ to the true value Γ_ρ.

Initial conditions of (17) are assumed to range on a fixed compact set $\mathcal{B}$ of $\mathcal{C}$. It is known from Proposition 3 that the positive orbit of $\mathcal{B}$ under the flow of (17) is bounded and hence the ω-limit set [10] of $\mathcal{B}$, denoted $\omega(\mathcal{B})$, is a non-empty compact invariant set.

Proposition 5: [9] Let $\mathcal{B}$ be a compact set of $\mathcal{C}$ satisfying $\mathcal{A} \subset \mathrm{int}(\mathcal{B})$. Suppose Assumptions 1 and 2 hold. Then $\omega(\mathcal{B}) = \mathcal{A}$.

We complete the analysis of the asymptotic properties of the reduced system by showing that the set $\mathcal{A}$ is also *locally exponentially* stable. To this end, we recall that if Assumption 3 holds, the exogenous input $w(t) = e^{S_\rho t}w_0$ is a periodically-varying function. Thus, the lower equation of (17) can be seen as a periodically-varying system

$$\dot{p} = f(w(t), p) \tag{23}$$

having an equilibrium at $p = 0$. The linear approximation of this system at $p = 0$ is a linear system

$$\dot{p} = A(w(t))p \tag{24}$$

which, in what follows, is proven to be exponentially stable. This – due to standard results – provides the desired property.

[9]The proofs of Propositions 4 and 5 are omitted for reasons of space and can be provided upon specific request.

[10]See Definition B.4 in [23].

Proposition 6: Suppose Assumptions 1, 2 and 3 hold. Then, the equilibrium $p = 0$ of (23) is locally exponentially stable, uniformly in w_0.

Proof: Let the solution Σ of (11) be partitioned as $\Sigma = \mathrm{col}(\Sigma_1, \ldots, \Sigma_m)$ in which Σ_i is a square $d \times d$ matrix for each $1 \leq i \leq m$. System (24) is the periodically-varying linear system

$$\begin{array}{rcl} \dot{\tilde{\mathbf{x}}} &=& \mathbf{A}\tilde{\mathbf{x}} + \mathbf{B}[I_m \otimes \tilde{\Gamma}_0]\Sigma w(t) \\ \dot{\tilde{\Gamma}}_0^{\mathrm{T}} &=& \begin{pmatrix} \Sigma_1 w(t) & \cdots & \Sigma_m w(t) \end{pmatrix} \mathbf{C}\tilde{\mathbf{x}}\,, \end{array} \tag{25}$$

which can be seen as the negative feedback interconnection (i.e. $\bar{u}_1 = \bar{y}_2$ and $\bar{u}_2 = -\bar{y}_1$) of the strictly passive system

$$\begin{array}{rcl} \dot{\tilde{\mathbf{x}}} &=& \mathbf{A}\tilde{\mathbf{x}} + \mathbf{B}\bar{u}_1 \\ \bar{y}_1 &=& -\mathbf{C}\tilde{\mathbf{x}} \end{array}$$

and of the lossless periodically-varying system

$$\begin{array}{rcl} \dot{\tilde{\Gamma}}_0^{\mathrm{T}} &=& \begin{pmatrix} \Sigma_1 w(t) & \cdots & \Sigma_p w(t) \end{pmatrix} \bar{u}_2 \\ \bar{y}_2 &=& [I_m \otimes \tilde{\Gamma}_0]\Sigma w(t)\,. \end{array}$$

Using the same arguments used in the the proof of Proposition 3 we can conclude that $\tilde{\mathbf{x}}(t) \to 0$ as $t \to \infty$, which in turn implies $\lim_{t\to\infty}[I_m \otimes \tilde{\Gamma}_0]\Sigma w(t) = 0$.

