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Closed-Loop Model Reduction by Moment Matching for Linear Systems

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Abstract— **We study the model reduction by moment matching problem for linear systems in a closed-loop configuration. First we show that the moments of a linear system can be expressed in a form that is independent of the structure of the signal generator. Then we define a class of reduced-order models that can replicate the steady-state response of the original system from input-output data. Finally, we demonstrate the applicability of the results using two simple numerical examples.**

I. Introduction

Model reduction consists in the simplification of the dynamics of a complex system to obtain a *simpler model* that can approximate the behavior of the original system under given conditions. The model reduction problem has been widely studied in the past decades because of its rich mathematical structure and applicability in diverse areas of science and engineering [1], for example in the modeling of mechanical systems, for which rigid-body models are often derived by ignoring the effects of flexible modes [2]; in large-scale integrated circuit design, consisting of billions of transistors [3]; in weather forecast models, using large amounts of atmospheric data [4]; and more recently in disease-modeling, to better understand the spread of global epidemics [5]. While most systems encountered in applications are nonlinear, significant attention has been devoted to the model reduction of linear systems. Within this context, the Hankel operator has been used to determine the approximation error between the reduced-order model and the underlying full-order system [6], leading to the celebrated balancing realization problem [7]; whereas H_2 and H_∞ norms of the error system have been utilized to compute approximation errors [8]–[11], and have helped in computing useful reduced-order models. More recently, the Loewner framework has been used to obtain reduced-order models for linear time-invariant and linear time-varying systems [12], [13]. Another class of methods of particular interest are the so-called moment matching methods, which allow the construction of reduced-order models such that the error between the underlying system and the reduced-order model is zero at some points of the complex plane [14]–[16]. To the best of our knowledge in all the above references and the references therein, the system to be reduced is considered in open-loop and/or is driven by user-selected signals. An important yet largely missed caveat is that small modeling errors in the open-loop do not necessarily imply small modeling errors in the closed-loop [17]. While model reduction for systems in closed-loop with a given controller has been given some attention in [17]–[20], the above mentioned aspect of reduced-order modeling has not been extensively studied as its open-loop counterpart. In particular, moment matching methods have largely focused on obtaining reduced-order models in scenarios where full information about the signal generator is readily available. This is generally the case if the system under consideration is in an open-loop setting. However, in closed-loop configuration, the output of the controller may be the only signal that is measurable.

To deal with the model reduction problem in a closed-loop architecture, we exploit the ideas in [15], [16], in which the underlying system is driven by an explicit signal generator. It is important to note, however, that the definition of moment utilizing explicit signal generators is dependent on the structure of the signal generator, and this may not be known if the input signal is generated by a controller. To deal with this issue, the goal of this work is to provide a notion of moment that is independent of the structure of the signal generator, thus allowing to define moments for systems in closed-loop with a controller.

The rest of the paper is structured as follows. The steady-state notion of moments for implicit and explicit signal generators is briefly summarized in Section II. The signal generator independent notion of moment is given in Section III. Reduced-order models using this newly formed definition are then introduced and their construction illustrated with two examples in Section IV. Finally, some concluding remarks are presented in Section V.

Notation: We use standard notation. $\mathbb{R}_{\geq 0}$ denotes the set of nonnegative real numbers, $\mathbb{R}_{>0}$ denotes the set $\mathbb{R}_{\geq 0} \setminus \{0\}$, $\mathbb{C}_{<0}$ denotes the set of complex numbers with negative real part, and $\mathbb{C}_{\geq 0}$ denotes the set $\mathbb{C} \setminus \mathbb{C}_{<0}$. The symbol I denotes the identity matrix of appropriate dimensions, $\sigma(A)$ denotes the spectrum of the square matrix A, and $\|A\|$ denotes the induced Euclidean matrix norm of the matrix A.

II. Preliminaries

In this section, we first recall the steady-state based description of moment as described in [14], [21]. We then recall the definition of moment using explicit signal generators given in [15], [16]. Finally, we introduce a formulation that allows the extension of the definition of moment to a closed-loop setting.

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A. The steady-state notion of moment

Consider a continuous-time, single-input, single-output system described by the equations

$$\begin{aligned}\dot{x} &= Ax + Bu,\\ y &= Cx,\end{aligned} \tag{1}$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$, and $C \in \mathbb{R}^{1\times n}$. Let the associated transfer function $W(s) = C(sI-A)^{-1}B$ be minimal, i.e., the triple (C, A, B) is controllable and observable.

