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A Unified Framework for Convergence Analysis in Social Networks

Ti-Chung Lee, *Senior Member*, *IEEE*, Jun-Kai Huang, and Youfeng Su, *Member*, *IEEE*

Abstract—This paper proposes a unified framework for the stability analysis of discrete-time nonlinear systems from social networks, including the Friedkin-Johnsen opinion model, two opinion dynamics models in the study of social power, and a general nonlinear polar opinion model. Three novel convergence results are proposed to treat various conditions based on LaSalle invariance principle. Several applications are provided to illustrate the power of the proposed framework.

I. Introduction

A typical social network is composed by social agents and their opinions, where agents interactive and evolve their opinions across the network, influencing and/or being influenced by one another. The recent years have witnessed a significant tendency of various agent-based opinion dynamics models, see the surveys [17, 18, 23], and the references therein. Opinion dynamics models refers to a description of how individuals' and groups' opinions, behaviors, and decisions change through interactions with the opinions of the others. Convergence analyses of opinion dynamics models are important because they characterize that local interactions result in various phenomena of the whole network.

The most well-known models are the DeGroot model [4] and its continuous-time counterpart known as the Abselson model [1], both of which describe how individual opinions are updated through a convex combination of their own and others' opinions, aiming to achieve consensus or agreement among agents. More consensus studies could be found in [9, 13, 14, 15, 19] for cooperative networks and in [2, 20] for cooperative-antagonistic networks. However, this class of opinion dynamics models lacks the analytical capability to address the impact of agents defending their own opinions on system changes. In order to account for different susceptibilities, the Friedkin-Johnsen model [5, 6] expands upon the DeGroot model by introducing a diagonal matrix representing the open attitude, where stubbornness of agents is considered. Obviously, the convergence analyses of DeGroot model could be covered by that of Friedkin-Johnsen model by viewing no stubbornness. Further variations such as the concatenated Friedkin-Johnsen model [22] and the multidimensional Friedkin-Johnsen model [16] focused on the evolvement of agents' opinions over an infinite sequence of interdependent issues and over multiple logic-constrained issues, respectively. Immersing the DeGroot model into the Friedkin-Johnsen model leads to the so-called DeGroot-Friedkin model, where have the capability of formulating the processes of reflected-appraisal mechanism and reflecting the social powers [10, 25]. The studies of DeGroot-Friedkin model subject to stubbornness could be seen in [24]. Susceptibility in all these Friedkin type models is determined solely by agents' initial opinions and does not change during the whole evolution procedure. So it is more reasonable to consider the agents' current attitude at hand. Toward this issue, a general nonlinear polar opinion model was proposed recently [3] for cooperative networks, and variations for cooperative-antagonistic networks and time-varying networks were reported in [26] and [11], respectively.

This paper summarizes a class of discrete-time nonlinear systems and verifies that opinion dynamics models such as DeGroot model, Friedkin-Johnsen model, and DeGroot-Friedkin model could fall (or be converted) into the category of this class of systems. Our main contribution is to investigate a comprehensive stability analysis of such a class of nonlinear systems and hence provide a unified framework in convergence analyses of all these mentioned opinion dynamics models. Indeed, three novel convergence results are proposed to treat various conditions based on LaSalle invariance principle so that they can be applied to more general systems beyond linear systems. Applications on Friedkin-Johnsen model and DeGroot-Friedkin model are clearly reported, showing the power of our framework. It is further emphasized that the proposed framework could also be applied to a discrete-time version of nonlinear polar opinion model.

Notations: $\Re$ is the set of real numbers, $Z_+ = \{0,1,\cdots\}$, $Z_p = \{1,\cdots,p\}$. For any subset $S \subseteq \Re$, $S^p = \{(u_1,\cdots,u_p)^T \,|\, u_i \in S, \forall i \in Z_p\}$ and $S^{p\times q}$ consists of all $p\times q$ matrices with their entries belonging to *S*. For any $u = (u_1,\cdots,u_p)^T \in \Re^p$, $\|u\| = \sqrt{u_1^2 + \ldots + u_p^2}$, u_i is its $i-th$ entry and with $J = \{ i_1 < \cdots < i_k \} \subseteq Z_p$, $u_J = (u_{i_1},\cdots,u_{i_k})^T \in \Re^k$. When $p=1$, the notation $|u| = \|u\|$ is also used and $\|u\|_\infty = \max(|u_1|,|u_2|,\cdots,|u_p|)$. $B_\infty(r) = [-r,r]^p$ for any $r>0$.

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T. C. Lee and Jun-Kai Huang are with Department of Electrical Engineering, National Sun Yat-sen University, Kaohsiung, Taiwan 80424, R.O.C (e-mail: tc1120@ms19.hinet.net; finkaiking@gmail.com).