The same arguments used in the the proof of Proposition 4 prove that, if Assumptions 1 and 2 hold, the latter implies that also $\tilde{\Gamma}_0^{\mathrm{T}}(t) \to 0$ as $t \to \infty$ (i.e. the lossless system is observable). Thus, all trajectories of (24) asymptotically decay to zero. Let $\Phi(t, t_0)$ denote the state transition matrix of (24) and let T be such that $w(t+T) = w(t)$ for all $t \in \mathbb{R}$. It is known that the asymptotic properties of a periodically-varying system like (24) are determined by the eigenvalues of the matrix $\Phi(T, 0)$. In particular, the trajectories of (24) asymptotically decay to zero if and only if all eigenvalues of $\Phi(T, 0)$ lie inside the open unit disc. Therefore, we can claim the existence of a number $0 < \delta < 1$ with the property that all eigenvalues of $\Phi(T, 0)$ are in $\mathbb{C}_\delta = \{\lambda \in \mathbb{C} : |\lambda| \leq \delta\}$.[11] This being the case, it follows that if $\tilde{A}$ is a matrix satisfying $e^{\tilde{A}t} = \Phi(T, 0)$, its eigenvalues have a real part that does not exceed a fixed negative number (independent of w_0). Then, according to well-know properties of periodically-varying linear systems, it is concluded that, for some $\lambda > 0$ and $k > 0$, the state transition matrix of (24) satisfies $\|\Phi(t, t_0)\| \leq ke^{-\lambda(t-t_0)}$ for all $t \geq t_0$ and all $w_0 \in \mathcal{W}$. This being the case, the proposition follows from Theorems 4.12 and 4.13 of [24]. ■

VI. Back to the full system

We return now to the full two-time scale system consisting of (15) and (13). Setting $\mathbf{p} = \mathrm{col}(w, p)$ we can can express system (15) in compact form as

$$\dot{\mathbf{p}} = M(\mathbf{p}) + Ny \tag{26}$$

in which $M(\mathbf{p})$ is the right-hand side of (16) and $N \in \mathbb{R}^{(d+n_p)\times m}$. System (13), on the other hand, can be seen

[11]Observe that, in the present case, the matrix $\Phi(T, 0)$ depends on the initial condition w_0 of the exosystem. Now, recall that $|\det[\Phi(T, 0)]| = \Pi_{i=1}^{n_p}|\lambda_i|$ in which $\lambda_1, \ldots, \lambda_{n_p}$ are the eigenvalues of $\Phi(T, 0)$. If all such eigenvalues are inside the open unit disc, then $|\det[\Phi(T, 0)]| < 1$. Since $|\det[\Phi(T, 0)]|$ is a continuous function of w_0 and the latter ranges over the compact set $\mathcal{W}$, we deduce the existence of a number $0 < \delta < 1$ such that $|\det[\Phi(T, 0)]| \leq \delta$ for all $w_0 \in \mathcal{W}$ and this proves the claim.

as a system of the form

$$\dot{y} = K(\mathbf{p}, y) + v\,, \qquad \text{where} \quad v = \mu^{-1} A_0 y. \qquad (27)$$

With $\mathcal{A}$ defined as in (18), let initial conditions $(\mathbf{p}(0), y(0))$ be taken in a compact set $\mathcal{B} \times Y$, with $\mathcal{B}$ a compact set that contains $\mathcal{A}$ in its interior and $Y = \{y \in \mathbb{R}^m : \|y\| \leq R\}$. We have shown in the previous section that, under appropriate hypotheses, in the system $\dot{\mathbf{p}} = M(\mathbf{p})$, the invariant set $\mathcal{A}$ is stable in the sense of Lyapunov and $\omega(\mathcal{B}) = \mathcal{A}$. A consequence [12] of the latter property is that, for each $\alpha > 0$ and $\epsilon > 0$ there exists $T > 0$ such that $\mathrm{dist}(\mathbf{p}(0), \mathcal{A}) \leq \alpha$[13] implies $\mathrm{dist}(\mathbf{p}(t), \mathcal{A}) \leq \epsilon$ for all $t \geq T$. Inspection of (13) also reveals that $K(\mathbf{p}, 0)$ is zero for all $\mathbf{p} \in \mathcal{A}$, and A_0 is a Hurwitz matrix. The system thus defined has the same structure of system (19) of [27] and assumptions identical to those considered in Theorem 3 of [27] hold. Thus, appealing to the results indicated in that Theorem, it can be concluded as follows.