Definition 1: The 0-moment of system (1) at $s_i \in \mathbb{C}\setminus\sigma(A)$ is the complex number $\eta_0(s_i) = C(s_iI - A)^{-1}B$. The k-moment of system (1) at $s_i \in \mathbb{C}\setminus\sigma(A)$ is the complex number

$$\eta_k(s_i) = \frac{(-1)^k}{k!}\left[\frac{d^k}{ds^k}\big(C(sI-A)^{-1}B\big)\right]_{s=s_i}.$$

Remark 1: The k-moment of system (1) at s_i coincides with the k-th coefficient of the Laurent series expansion of W in a neighborhood of s_i, provided that it exists [21].

Remark 2: The reduced-order models obtained from this notion of moment is such that the resulting transfer function and its derivatives take the same values as W and its derivatives at all s_i.

The notion of moment for system (1) can also be described in terms of a Sylvester equation, as outlined below.

Lemma 1: [14] Consider system (1), let $s_i \in \mathbb{C}$ be such that $s_i \notin \sigma(A)$, for all $i = 1, \cdots, n$. Then there exists a one-to-one relation between the moments $\eta_0(s_1), \cdots, \eta_{k_1-1}(s_1), \cdots, \eta_0(s_\eta), \cdots, \eta_{k_\eta-1}(s_\eta)$ and the matrix $C\Pi$, where Π is the unique solution of the Sylvester equation

$$A\Pi + BL = \Pi S, \tag{2}$$

with $S \in \mathbb{R}^{\nu\times\nu}$ a non-derogatory matrix with characteristic polynomial

$$p(s) = \prod_{i=1}^{\eta}(s - s_i)^{k_i}, \tag{3}$$

with $\nu = \sum_{i=1}^{\eta} k_i$, and the pair (L, S) is observable.

This formulation allows a relationship to be established, through the Sylvester equation, between the moments of a system and its steady-state output response, as outlined next.

Theorem 1: [14] Consider system (1). Let $s_i \in \mathbb{C}$ be such that $s_i \notin \sigma(A)$, for all $i = 1, \cdots, n$, and assume that $\sigma(A) \subset \mathbb{C}_{<0}$. Let $S \in \mathbb{R}^{\nu\times\nu}$ be any non-derogatory matrix with characteristic polynomial as defined in (3). Consider the interconnection of system (1) with the signal generator

$$\begin{aligned}\dot{\omega} &= S\omega,\\ u &= L\omega,\end{aligned} \tag{4}$$

such that the pair (L, S) is observable and the triple $(L, S, \omega(0))$ is minimal. Then there exists a one-to-one relation between the moments $\eta_0(s_1), \cdots, \eta_{k_1-1}(s_1), \cdots, \eta_0(s_\eta), \cdots, \eta_{k_\eta-1}(s_\eta)$ and the steady-state output of the interconnected system (1) and (4).

The reduced-order models obtained from this notion of moment is such that the steady-state output response of the reduced model is equal to the steady-state output response of system (1). This interpretation allows the model reduction problem to be transformed from a problem of interpolation of points to a problem of interpolation of signals.

B. Moment of a system driven by explicit signal generators

In several applications, such as those arising in modern power electronics, a differential representation of the signal generator is not available. Instead, the signal generator is usually described *explicitly* by the equations

$$\begin{aligned}\omega(t) &= \Lambda(t, t_0)\omega_0,\\ u &= L\omega,\end{aligned} \tag{5}$$

with $\Lambda(t, t_0) \in \mathbb{R}^{\nu\times\nu}$ such that $\Lambda(t_0, t_0) = I$, and $L \in \mathbb{R}^{1\times\nu}$ [16]. Such a representation describes a very general class of signal generators and includes the *implicit* representation (4).

For this class of signal generators, the results described in Section II-A cannot be readily used [16], and the steady-state notion of moment needs to be redefined.

Consider system (1) driven by the signal generator (5). To establish the existence of the steady-state output response of system (1), the following hypotheses are introduced on the class of input signals given by (5).

Assumption 1: The vector-valued function ω in (5) has a strictly proper Laplace transform with non-negative poles.

Assumption 2: The matrix-valued function Λ is non-singular for all $t \geq t_0$.

In addition, assume that there exists a set $\mathcal{T} \subset \mathbb{R}_{\geq 0}$ in which Λ is differentiable with respect to t, and consider the time-varying system described by the equation

$$\dot{z}(t) = G(t)^T z(t), \tag{6}$$

with $G(t) = -\dot{\Lambda}(t, t_0)\Lambda(t, t_0)^{-1}$, and let $\Phi(t, t_0)$ be the transition matrix of system (6).

Assumption 3: The function G is piecewise continuous with respect to t. Moreover, there exists a $T \geq t_0$ and a polynomial p such that $\|\Phi(t, t_0)^T\| \leq p(t)$ for all $t \geq T$.