Y. Su is with College of Computer and Data Science, Fuzhou University, Fuzhou 350116, China (e-mail: yfsu@fzu.edu.cn).

For any subset $\Omega_0 \subseteq \Re^p$, $\|u\|_{\Omega_0} = \inf_{v\in\Omega_0}\|u-v\|$. For a finite set S, $\#S$ denotes the number of all elements. 1_p denotes the p-item vector of ones and $e_i \in \Re^p, \forall i \in Z_p$, with the i-th entry being 1 and the other entries being 0. A row-stochastic matrix $A \in \Re^{p\times p}$ is a nonnegative matrix with $A1_p = 1_p$. $diag(u_1,\cdots,u_p) \in \Re^{p\times p}$ is the diagonal matrix with its diagonal entries being $u_1,,\cdots,u_p$. For any $J = \{i_1 < \cdots < i_k\} \subseteq Z_p \cap Z_q$, $Z_q \setminus J = \{j_1 < j_2 < \cdots < j_{q-k}\}$ and $D = (d_{ij}) \in \Re^{p\times q}$, $D_J = (d_{i_l i_s})_{l,s=1}^k \in \Re^{k\times k}$ and $D'_J = (d_{i_l j_r})_{1\le l\le k, 1\le r\le (q-k)} \in \Re^{k\times(q-k)}$. For a function $x: Z_+ \to \Re^p$,

$$\Omega(x) = \{u \in \Re^p \,|\, \exists t_n \to \infty \text{ such that } \lim_{n\to\infty} x(t_n) = u\}$$

is the ω-limit set of x.

II. Preliminaries

In this section, the studied system is introduced and motivated. Boundedness of solutions is proven. Then, several useful results including the well-known LaSalle invariance principle and Brouwer fixed-point theorem are also provided.

A. The studied system and motivation

For a closed set $X \subseteq \Re^p$, consider the following discrete-time system:

$$x^+ = F(x)x \qquad x \in X \tag{1}$$

where x is the state, $F: X \to \Re^{p\times p}$ is a matrix-valued continuous function. A function $x: Z_+ \to X$ is said to a solution of (1), if $x(k+1) = F(x(k))x(k)$ for any $k \in Z_+$.

System (1) includes at least three typical systems from social networks as discussed below.

The fist opinion model: Recall the so called Friedkin and Johnsen opinion model as follows [5]:

$$z^+ = \Theta Az + (I - \Theta)z(0) \tag{2}$$

where $z \in \Re^p$ is the state, $A \in \Re^{p\times p}$, which corresponds to a social network, is a row-stochastic matrix, and $\Theta = diag(\zeta_1, \zeta_2, \cdots, \zeta_p)$ where $0 \le \zeta_i \le 1, i \in Z_p$, represents the stubborn condition of each agent.

Consider an equilibrium point z_* of (2) as follows:

$$z_* = \Theta A z_* + (I - \Theta)z(0). \tag{3}$$

As we shall see below, under certain connectivity condition (see (C) in Section IV.A), ΘA is Schur stable and hence $(I - \Theta A)$ is nonsingular. Thus,

$$z_* = (I - \Theta A)^{-1}(I - \Theta)z(0). \tag{4}$$

Let $x = z - z_*$ be the error state and the error system can be written into the form of (1) with $F = \Theta A$.

The second model relative to social power: Consider the so-called DeGroot-Friedkin model (see Lemma 2.2 of [10]):

$$z^+ = \sum_{i=1}^{p+1}\Big[\frac{1}{\sum_{j=1}^{p+1}\gamma_j/(1-z_j)}\Big]\frac{\gamma_i}{(1-z_i)}e_i \tag{5}$$

where $p \ge 2$, $z \in \Re^{p+1}$ is a state with $z_i \ge 0, \forall i \in Z_{p+1}$, $\sum_{i=1}^{p+1} z_i = 1$, $\sum_{i=1}^{p+1}\gamma_i = 1$ and

$$diag(\gamma_1, \gamma_2, \cdots, \gamma_{p+1}) \tag{6}$$

is positive definite. Such a system comes from the study of social power [25]. Let

$$\Delta = \{u \in [0,1]^{p+1} \,|\, \textstyle\sum_{i=1}^{p+1} u_i = 1\},\ E = \{e_1, e_2, \cdots, e_{p+1}\}. \tag{7}$$

System (5) cannot be defined for any $u = e_j \in E$ and

$$\lim_{z\to e_j, x\in\Delta}\frac{\sum_{i=1}^{p+1}\gamma_i/(1-z_i)e_i}{\sum_{i=1}^{p+1}\gamma_i/(1-z_i)} = e_j \qquad \forall j \in Z_{p+1}. \tag{8}$$

So it is reasonable to define a function $H: \Delta \to \Delta$ as

$$H(u) = \begin{cases}\sum_{i=1}^{p+1}\dfrac{\gamma_i/(1-u_i)}{\sum_{j=1}^{p+1}\gamma_j/(1-u_j)}e_i, u \in \Delta\setminus E \\ e_i, u = e_i \in E.\end{cases} \tag{9}$$

By definition, H is a continuous function on Δ and the system (5) can be extended to the following system:

$$z^+ = H(z). \tag{10}$$

Moreover, each element in E is an equilibrium point. In the later development, we will see that in case of $p \ge 2$, E is the set of all equilibrium points if and only if there is a $i_0 \in Z_{p+1}$ such that $\gamma_{i_0} \ge 1/2$ [10]. For simplicity, we assume that without lose of generality, $i_0 = p+1$ and show how to convert such a system into the form (1).