Proposition 7: There exists a number μ^{**} such that, if $0 < \mu \leq \mu^{**}$, all trajectories of the system (26), (27) are bounded, $\lim_{t\to\infty} \mathrm{dist}(\mathbf{p}(t), \mathcal{A}) = 0$, which implies $\lim_{t\to\infty} p(t) = 0$, and $\lim_{t\to\infty} y(t) = 0$. In particular, $\lim_{t\to\infty} \tilde{\mathbf{x}}(t) = 0$ which in turn implies $\lim_{t\to\infty} e(t) = 0$. Thus, the goal of asymptotic output regulation is achieved.

VII. Completing the Design

The controller achieving (adaptive) output regulation is a controller of the form

$$\begin{aligned} \dot{\eta} &= F\eta + G[(I_m \otimes \hat{\Gamma}_0)\eta + z_2] \\ \dot{\hat{\Gamma}}_0^{\mathrm{T}} &= \begin{pmatrix} \eta_1 & \cdots & \eta_m \end{pmatrix} z_2 \\ u &= -k[(I_m \otimes \hat{\Gamma}_0)\eta + z_2] \end{aligned}$$

which is driven by the "partial state" z_2. Normally, the latter is not directly available for feedback. However, it can be estimated by means of a "high-gain" observer driven by the regulated output e. Details of the design are somewhat standard and are not repeated here. It should be stressed, thought, that since the convergence of trajectories of system (26)–(27) to the compact invariant set $\mathcal{A} \times \{0\}$ is locally exponential, the error in the estimation of z_2 achieved by means of a high-gain observer asymptotically decays to zero and hence the property of asymptotic convergence of $e(t)$ to 0 is conserved.

VIII. Conclusions

The problem of adaptive output regulation of a minimum-phase MIMO LTI system with unmodeled input dynamics is solved using a post-processing tunable internal model. As it was shown, the proposed approach guarantees convergence to zero of the regulation error as well as of the parameter estimation error.

References

[1] B. Francis, O. A. Sebakhy, and W. M. Wonham, "Synthesis of multivariable regulators: The internal model principle," *Applied Mathematics and Optimization*, vol. 1, pp. 64–86, Mar. 1974.

[2] M. Bin, J. Huang, A. Isidori, L. Marconi, M. Mischiati, and E. Sontag, "Internal models in control, bioengineering, and neuroscience," *Annual Review of Control, Robotics, and Autonomous Systems*, vol. 5, no. 1, pp. 55–79, 2022.

[3] E. Davison, "The robust control of a servomechanism problem for linear time-invariant multivariable systems," *IEEE Transactions on Automatic Control*, vol. 21, no. 1, pp. 25–34, 1976.

[4] A. Pyrkin and A. Isidori, "Adaptive output regulation of right-invertible MIMO LTI systems, with application to vessel motion control," *European Journal of Control*, vol. 46, pp. 63–79, 2019.

[5] J. Abedor, K. Nagpal, P. Khargonekar, and K. Poolla, "Robust regulation in the presence of norm-bounded uncertainty," *IEEE Transactions on Automatic Control*, vol. 40, no. 1, pp. 147–153, 1995.

[6] A. A. Stoorvogel, A. Saberi, and P. Sannuti, "Performance with regulation constraints," *Automatica*, vol. 36, no. 10, pp. 1443–1456, 2000.

[7] H. Köroğlu and C. W. Scherer, "An LMI approach to H_∞ synthesis subject to almost asymptotic regulation constraints," *Systems & Control Letters*, vol. 57, no. 4, pp. 300–308, 2008.

[8] A. Isidori and C. Byrnes, "Output regulation of nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 35, no. 2, pp. 131–140, 1990.