Assumption 1 is a standard condition required for the existence of a well-defined steady-state response for system (1) driven by the signal generator (5) [15], [16], [22]. Assumption 2 is required to obtain a unique solution, ω, from (5). Finally, Assumption 3 guarantees that the norm of z in system (6) does not diverge exponentially [23]. This allows the steady-state response of system (1) driven by the signal generator (5), denoted by[1] x_s, to be written as $x_s(t) = \Pi(t)\omega(t)$ for all $t \geq 0$ and for some matrix-valued function Π. In addition, the piecewise continuity of G guarantees that the steady-state response is unique. This allows the definition of the steady-state notion of moment for systems driven by explicit signal generators, as outlined in the next statement.

[1] $x_s(t) \in \mathbb{R}^n$ is the steady-state response of $x(t) \in \mathbb{R}^n$ if for any initial condition, $x(t_0)$, $\lim_{t\to\infty} x(t) = x_s(t)$.

Theorem 2: [15], [16] Consider the interconnection of system (1) and the signal generator (5). Suppose that Assumptions 2 and 3 hold, $\sigma(A) \subset \mathbb{C}_{<0}$ and $\Lambda(t,t_0)$ is differentiable almost everywhere. Let Π, defined as

$$\Pi(t) = \left(e^{A(t-t_0)}\Pi(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}BL\Lambda(\tau,t_0)d\tau \right)\Lambda(t,t_0)^{-1} \tag{7}$$

denote a family of matrix-valued functions parameterized in $\Pi(t_0) \in \mathbb{R}^{n\times\nu}$. Then there exists a unique Π_∞ such that for any $\Pi(t_0)$, $\lim_{t\to\infty}\Pi(t) - \Pi_\infty(t) = 0$, where Π_∞ is the solution of equation (7) with $\Pi(t_0) = \Pi_\infty(t_0)$. Moreover, if $x(t_0) = \Pi_\infty(t_0)\omega(t_0)$, then $x(t) - \Pi_\infty(t)\omega(t) = 0$ for all $t \geq t_0$, and the set $\mathcal{M}_\infty = \{(x,\omega) \in \mathbb{R}^{n+\nu} \,|\, x(t) = \Pi_\infty(t)\omega(t)\ \}$ is attractive.

Remark 3: The definition of the function Π_∞ can be given as in (7) or, alternatively, as the unique solution of

$$\dot{\Pi}(t) = A\Pi(t) + BL - \Pi(t)\dot{\Lambda}(t,t_0)\Lambda(t,t_0)^{-1}, \tag{8}$$

with the initial condition $\Pi(t_0) = \Pi_\infty(t_0)$ [15], [16]. Moreover, when the system is driven by an explicit signal generator, the matrix Π_∞ is a function of time, unlike when we have an implicit model of the signal generator.

Definition 2: [15], [16] Consider the interconnection of system (1) and the signal generator (5). Suppose that Assumptions 1, 2, and 3 hold, and that $\sigma(A) \subset \mathbb{C}_{<0}$. The function $C\Pi_\infty\,\omega$, where Π_∞ is the solution of equation (7) with $\Pi(t_0) = \Pi_\infty(t_0)$, is defined as the *moment of system* (1) *at* Λ.

Thus, using the integral-matrix equation (7), or the differential equation (8), the steady-state notion of moment has been extended to a class of input signals originating from signal generators that do not necessarily have a differential or implicit representation. Finally, note that the frameworks discussed in II-A and II-B are readily applicable to the multi-input, multi-output case [14].

C. Problem Formulation

The results described in Section II-B require the knowledge of the matrix-valued function Λ and of the matrix L for constructing the input signal u. However, both of these quantities may not be known *a-priori*. This is the case, for example, if the system to be reduced operates in a closed-loop setting. It is, therefore, more realistic to model the control signal u as the direct output, ω, of an explicit signal generator, such as the one described in (5). In addition, to accommodate multiple-input, multiple-output systems, we need to adjust the frameworks presented so far, as discussed briefly at the end of Section II-B. For these reasons, consider the following formulation.

Consider a continuous-time, multiple-input, multiple-output system described by the equations

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned} \tag{9}$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{p\times n}$. In addition, consider the signal generator

$$\begin{aligned} \omega(t) &= \Lambda(t,t_0)\omega_0, \\ u(t) &= \omega(t), \end{aligned} \tag{10}$$

with $\omega(t) \in \mathbb{R}^m$, $\omega_0 \in \mathbb{R}^m$, and $\Lambda(t,t_0) \in \mathbb{R}^{m\times m}$. Note that the signal generator (10) is a restriction of the signal generator (5) with $L = I \in \mathbb{R}^{m\times m}$. The output of the signal generator (10) models the control input that a system would receive in closed-loop operation. Although the signal generator (10) still depends on Λ, the moment of the system can be expressed in a form that is independent of Λ, as will be shown in the next section. Finally, we note here that Assumptions 1, 2, and 3 readily extend to the multiple-input, multiple-output case.