Indeed, let $c_i = 2\gamma_i, \forall i \in Z_{p+1}$, and

$$\sum_{i=1}^{p} c_i = 2 - 2\gamma_{p+1} \le 1 \le c_{p+1}, \forall i \in Z_p. \tag{11}$$

Then, for every solution $z: Z_+ \to \Re^{p+1}$ of (10), under the map

$$\pi: (u_1, \cdots, u_{p+1})^T \in \Re^{p+1} \mapsto (u_1/c_1, \cdots, u_p/c_p)^T \in \Re^p, \tag{12}$$

the composition function $x: k \in Z_+ \mapsto \pi(z(k)) \in X$ is a solution of the form (1) where

$$X = \{(z_1, z_2, \cdots, z_p)^T \in [0,\infty)^p \,|\, c_1 z_1 + c_2 z_2 + \cdots + c_p z_p \le 1\}, \tag{13}$$

and for any $u \in X$, $i, j \in Z_p$ with $j \ne i$,

$$F_{ij}(u) = c_j\alpha(u),\ \ F_{ii}(u) = \begin{cases} c_i\alpha(u)(1 + \dfrac{\sum_{l=1}^{p} c_l u_l}{1 - c_i u_i}), c_i u_i < 1 \\ 1, c_i u_i = 1\end{cases} \tag{14}$$

with

$$\alpha(u) = \begin{cases}\dfrac{1}{c_{p+1} + (\sum_{l=1}^{p} c_l u_l)(\sum_{l=1}^{p} c_l/(1 - c_l u_l))}, c_i u_i < 1, \forall i \in Z_p \\ 0, c_i u_i = 1 \text{ for some } i \in Z_p.\end{cases} \tag{15}$$

Lemma 1 shows this argument with its proof omitted here.

Lemma 1. Consider the function H defined as in (9) where $p \ge 2$, $\gamma_{p+1} \ge 1/2$, $\sum_{i=1}^{p+1}\gamma_i = 1$ and $\gamma_i > 0, \forall i \in Z_p$. Let $F: X \to \Re^p$ be defined as in (13)-(15). Then, α and F are both continuous, $\pi(\Delta) = X$ and

$$F(\pi(\tilde{u})) = \pi(H(\tilde{u})) \quad \forall \tilde{u} \in \Delta.$$

The third model: Consider system (10) where H is defined as in (9) with $\gamma_i < 1/2 \quad \forall i \in Z_{p+1}$. Firstly, we assume that there exists an extra equilibrium point $z^* \in \Delta\setminus E$, which will be discussed later.

Let $x = z - z^*$ be the error state and

$$y_i = 1/(1-z_i^*),\ \ \hat{y}_i = \sum_{j=1,j\neq i}^{p+1}\gamma_j/(1-z_j^*) = \sum_{j=1,j\neq i}^{p+1}\gamma_j y_j\ . \quad (16)$$

Then, for any solution z of (10), $x = z - z^*$ is a solution of (1) where for some $\varepsilon_0 > 0$,

$$X = (B_\infty(1-\varepsilon_0)\cap\Delta) - z^* \quad (17)$$

is a closed set and for any $i \in Z_{p+1}$,

$$F_{ii} = \frac{y_i\hat{y}_i\bar{z}_i}{(\gamma_i y_i + \hat{y}_i)\sum_{l=1}^{p+1}\bar{z}_l},\ F_{ij} = \frac{-\gamma_i y_i y_j \bar{z}_j}{(\gamma_i y_i + \hat{y}_i)\sum_{l=1}^{p+1}\bar{z}_l}, \forall j \neq i, \quad (18)$$

where for each $i \in Z_{p+1}$, $\bar{z}_i = \gamma_i/(1-z_i) = \gamma_i/(1-x_i+z_i^*)$ is viewed as a function of x.

Remark 1. When $v_L = (\gamma_1,\gamma_2,\cdots,\gamma_{p+1})^T$ is the unique left positive eigenvector of a Laplace matrix L of a strongly connected digraph G with $\sum_{i=1}^{p+1}\gamma_i = 1$, it can be shown that either there exists $i_0 \in Z_{p+1}$ such that $\gamma_{i_0} = 1/2$ or $\gamma_i < 1/2$ for all $i \in Z_{p+1}$ [10]. ■

B. The boundedness of solutions

The boundedness of solutions of (1) is guaranteed here. To this end, the following assumption is needed:

(A1) For some $r_0 > 0$ and any $i \in Z_p$, $\sum_{j=1}^{p}\left|F_{ij}(u)\right| \le 1$ for any $u \in X \cap B_\infty(r_0)$ where $F_{ij}(u)$ is the (i, j) entry of $F(u)$.