[9] J. Huang and W. J. Rugh, "On a nonlinear multivariable servomechanism problem," *Automatica*, vol. 26, no. 6, pp. 963–972, 1990.

[10] J. Huang, "Asymptotic tracking and disturbance rejection in uncertain nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 40, no. 6, pp. 1118–1122, 1995.

[11] M. Bin, D. Astolfi, and L. Marconi, "About robustness of control systems embedding an internal model," *IEEE Transactions on Automatic Control*, vol. 68, no. 3, pp. 1306–1320, 2023.

[12] M. Bin, D. Astolfi, L. Marconi, and L. Praly, "About robustness of internal model-based control for linear and nonlinear systems," in *2018 IEEE Conference on Decision and Control (CDC)*, 2018, pp. 5397–5402.

[13] A. Serrani, A. Isidori, and L. Marconi, "Semi-global nonlinear output regulation with adaptive internal model," *IEEE Transactions on Automatic Control*, vol. 46, no. 8, pp. 1178–1194, 2001.

[14] A. A. Bobtsov, D. Efimov, A. A. Pyrkin, and A. Zolghadri, "Switched algorithm for frequency estimation with noise rejection," *IEEE Transactions on Automatic Control*, vol. 57, no. 9, pp. 2400–2404, 2012.

[15] S. Aranovskiy and L. B. Freidovich, "Adaptive compensation of disturbances formed as sums of sinusoidal signals with application to an active vibration control benchmark," *European Journal of Control*, vol. 19, no. 4, pp. 253–265, 2013.

[16] L. Liu, Z. Chen, and J. Huang, "Parameter convergence and minimal internal model with an adaptive output regulation problem," *Automatica*, vol. 45, no. 5, pp. 1306–1311, 2009.

[17] M. Bin, L. Marconi, and A. R. Teel, "Adaptive output regulation for linear systems via discrete-time identifiers," *Automatica*, vol. 105, pp. 422–432, 2019.

[18] M. Bin and L. Marconi, ""class-type" identification-based internal models in multivariable nonlinear output regulation," *IEEE Transactions on Automatic Control*, vol. 65, no. 10, pp. 4369–4376, 2020.

[19] P. Bernard, M. Bin, and L. Marconi, "Adaptive output regulation via nonlinear Luenberger observer-based internal models and continuous-time identifiers," *Automatica*, vol. 122, p. 109261, 2020.

[20] M. Bin, P. Bernard, and L. Marconi, "Approximate nonlinear regulation via identification-based adaptive internal models," *IEEE Transactions on Automatic Control*, vol. 66, no. 8, pp. 3534–3549, 2021.

[21] V. Nikiforov and D. Gerasimov, *Adaptive Regulation: Reference Tracking and Disturbance Rejection*. Springer International Publishing, 2022.

[22] B. Brogliato, R. Lozano, B. Maschke, and O. Egeland, *Dissipative Systems Analysis and Control: Theory and Applications*, 3rd ed. Springer International Publishing, 2020.

[23] A. Isidori, *Lectures in Feedback Design for Multivariable Systems*. Springer International Publishing, 2017.

[24] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Prentice Hall, 2002.

[25] N. P. Bhatia and G. P. Szegö, *Stability Theory of Dynamical Systems*, 1st ed. Springer-Verlag, 1970.

[26] J. K. Hale, L. T. Magalhães, and W. M. Oliva, *Dynamics in Infinite Dimensions*. Springer New York, 2002.

[27] L. Marconi, L. Praly, and A. Isidori, "Output stabilization via nonlinear Luenberger observers," *SIAM Journal on Control and Optimization*, vol. 45, no. 6, pp. 2277–2298, 2007.

[12] See, e.g. [25], [26], [23].

[13] The notation $\mathrm{dist}(\mathbf{p}(0), \mathcal{A})$ denotes the distance of a point $\mathbf{p}(0)$ from a set $\mathcal{A}$ (see section B.4 in Appendix B of [23] for details).