III. Definition of Moment in Closed-Loop

In this section, we extend the definition of moment of system (9) to the case in which the signal generator is described by (10), but Λ is not known, i.e., u is the only measured signal. To this end, we begin by showing that the moment of the system can be expressed in a form that is agnostic of the structure of the signal generator (10). More precisely, we show that the moment of the system, denoted by a matrix-valued function Π, can be expressed independently of Λ using the following result.

Theorem 3: Consider the interconnection of system (9) and the signal generator (10). Suppose that Assumptions 2 and 3 hold, and $\sigma(A) \subset \mathbb{C}_{<0}$. Let Π, defined as

$$\dot{\overbrace{\Pi(t)u(t)}} = \big(A\Pi(t) + B\big)u(t), \tag{11}$$

be a family of matrix-valued functions parameterized in $\Pi(t_0) \in \mathbb{R}^{n\times m}$. Then there exists a unique Π_∞ such that for any $\Pi(t_0)$, $\lim_{t\to\infty}(\Pi(t) - \Pi_\infty(t))u(t) = 0$, where Π_∞ is the solution of equation (11) with $\Pi(t_0) = \Pi_\infty(t_0)$. In addition, if $x(t_0) = \Pi_\infty(t_0)\omega_0$, then $x(t) = \Pi_\infty(t)u(t)$ for all $t \geq t_0$, and the set $\mathcal{M}_\infty = \{(x,u) \in \mathbb{R}^{n+m} \,|\, x(t) = \Pi_\infty(t)u(t)\}$ is attractive.

Proof: Since Assumption 2 holds, equation (7) with the signal generator (10) can be written as

$$\Pi(t)\Lambda(t,t_0) = e^{A(t-t_0)}\Pi(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}B\Lambda(\tau,t_0)d\tau.$$

Multiplying both sides by ω_0 yields

$$\Pi(t)\omega(t) = e^{A(t-t_0)}\Pi(t_0)\omega_0 + \int_{t_0}^{t} e^{A(t-\tau)}B\omega(\tau)d\tau,$$

and using the relation $u(t) = \omega(t)$ yields

$$\Pi(t)u(t) = e^{A(t-t_0)}\Pi(t_0)\omega_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau. \tag{12}$$

Let $\mathcal{T} \subset \mathbb{R}_{\geq 0}$ be a set in which Πu is differentiable. Differentiating both sides of (12) over $\mathcal{T}$ yields

$$\begin{aligned}\dot{\overbrace{\Pi(t)u(t)}} &= Ae^{A(t-t_0)}\Pi(t_0)\omega_0 + \frac{d}{dt}\left(\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau\right)\\ &= Ae^{A(t-t_0)}\Pi(t_0)\omega_0 + Bu(t) +\\ &\quad A\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau\\ &= A\big(e^{A(t-t_0)}\Pi(t_0)\omega_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau\big) +\\ &\quad Bu(t)\\ &= A\Pi(t)u(t) + Bu(t)\\ &= \big(A\Pi(t) + B\big)u(t).\end{aligned}$$

Let Π_1 and Π_2 be the solutions of (11) with initial conditions $\Pi_1(t_0)$ and $\Pi_2(t_0)$, respectively, and define the error $E(t) = \Pi_1(t)u(t) - \Pi_2(t)u(t)$. Then,

$$\begin{aligned}\dot{E}(t) &= \dot{\overbrace{\Pi_1(t)u(t)}} - \dot{\overbrace{\Pi_2(t)u(t)}}\\ &= \big(A\Pi_1(t) + B\big)u(t) - \big(A\Pi_2(t) + B\big)u(t)\\ &= A\big(\Pi_1(t)u(t) - \Pi_2(t)u(t)\big)\\ &= AE(t),\end{aligned}$$

which leads to

$$E(t) = e^{A(t-t_0)}E(t_0).$$

Since $\sigma(A) \subset \mathbb{C}_{<0}$, E converges to zero. This implies that for a given u, there exists a unique solution Π_∞ to which all solutions of (11) converge. More precisely, for any $\Pi(t_0)$, there exists a $\Pi_\infty(t_0)$ such that $\lim_{t\to\infty}(\Pi(t) - \Pi_\infty(t))u(t) = 0$.