Proposition 1. Consider system (1) where (A1) holds. Then, every solution $x: Z_+ \to X$ of (1) with $\|x(0)\|_\infty \le r_0$ satisfies

$$\|x(k+1)\|_\infty \le \|x(k)\|_\infty \le r_0 \quad \forall k \in Z_+. \quad (19)$$

Particularly,

$$\|x(k)\| \le \sqrt{p}\|x(k)\|_\infty \le \sqrt{p}\|x(0)\|_\infty \le \sqrt{p}\|x(0)\|\ \forall k \in Z_+.$$

Proof of Proposition 1: By (A1),

$$\left|\sum_{j=1}^{p}F_{ij}(u)u_j\right| \le \sum_{j=1}^{p}\left|F_{ij}(u)\right|\|u\|_\infty \le \|u\|_\infty \quad (20)$$

for any $i \in Z_p$ and $u \in X \cap B_\infty(r_0)$. So $\|F(u)u\|_\infty \le \|u\|_\infty$ for any $u \in X \cap B_\infty(r_0)$. The remainder is trivial. ■

C. Several useful results

This section proposes several prepared results. The first one is the well-known LaSalle invariance principle [11, 21].

Proposition 2. Consider system (1). Let $x: Z_+ \to X$ be a bounded solution and Ω_0 a nonempty closed set of $\Re^p$. Suppose there exists a continuous function $V: X \to [0,\infty)$ such that it is nonincrasing along the solution x, i.e.,

$$V(x(k+1)) \le V(x(k)) \quad \forall k \in Z_+, \quad (21)$$

and for any solution $\bar{x}: Z_+ \to \Omega(x)$ satisfying $V(\bar{x}(k+1)) = V(\bar{x}(k)), \forall k \in Z_+$, we have $\bar{x}(0) \in \Omega_0$. Then, $\lim_{k\to\infty}\|x(k)\|_{\Omega_0} = 0$. ■

Next the Brouwer fixed-point theorem is recalled. We refer readers to [8] for an elementary proof.

Proposition 3. Let $f: [0,1]^p \to [0,1]^p$ be continuous. Then, there exists $x_* \in [0,1]^p$ such that $f(x_*) = x_*$. ■

Finally, the following simple and useful criterion is proposed. The proof is trivial and left for readers.

Lemma 2. If $\sum_{i=1}^{p}a_i \le \sum_{i=1}^{p}b_i$ and $a_i \le b_i$ for any $i \in Z_p$, then $a_i = b_i$ for any $i \in Z_p$. ■

III. Main Results

This section proposes three main results under some extra conditions to guarantee the convergence of solutions.

A. Exponential convergence under a stronger condition

The following stronger condition than (A1) is made.

(A2) There exists $0 < \delta_0 < 1$ such that for any $i \in Z_p$ and $u \in X$, $\sum_{j=1}^{p}\left|F_{ij}(u)\right| \le \delta_0$.

With (A2), Theorem 1 shows the exponential convergence of any solution $x: Z_+ \to X$ of (1).

Theorem 1. Considered system (1). Suppose (A2) holds. Then, there exist $a > 0$ and $b > 0$ such that the inequality

$$\|x(k)\| \le ae^{-bk}\|x(0)\| \quad \forall k \in Z_+ \quad (22)$$

holds for every solution $x: Z_+ \to X$ of (1).

Proof of Theorem 1: By (A2),

$$\left|\sum_{j=1}^{p}F_{ij}(u)u_j\right| \le \sum_{j=1}^{p}\left|F_{ij}(u)\right|\|u\|_\infty \le \delta_0\|u\|_\infty \quad (23)$$

for any $i \in Z_p$ and $u \in X$. So $\|F(u)u\|_\infty \le \delta_0\|u\|_\infty$ for any $u \in X$. Let $a = p^{1/2} > 0$ and $b = -\ln(\delta_0) > 0$. Then,

$$\|x(k)\| \le \sqrt{p}\|x(k)\|_\infty \le \sqrt{p}\ \delta_0^k\|x(0)\|_\infty \le ae^{-bk}\|x(0)\|\ \forall k \in Z_+. ■$$

B. Asymptotic convergence under a graph condition

This section proposes a graph-like condition to guarantee asymptotic stability.

To this end, the following "connectivity" condition is assumed:

(G) For any positive constant c and any solution $x: Z_+ \to X$ of (1) with $\|x(k)\|_\infty = c, \forall k \in Z_+$, there exist a sufficiently large integer N_0 and a non-empty set $J_0 \subseteq Z_p$ such that the following conditions hold:

(G1) For any $j \in Z_p \setminus J_0$ with $\left|x_j(k_j)\right| = c$ for some integer $k_j > N_0$, there exist $n \le N_0, i \in J_0$ and a finite sequence

$$(i = i_1, i_2), (i_2, i_3), \cdots (i_n, j = i_{n+1})$$

such that $F_{i_{(l+1)}i_l}(x(k_j + l - n - 1)) \neq 0$ for any $l \in Z_n$.

(G2) For any $u \in X$ with $\|u\|_\infty = c$ and $\|F(u)u\|_\infty = c$, $\left|\sum_{l=1}^{p}F_{il}(u)u_l\right| < c$ for any $i \in J_0$.