By Assumption 2, the unique solution of system (9) with input u defined by (10) is

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau.$$

Let $x(t_0) = \Pi_\infty(t_0)\omega_0$. Thus, $x(t)$ can be written as

$$x(t) = e^{A(t-t_0)}\Pi_\infty(t_0)\omega_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau.$$

From (12), we see that $x(t) = \Pi_\infty(t)u(t)$ for all $t \geq t_0$. The attractivity of $\Pi_\infty(t)$ and the invariance of $x(t) = \Pi_\infty(t)u(t)$ imply that the set $\mathcal{M}_\infty$ is attractive. ■

Corollary 3.1: Suppose that Assumptions 1, 2, and 3 hold. Then the function $\Pi_\infty u$ is the *steady-state response* x_s *of* x, i.e., for any $x(t_0)$ and ω_0, $\lim_{t\to\infty} x(t) - \Pi_\infty(t)u(t) = 0$.

Remark 4: The differential equation (11) or the integral equation (12) play the role of the Sylvester equation (2) when system (9) is driven by an implicit signal generator (4). To see this, note that for an implicit signal generator (4) represented as (10), we have

$$\dot{\omega}(t) = S\omega(t) \implies \dot{u}(t) = Su(t).$$

Using this result in (11), we have

$$\begin{aligned}\big(A\Pi(t) + B\big)u(t) =&\dot{\overbrace{\Pi(t)u(t)}}\\ =&\dot{\Pi}(t)u(t) + \Pi(t)\dot{u}(t)\\ =&\dot{\Pi}(t)u(t) + \Pi(t)Su(t),\end{aligned}$$

which can be written as

$$\big(A\Pi(t) + B - \Pi(t)S\big)u(t) = \dot{\Pi}(t)u(t). \tag{13}$$

From Remark 3, we know that Π is a constant when the system is driven by an implicit signal generator, implying $\dot{\Pi} = 0$. Using this in (13) we have, for any $u(t)$,

$$A\Pi + B - \Pi S = 0,$$

which is indeed the Sylvester equation (2).

Definition 3: Consider the interconnection of system (9) and the signal generator (10). Suppose that Assumptions 1, 2, and 3 hold, and $\sigma(A) \subset \mathbb{C}_{<0}$. The function $C\Pi_\infty u$, where Π_∞ is the solution of equation (11) with $\Pi(t_0) = \Pi_\infty(t_0)$, is the moment of system (9) at u.

Corollary 3.2: Consider the interconnection of system (9) and the signal generator (10). Suppose that Assumptions 1, 2, and 3 hold, and $\sigma(A) \subset \mathbb{C}_{<0}$. Then the moment of system (9) at u coincides with the steady-state output response of the interconnected system.

Proof: Since Assumptions 1, 2, and 3 hold, and $\sigma(A) \subset \mathbb{C}_{<0}$, the steady-state response of system (9) is well-defined. In addition, the relation $x_s(t) = \Pi_\infty(t)u(t)$, where Π_∞ is the unique solution of equation (11) with $\Pi(t_0) = \Pi_\infty(t_0)$, holds for all t. Furthermore, by Theorem 3, the set $\mathcal{M}_\infty$ is attractive, and the steady-state output response of the interconnected system is $C\Pi_\infty u$, which by definition is the moment of the system. ■

IV. Reduced-Order Modeling

In this section, we introduce reduced-order models achieving moment matching by utilizing the notion of moment presented in Section III.

Definition 4: Consider the interconnection of system (9) and the signal generator (10). Suppose that Assumptions 1, 2, and 3 hold, and $\sigma(A) \subset \mathbb{C}_{<0}$. Then the system described by the equations

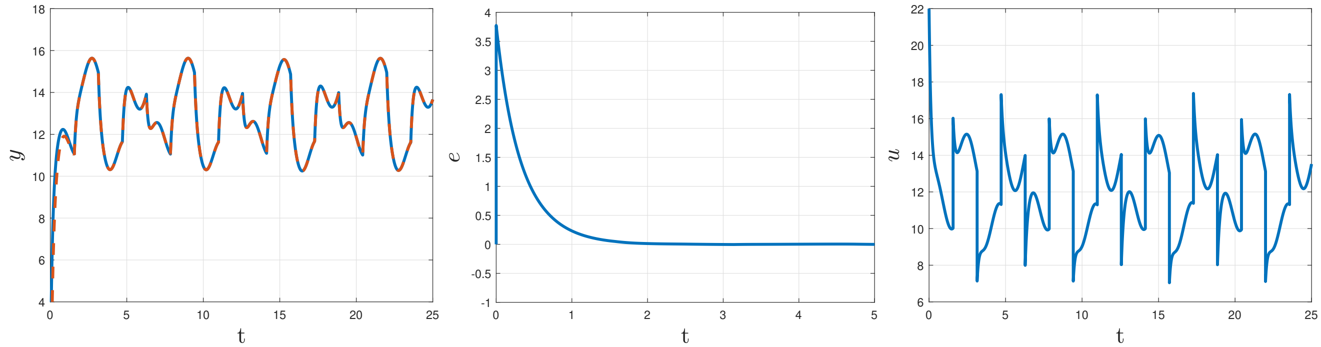
$$\begin{aligned}\xi(t) &= F(t,t_0)\xi_0 + \int_{t_0}^{t} G(t-\tau)u(\tau)d\tau,\\ \psi(t) &= H(t)\xi(t),\end{aligned} \tag{14}$$

with $\xi(t) \in \mathbb{R}^\rho$, $\psi(t) \in \mathbb{R}^p$, $F(t,t_0) \in \mathbb{R}^{\rho\times\rho}$, $G(t) \in \mathbb{R}^{\rho\times m}$ is a *model* of system (9) at (10) if there exists a unique solution P_∞ of the integral equation

$$P(t)u(t) = F(t,t_0)P(t_0)\omega_0 + \int_{t_0}^{t} G(t-\tau)u(\tau)d\tau, \tag{15}$$

or of the differential equation

$$\begin{aligned}\dot{\overbrace{P(t)u(t)}} = \dot{F}(t,t_0)P(t_0)\omega_0 + G(0)u(t) +\\ \int_{t_0}^{t} \dot{G}(t-\tau)u(\tau)d\tau,\end{aligned} \tag{16}$$



(a) (b) (c)

Fig. 1. Time histories of (a) the output of the full-order model (solid blue line) and the output of the reduced-order model (dashed orange line); (b) the error between the output of the full-order model and the output of the reduced-order model; and (c) the control input u, for Example 1. Note that the error plot has a different time-scale in order to highlight the transient behavior of the reduced-order model.

with $P(t_0) = P_\infty(t_0)$, such that for any $P(t_0)$, $\lim_{t\to\infty}(P(t) - P_\infty(t))u(t) = 0$ and

$$C\Pi_\infty(t) = H(t)P_\infty(t), \tag{17}$$

where Π_∞ is the unique solution of equation (11) with $\Pi(t_0) = \Pi_\infty(t_0)$. Furthermore, the system (14) is a *reduced-order model* of system (9) at (10) if $\rho < n$.

Remark 5: To guarantee the existence of the reduced-order model (14), the following conditions need to be satisfied. First, the reduced-order model has to be asymptotically stable [24]. This guarantees the existence of the moment of system (14), i.e., the existence of the matrix-valued function P_∞. Second, the matrix-valued function P_∞ needs to have full column rank, m, for all $t \in \mathbb{R}_{\geq 0}$. This ensures that (17) has a solution. As a consequence of this rank condition, we obtain $\rho \geq m$, that is the reduced-order model has to have minimum order m.

We now describe an approach that allows the simplification of the family of reduced-order models (14). To this end, consider the selection

$$\begin{aligned} F(t,t_0) &= e^{\tilde{F}(t-t_0)}, \\ G(t) &= e^{\tilde{F}t}\tilde{G}, \end{aligned} \tag{18}$$

for some $\tilde{F}$ and $\tilde{G}$. This allows representing the state-transition equation of (14) as a linear time-invariant system, which in turn allows easily enforcing additional constraints.

Corollary 3.3: The system described by the equations (14), with $F(t,t_0)$ and $G(t)$ described by (18), is a model of the system (9) at (10) if there exists a unique solution P_∞ of the differential equation

$$\overbrace{P(t)u(t)}^{\cdot} = \left(\tilde{F}P(t) + \tilde{G}\right)u(t), \tag{19}$$

with $P(t_0) = P_\infty(t_0)$, such that for any $P(t_0)$, $\lim_{t\to\infty}(P(t) - P_\infty(t))u(t) = 0$ and

$$C\Pi_\infty(t) = H(t)P_\infty(t), \tag{20}$$

where Π_∞ is the unique solution of equation (11) with $\Pi(t_0) = \Pi_\infty(t_0)$.

We conclude this section with two numerical examples.