Remark 2. Condition (G1) means that any node j attaining the maximal absolute value of all partial states at a sufficiently large time instant must be affected directly or indirectly (corresponding to the condition $F_{i_{(l+1)}i_l}(x(k_j + l - n - 1)) \neq 0$) within N-steps by a node i in a root-like node set J_0, while condition (G2) requires that with the maximal node set defined as $M(k) = \{i \in Z_p \mid \left|x_i(k)\right| = c\}$ for each $k \in Z_+$, J_0 cannot be a common (invariant) subset of $M(k), \forall k \in Z_+$, i.e., the size (absolute value) of the partial state x_i with $i \in J_0$ must shrink when it attains the maximal absolute value of all partial states. ■

Remark 3. In addition to (A1), it is possible to show that (G2) is equivalent to the following conditions:

(G2') For any $u \in X$ with $\|F(u)u\|_\infty = c$ and $\|u\|_\infty = c$, and any $i \in J_0$, one of the following conditions must hold:

a) $\left|\sum_{l=1}^{p} F_{il}(u)u_l\right| < c$.

b) $\sum_{l=1}^{p} |F_{il}(u)||u_l| < c$.

c) $\sum_{l=1}^{p} |F_{il}(u)| < 1$. ■

Based on (A1) and (G), the following result holds.

Theorem 2. Considered system (1). Suppose (A1) and (G) hold. Then, every solution $x: Z_+ \to X$ of (1) with $\|x(0)\|_\infty \le r_0$ satisfies $\lim_{k\to\infty} x(k) = 0$.

Proof of Theorem 2: Let $V = \|x\|_\infty \ge 0$ be a Lyapunov candidate. According to Proposition 1, it is nonincrasing along the solution *x*, i.e., (19) holds. In view of LaSalle invariance principle (Proposition 2) with $\Omega_0 = \{0\}$, it remains to check that for any solution

$$\bar{x}: Z_+ \to \Omega(x) \subseteq X \cap B_\infty(r_0)$$

satisfying $\|\bar{x}(k)\|_\infty = \|\bar{x}(0)\|_\infty = c, \forall k \in Z_+$, we have $\bar{x}(0) = 0$. When $c = 0$, $\bar{x}(0) = 0$ and the theorem is done. In the following, if $c > 0$, a contradiction will be found.

Let $M_k = \left\{i \in Z_p \middle| |\bar{x}_i(k)| = c\right\}$ for any $k \in Z_+$. Based on (A1), for any $i \in M_{k+1}$, the following inequalities hold:

$$c = |\bar{x}_i(k+1)| = \left|\sum_{j=1}^{p} F_{ij}(\bar{x}(k))\bar{x}_j(k)\right| \le \sum_{1\le j\le p \& j\ne i} |F_{ij}(\bar{x}(k))||\bar{x}_j(k)|$$
$$\le \sum_{1\le j\le p \& j\ne i} |F_{ij}(\bar{x}(k))|\|\bar{x}(k)\|_\infty \le \|\bar{x}(k)\|_\infty = c. \tag{24}$$

All inequalities then become equalities. When $F_{ij}(\bar{x}(k)) \ne 0$, it holds that $|\bar{x}_j(k)| = \|\bar{x}(k)\|_\infty = c$ by Lemma 2 and $j \in M_k$.

Let N_0 and J_0 be the integer and the set given in (G) (with respect to $\bar{x}$), respectively. We show that $M_{k+1} \cap J_0$ is empty for any $k \in Z_+$. Indeed, for any $i \in M_{k+1}, u = \bar{x}(k) \in X$ satisfies $\|u\|_\infty = c$,

$$\|F(u)u\|_\infty = \|\bar{x}(k+1)\|_\infty = c, \ \left|\sum_{l=1}^{p} F_{il}(u)u_l\right| = |\bar{x}_i(k+1)| = c.$$

Thus, $i \notin J_0$ by (G2). So $M_{k+1} \cap J_0$ is empty for any $k \in Z_+$.

Employing (G1), for any $j \in M_{N_0+1}$, there exists a finite sequence $(i = i_1, i_2), (i_2, i_3), \cdots (i_n, j = i_{n+1})$ with $n \le N_0, i \in J_0$, and $F_{i_{(l+1)} i_l}(x(N_0 + l - n)) \ne 0$ for any $l \in Z_n$. Again by the previous discussion and using induction, $i_l \in M_{N_0+l-n}$ for any $l \in Z_{n+1}$. Particularly, $i = i_1 \in M_{N_0-n+1} \cap J_0$, a contradiction.

Therefore $c = 0$ and the result follows Proposition 2. ■

C. Asymptotic convergence when all diagonal entries are positive

This section assumes the following stronger condition:

(A3) For some $r_0 > 0$ and any $i \in Z_p$, $F_{ii}(v) > 0$ for any $v \in X \cap B_\infty(r_0) \setminus \{0\}$ and $\sum_{j=1}^{p} |F_{ij}(u)| \le 1$ for any $u \in X \cap B_\infty(r_0)$.