Example 1: Consider the interconnection of system (9) and the signal generator (10). The state-space matrices of the full-order system are generated using the `rss` command in MATLAB, and are given by

$$A = \begin{bmatrix} -2.6450 & 0.3960 & -0.4228 & -0.5738 \\ -0.0344 & -2.5707 & 0.6920 & 0.1247 \\ -0.5566 & -0.0248 & -3.3686 & -1.0689 \\ -0.5948 & 0.5815 & -0.8918 & -3.4373 \end{bmatrix},$$
$$B = \begin{bmatrix} -0.2298 & -1.4617 & -2.8823 & -0.0475 \end{bmatrix}^T,$$
$$C = \begin{bmatrix} -0.4625 & -0.5766 & -0.8460 & -1.8172 \end{bmatrix}.$$

The reduced-order model (18) is chosen with

$$\tilde{F} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \; \tilde{G} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T. \tag{21}$$

Note that the full-order system is both controllable and observable. The reference signal to the full-order system is a combination of sinusoids with two angular frequencies, a chirp signal, and a square wave signal. The input to the plant is from a PID controller, which in turn receives as input the error between the reference signal and the output. Figure 1 shows the time histories of the outputs of the full-order and of the reduced-order models, respectively. Consistent with the results presented thus far, the steady-state output responses of both these models match closely.

Now consider feedback linearizable nonlinear systems with bounded trajectories. These can be expressed as a linear system with matched nonlinearities. For such systems, the modeling perspective presented in the preceding sections is readily applicable. To demonstrate this, consider the following example.

Example 2: The dynamics of a forced Van der Pol oscillator can be described by the differential equation

$$\frac{d^2x}{dt^2} - \mu(1-x^2)\frac{dx}{dt} + x - \mathcal{A}\,cos\gamma t = 0. \tag{22}$$

This system can be represented as the interconnection of system (9) and the signal generator (10) using the state-space equation

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -\mu \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u,$$

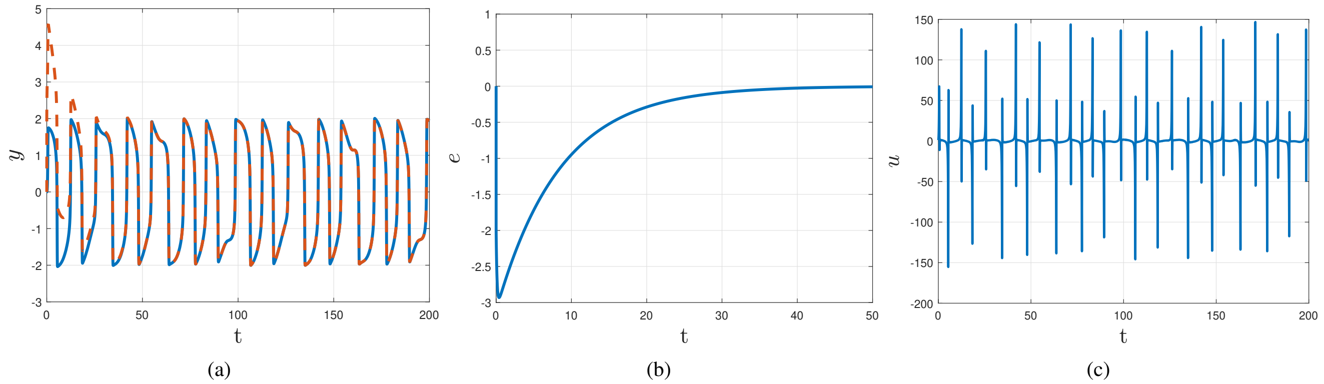


Fig. 2. Time histories of (a) the output of the nonlinear system (solid blue line) and the output of the reduced-order model (dashed orange line); (b) the error between the output of the nonlinear system and the output of the reduced-order model; and (c) the control input u, for Example 2. Note that the error plot has a different time-scale in order to highlight the transient behavior of the reduced-order model.

where $u = \mathcal{A}\,cos\gamma t + 2\mu\dot{x} - \mu x^2\dot{x}$. For this example, we choose $\mathcal{A} = 1.2$ and $\gamma = 2\pi/10$. In addition, we choose $\mu = 8.53$, which renders (22) chaotic [25]. The output of this nonlinear system is chosen to be x. Finally, we choose (21) to be our "reduced-order" model (see also (14)). Note that our objective in this example is not to obtain a model that has fewer states than the underlying system, instead it is to illustrate that the nonlinear system under consideration can be represented by a linear system with a time-varying output map. Figure 2 shows the time histories of the outputs of the nonlinear system and of the reduced-order model, respectively.

V. Conclusion

We have defined the steady-state notion of moment of a system in a closed-loop setting such that the moment is independent of the form of the signal generator. We have then defined a class of reduced-order models, the steady-state output response of which converges to that of the underlying system. These results have been illustrated using two numerical examples. The first example illustrates our formulation in a linear setting, whereas the second example exemplifies the applicability of our approach beyond linear systems. The transient behavior of the reduced-order model can be improved by strategically exploiting the structure of the matrices in the state-transition equation of the reduced-order model. We note that this approach can be very useful for model reduction in a data-driven setting. Future work will include exploring a more general notion of moment for nonlinear systems along with applications to control design.