Under the following "observability-like" condition, asymptotic stability can then be established:

(E) For any $J \subseteq Z_p$, the set $D_J \subseteq \{0\}$ where

$$D_J = \{u \in \Omega(x) | (F(u))_J u_J = u_J, \ (F(u))'_J = 0, \ |u_i| = \|u\|_\infty, \ \forall i \in J\}. \tag{25}$$

Now the following result can be proposed, which together with Proposition 1, shows that the origin is globally asymptotically stable w.r.t. *X*.

Theorem 3. With a closed set $X \subseteq \Re^p$, consider the system (1). Suppose (A3) and (E) hold. Then, every solution $x: Z_+ \to X$ of (1) satisfies $\lim_{k\to\infty} x(k) = 0$.

Proof of Theorem 3: Let $V = \|x\|_\infty \ge 0$. It is nonincrasing along the solution *x*. In view of LaSalle invariance principle (Proposition 2) with $\Omega_0 = \{0\}$, it remains to check that for any solution $\bar{x}: Z_+ \to \Omega(x)$ satisfying

$$\|\bar{x}(k)\|_\infty = \|\bar{x}(0)\|_\infty = c, \forall k \in Z_+,$$

we have $\bar{x}(0) = 0$. When $c = 0$, $\bar{x}(0) = 0$ and the theorem is done. If $c > 0$, a contradiction will be found.

Let $M_k = \left\{i \in Z_p \middle| |\bar{x}_i(k)| = c\right\}$ for any $k \in Z_+$. We first claim that for any $k \in Z_+$, $M_{k+1} \subseteq M_k$. Indeed, if $i \in M_{k+1}$, $\|\bar{x}(k)\|_\infty = c > 0$ implies $\bar{x}(k) \ne 0, F_{ii}((\bar{x}(k))) > 0$ and

$$c = |\bar{x}_i(k+1)| = \left|\sum_{j=1}^{p} F_{ij}(\bar{x}(k))\bar{x}_j(k)\right| \le F_{ii}(\bar{x}(k))|\bar{x}_i(k)|$$
$$+ \sum_{j=1, j\ne i}^{p} |F_{ij}(\bar{x}(k))||\bar{x}_j(k)| \le \|\bar{x}(k)\|_\infty = c \tag{26}$$

based on (A3). By Lemma 2, we then have $|\bar{x}_i(k)| = c = |\bar{x}_i(k+1)|$. Hence $M_{k+1} \subseteq M_k$ for any $k \in Z_+$ and the claim is true.

Since $M_k \subseteq Z_p$ is a finite set, there exists $k_* \in Z_+$ such that $M_k = M_{k_*}$, $\forall k \ge k_*$. Again by (26) and Lemma 2, for any $i \in M_{k_*}$, if $F_{ij}(\bar{x}(k_*)) \ne 0$, then $|\bar{x}_j(k_*)| = c$ and so $j \in M_{k_*}$. This indicates $(F(\bar{x}(k_*)))'_{M_{k_*}} = 0$. On the other hand, when $\bar{x}_i(k_*+1) \ne \bar{x}_i(k_*)$, $\bar{x}_i(k_*+1) = -\bar{x}_i(k_*)$ (due to $M_{k_*+1} = M_{k_*}$) and

$$\sum_{j=1, j\ne i}^{p} |F_{ij}(\bar{x}(k_*))||\bar{x}_j(k_*)| \ge |\bar{x}_i(k_*+1) - F_{ii}(\bar{x}(k_*))\bar{x}_i(k_*)| > c$$

violates (26). Therefore $\bar{x}_i(k_*+1) = \bar{x}_i(k_*)$ and

$$\bar{x}_{M_{k_*}}(k_*) = \bar{x}_{M_{k_*}}(k_*+1) = F_{M_{k_*}}(\bar{x}(k_*))\bar{x}_{M_{k_*}}(k_*). \tag{27}$$

So $\bar{x}(k_*) \in D_{M_{k_*}} \subseteq \{0\}$ by (E), a contradiction appears.

Consequently, $\lim_{k\to\infty} x(k) = 0$ according to Proposition 2. ■

Remark 4. In some important cases, we have $(F(u))_J = (\hat{F}(u_J))_J$ and $(F(u))'_J = (\hat{F}(u_J))'_J$ for some matrix-valued function $\hat{F}$. When this condition holds, under (A3), it can be seen that D_J is an invariant set and for each $u \in D_J$, u_J is a fixed (equilibrium) point of the subsystem $x_J^+ = (F(x)x)_J = (\hat{F}(x_J))_J x_J$. Consequently, (E) becomes a necessary condition to guarantee the convergence of state to zero. ■

IV. Applications to Three Important Models

Convergence analysis of the three models presented in Section II is performed in this section.

A. The linear opinion model with stubborn agents

This section studies system (2) under the following connectivity condition where $J_0 = \{i \in Z_p | 0 \le \zeta_i < 1\}$ contains all possible stubborn agents *i*, which have $\zeta_i = 0$.

(C) For any $j \in Z_p \setminus J_0 = \{i \in Z_p | \zeta_i = 1\}$, there exists a finite sequence $(i = i_1, i_2), (i_2, i_3), \cdots (i_n, j = i_{n+1})$ with $i \in J_0$ and $A_{i_{(l+1)} i_l} \neq 0$ for any $l \in Z_n$.

Based on Theorem 2, the following result can be proposed. To save space, its proof is omitted here.

Theorem 4. Consider system (2). Suppose (C) holds. Then, ΘA is Schur stable, $(I - \Theta A)$ is nonsingular and every solution $z : Z_+ \to \Re^p$ of (2) satisfies

$$\lim_{k\to\infty} z(k) = (I - \Theta A)^{-1}(I - \Theta)z(0). \qquad (28)$$

Remark 5. In case of $\zeta_i = 0$, $z_i(k) \equiv z_i(0)$ and the attitude of this agent does not change. Such an agent is called a stubborn agent. If it doses not send any message to the other agents, convergence (agreement) is hard to be guaranteed. Roughly speaking, condition (C) requires that the information of stubborn agents ($\zeta_i = 0$) or partially stubborn agents ($\zeta_i < 1$) must pass their message to non-stubborn agents ($\zeta_i = 1$) through networks directly or indirectly (by a directed path) such that convergence (agreement) can be guaranteed. It is a necessary condition to guarantee that ΘA is Schur stable, see [22] for further discussion. ■

Remark 6. The assumption of *A* being row-stochastic can be relaxed. All we need is the following condition:

(A0) For any $i \in Z_p$, $\sum_{j=1}^{p} |A_{ij}| \leq 1$.

Particularly, the result could be extended to the case of signed graphs. ■

B. DeGroot-Friedkin model with some gamma coefficient not less than 0.5

This section studies system (10) where *H* is defined as in (9), $p \geq 2$ and there is $i_0 \in Z_{p+1}$ such that $\gamma_{i_0} \geq 1/2$.

To simply the discussion, we assume $i_0 = p+1$ and consider the error system (1) where $x = \pi \circ z$ with π being defined as in (12), *X* is the closed set defined in (13), and the (i, j) entry F_{ij} of *F* is defined as in (14)-(15).

The following technique lemma is needed [10].

Lemma 3. Consider system (10) and a solution $z : Z_+ \to \Delta$. Assume that $p \geq 2$, $r_{p+1} \geq 1/2$ and $0 \leq z_i(0) < 1$ for any $i \in Z_{p+1}$. Then, $0 \leq z_{p+1}(k) < z_{p+1}(k+1) < 1$ for any $k \in Z_+$, and $z_i(k) \leq 1 - z_{p+1}(1) < 1$ for any $i \in Z_p$ and $k \geq 1$. ■

Now the following result can be proposed and its proof is omitted here due to a limited space.

Theorem 5. Considered system (10) and a solution $z : Z_+ \to \Delta$. Suppose that $z(0) \in \Delta \setminus \{e_1, e_2, \cdots, e_p\}$, $p \geq 2$, $\gamma_{p+1} \geq 1/2$. Then, $\lim_{k\to\infty} z(k) = e_{p+1}$. Particularly, $E = \{e_1, e_2, \cdots, e_{p+1}\}$ is the set of all equilibrium points.

Remark 7. Here we only consider the case of $p \geq 2$. When $p = 1$ and $\gamma_1 = \gamma_2 = 1/2$, system (10) becomes $z^+ = z$ and every point in Δ is an equilibrium point. On the other hand, if $p = 1$ and $\gamma_1 < 1/2 < \gamma_2$, the proof of Theorem 5 can be applied to show that for any $z(0) \in \Delta \setminus \{e_1\}$, $\lim_{k\to\infty} z(k) = e_2$. Similarly, in case of $p = 1$ and $\gamma_1 > 1/2 > \gamma_2$, $\lim_{k\to\infty} z(k) = e_1$ when $z(0) \in \Delta \setminus \{e_2\}$. So $E = \{e_1, e_2\}$ is the set of all equilibrium points. ■

C. DeGroot-Friedkin model with each gamma coefficient less than 0.5

In this section, system (10) is studied where *H* is defined as in (9), $p \geq 2$ and $\gamma_i < 1/2$ for any $i \in Z_{p+1}$.

By definition of the equilibrium point $z^* \in \Delta \setminus E$, the following equation holds:

$$\sum_{i=1}^{p+1} \gamma_i e_i / (1 - z_i^*) = \sum_{i=1}^{p+1} \gamma_i z^* / (1 - z_i^*). \qquad (29)$$

The following property can be derived from (29), see Appendix for a proof.

Lemma 4. For any $i \in Z_{p+1}$, let

$$y_i = 1/(1 - z_i^*), \ \hat{y}_i = \sum_{j=1, j\neq i}^{p+1} \gamma_j / (1 - z_j^*) = \sum_{j=1, j\neq i}^{p+1} \gamma_j y_j. \qquad (30)$$

Then, $y_i = 1 + (\gamma_i y_i / \hat{y}_i)$ for any $i \in Z_{p+1}$. ■

To show the convergence, Lemma 5 below is needed. Its proof can be found in [Theorem 4.1, 10].