References

[1] P. Benner, M. Ohlberger, A. Cohen, and K. Willcox, *Model reduction and approximation: theory and algorithms*. SIAM, 2017.

[2] P. Koutsovasilis and M. Beitelschmidt, "Comparison of model reduction techniques for large mechanical systems: a study on an elastic rod," *Multibody System Dynamics*, vol. 20, pp. 111–128, 2008.

[3] P. Benner, M. Hinze, and E. J. W. Ter Maten, *Model reduction for circuit simulation*. Springer, 2011, vol. 74.

[4] I. Horenko, R. Klein, S. Dolaptchiev, and C. Schütte, "Automated generation of reduced stochastic weather models i: simultaneous dimension and model reduction for time series analysis," *Multiscale Modeling & Simulation*, vol. 6, no. 4, pp. 1125–1145, 2008.

[5] M. J. Keeling and K. T. Eames, "Networks and epidemic models," *Journal of the royal society interface*, vol. 2, no. 4, pp. 295–307, 2005.

[6] K. Glover, "All optimal hankel-norm approximations of linear multivariable systems and their L^∞-error bounds," *International journal of control*, vol. 39, no. 6, pp. 1115–1193, 1984.

[7] B. Moore, "Principal component analysis in linear systems: Controllability, observability, and model reduction," *IEEE transactions on automatic control*, vol. 26, no. 1, pp. 17–32, 1981.

[8] A. Antoulas and A. Astolfi, "H_∞-norm approximation," *Unsolved Problems in Mathematical Systems and Control Theory*, p. 267, 2004.

[9] D. Kavranoğlu and M. Bettayeb, "Characterization of the solution to the optimal H_∞ model reduction problem," *Systems & Control Letters*, vol. 20, no. 2, pp. 99–107, 1993.

[10] X.-X. Huang, W.-Y. Yan, and K. L. Teo, "H_2 near-optimal model reduction," *IEEE Transactions on Automatic Control*, vol. 46, no. 8, pp. 1279–1284, 2001.

[11] S. Gugercin, A. C. Antoulas, and C. Beattie, "H_2 model reduction for large-scale linear dynamical systems," *SIAM journal on matrix analysis and applications*, vol. 30, no. 2, pp. 609–638, 2008.

[12] A. Mayo and A. Antoulas, "A framework for the solution of the generalized realization problem," *Linear algebra and its applications*, vol. 425, no. 2-3, pp. 634–662, 2007.

[13] J. D. Simard and A. Astolfi, "Loewner functions for linear time-varying systems with applications to model reduction," *IFAC-PapersOnLine*, vol. 53, no. 2, pp. 5623–5628, 2020.

[14] A. Astolfi, "Model reduction by moment matching for linear and nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 10, pp. 2321–2336, 2010.

[15] G. Scarciotti and A. Astolfi, "Characterization of the moments of a linear system driven by explicit signal generators," in *2015 American Control Conference (ACC)*. IEEE, 2015, pp. 589–594.

[16] ——, "Model reduction by matching the steady-state response of explicit signal generators," *IEEE Transactions on Automatic Control*, vol. 61, no. 7, pp. 1995–2000, 2015.

[17] L. A. Aguirre, "Validation of reduced-order models for closed loop applications," in *Proceedings of IEEE International Conference on Control and Applications*. IEEE, 1993, pp. 605–610.

[18] D. Xue and D. Atherton, "An optimal model reduction method for closed-loop systems," in *[1991] Proceedings of the 30th IEEE Conference on Decision and Control*. IEEE, 1991, pp. 2729–2730.

[19] G. Braileanu, "Closed-loop model reduction method," in *[1992] Proceedings of the 31st IEEE Conference on Decision and Control*. IEEE, 1992, pp. 2864–2865.

[20] R. B. Choroszucha, J. Sun, and K. Butts, "Closed-loop model order reduction and mpc for diesel engine airpath control," in *2015 American Control Conference (ACC)*. IEEE, 2015, pp. 3279–3284.

[21] A. C. Antoulas, *Approximation of large-scale dynamical systems*. SIAM, 2005.

[22] B. A. Francis, "The linear multivariable regulator problem," *SIAM Journal on Control and Optimization*, vol. 15, no. 3, pp. 486–505, 1977.

[23] R. W. Brockett, *Finite dimensional linear systems*. SIAM, 2015.

[24] H. K. Khalil, *Nonlinear control*. Pearson New York, 2015, vol. 406.

[25] Y. Ku and X. Sun, "Chaos in van der pol's equation," *Journal of the Franklin Institute*, vol. 327, no. 2, pp. 197–207, 1990.