Lemma 5. Consider System (10) where $\sum_{i=1}^{p+1} \gamma_i = 1$ and

$$0 < \gamma_i < 1/2, \forall i \in Z_{p+1}.$$

Then, there exists a sufficiently small $\varepsilon_0 > 0$ such that $H(B_\infty(1-\varepsilon) \cap \Delta) \subseteq B_\infty(1-\varepsilon) \cap \Delta$ for any $0 < \varepsilon \leq \varepsilon_0$. ■

Notice that for any sufficiently small $\varepsilon > 0$,

$$B_\infty(1-\varepsilon) \cap \Delta \subseteq \Re^{p+1}$$

is a convex compact set. Thus, it is possible to find a homeomorphism (see [8])

$$\eta : B_\infty(1-\varepsilon) \cap \Delta \to [0,1]^p.$$

According to Lemma 5 and the Brouwer fixed-point theorem (Proposition 3), there exists one element $x_* \in [0,1]^p$ such that $\eta(H(\eta^{-1}(x_*))) = x_*$. Let $z^* = \eta^{-1}(x_*)$. Then, $z^* = H(z^*) \in B_\infty(1-\varepsilon) \cap \Delta \subseteq \Delta \setminus E$ is an equilibrium point.

Now the following convergence result can be proposed based on Theorem 1 where the proof is omitted.

Theorem 6. Consider system (10). Suppose that $p \geq 2$ and $\gamma_i < 1/2$ for any $i \in Z_{p+1}$. Then, there exists exactly one equilibrium point $z^* \in \Delta \setminus E$, and every solution $z : Z_+ \to \Delta$ with $z(0) \in \Delta \setminus E$ satisfies the following inequality:

$$\|z(k) - z^*\| = \|x(k)\| \leq ae^{-bk} \|x(0)\| = ae^{-bk} \|z(0) - z^*\| \qquad (31)$$

for any $k \in Z_+$ and some $a > 0$, $b > 0$.

D. Possible extensions

Other than the considered three models, the proposed results could be applied to more systems.

For example, consider the following continuous-time opinion model [3]:

$$\dot{x} = -A(x)Lx \quad x \in X \qquad (32)$$

where $x \in X$ is the state, $L \in \Re^{p\times p}$ and $A(x) = diag(A_1(x), A_2(x), \cdots, A_p(x))$ is a positive semi-definite matrix-valued function. For any $i \in Z_p$, A_i is called the susceptibility function and is continuous. Moreover, $L = (l_{ij})$ is a matrix associated to a proper social network. Here we assume that it is a generalized opposing Laplace (diagonally dominated) matrix that by definition, satisfies the following inequality [20]:

(L) For any $i \in Z_p$, $l_{ii} \geq \sum_{j=1, j\neq i}^{p} |l_{ij}|$.

Using the first order approximation, the following Euler model can be obtained:

$$x((k+1)T) = (I - TA(x(kT))L)x(kT) \qquad (33)$$

where $T>0$ is the sampling time. The system (33) is in the form of (1) where

$$F(x) = I - TA(x)L. \qquad (34)$$

Moreover, the following lemma shows that (A3) holds, see Appendix for a proof.

Lemma 6. Suppose (L) holds. Then, $F: u \in X \mapsto I - TA(u)L$ satisfies (A3) for any $r_0 > 0$ and any sufficiently small sampling time T (depending on r_0). ■

The proposed result (Theorem 3) can then be applied to such systems. Details are omitted here due to a limited space.

V. Conclusions

A unified framework was proposed to analyze a class of discrete-time nonlinear system, which contains several important models appeared in the recent studies related to social networks. Although only a few cases were discussed here, it is possible to extend these results to more systems. For example, we are working on the nonlinear opinion model (33). Based on the same framework, some comprehensive results can been achieved and will be studied in our future work.

Appendix A: Proof of Lemma 4

Since $z^* = H(z^*)$, $z_i^* = \gamma_i y_i / (\gamma_i y_i + \hat{y}_i)$ implies that for any $i \in Z_{p+1}$,

$$y_i = \frac{1}{1 - z_i^*} = 1/(1 - \frac{\gamma_i y_i}{\gamma_i y_i + \hat{y}_i}) = 1 + (\gamma_i y_i / \hat{y}_i).$$

The proof is then done. ■

Appendix B: Proof of Lemma 6

Since X is closed and A_i is continuous, $X \cap B_\infty(r_0)$ is compact and $\max_{u \in X \cap B_\infty(r_0)} A_i(u) < \infty$. By choosing any sufficiently small sampling time T, $F_{ii}(u) = 1 - TA_i(u) l_{ii} > 0$ for any $i \in Z_p$ and $u \in X \cap B_\infty(r_0)$. According to condition (L), (A3) holds by the fact that for any $i \in Z_p$,

$$\sum_{j=1}^{p} |F_{ij}(u)| = 1 + TA_i(u)(-l_{ii} + \sum_{j=1, j\neq i}^{p} |l_{ij}|) \leq 1 \; \forall u \in X \cap B_\infty(r_0). \quad ■$$

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