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Adaptive Spatial PID and PD coupling in synchronization control of collocated infinite and finite dimensional systems

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Abstract— **This paper presents an entirely different approach for the synchronization of identical networked infinite dimensional systems. With a large class of infinite dimensional systems representing partial differential equations (PDEs), the concept of a functional form of the consensus protocol used for synchronization is applied here and incorporates spatial derivatives and spatial averages of the differences of the PDE states. This leads to spatial PD-type of consensus protocols for synchronization of PDEs. When the networked PDEs are tasked with following a leader, also described by a PDE of the same type, an added component of the controller is incorporated to ensure leader following. The proposed PD-coupling in the synchronization control of infinite dimensional systems attains a new form for the finite dimensional case, where now a temporal PID coupling in the consensus protocol is implemented. Simulation studies for both the infinite and the finite dimensional cases are included to demonstrate the effects of the non-traditional coupling in the synchronization control of networked systems.**

Index Terms— **Infinite dimensional systems; synchronization.**

I. Introduction

The synchronization of identical dynamical systems, whereby each networked controller incorporates a consensus protocol in order to enhance agreement (synchronization) and possibly leader-following (tracking) was primarily examined in the finite dimensional system arena, [1]. One of the first papers addressing the synchronization of identical PDEs was in [2], [3], [4]. The authors examined coupled advection-diffusion PDEs with diffusive coupling, in which the consensus protocol included the second spatial derivative of the pairwise state differences. It also examined the convergence properties of the networked PDEs to a desired state (leader following), also governed by an advection-diffusion PDE. An $\mathbb{H}^\infty$ analysis provided expressions for the consensus weights via the solution to an operator inequality.

Using control signals, transmitted though the spatial processes via an appropriate input operator was the next extension to synchronization of networked PDEs. The work in [5] introduced controllers that addressed the two objectives: leader following and agreement. The controllers assumed that each agent had access to their own states and the states of their communicating neighbors. An improvement to the earlier work was presented in [6] which examined both the optimal selection of the consensus weights and also the adaptation of the consensus weights. The minimization of the communication exchange imposed by the implementation of the consensus protocol was solved in [7] which considered boundary control, unknown disturbances and measurements. The synchronization controllers replaced the full states by the scalar outputs in the consensus protocols. The adaptation of the consensus weights and of the unknown disturbance was realized via the use of available signals (output signals). Additionally consensus protocols were also implemented in the adaptation laws of the consensus weights which resulted in an improved performance. A generalization of the framework in [7] to a class of positive real infinite dimensional system was presented in [8] which considered both the optimization and adaptation of the consensus weights. At the same time, a robust control approach was used in [9] by using the signum functions in the boundary controllers of identical diffusion PDEs with boundary local interactions, as a means to account for boundary perturbations.

Ways to enhance the performance of synchronization controllers included optimization and adaptation of the synchronization gains. Additional improvements were the inclusion of robust laws in the adaptive laws for the consensus weights and the inclusion of the parameter consensus in the adaptive laws for the consensus weights. Reduction of the cost of the synchronization controllers included the exchange of output signals (usually scalars) instead of full state signals and the optimization of the communication topology. The communication topology optimization for a related system of a single PDE with multiple collocated actuator-sensor pairs was examined in [10].

However, one aspect not considered in the performance improvement of the synchronization of networked PDE systems is the *type* of information exchanged. The consensus protocols include pairwise differences of the measurement signals; more useful information can be conveyed by spatial gradients of the pairwise differences or by spatial integrals of the pairwise differences. These spatial modifications mimic the time-analog in the PID-type controllers where the controller is proportional to the error, to the accumulated errors represented by the integral terms and the anticipatory action of the derivative component of the PID controller.

This paper is concerned with the various types of synchronization controllers and presents the *spatial PD coupling* that is used in the consensus protocols. Providing an additional improvement in the synchronization control of networked finite dimensional systems, it also proposes a PID coupling for the consensus protocol for finite dimensional systems.

Section II motivates the proposed functional form of the consensus protocol and provides conditions for which such a protocol is realizable using only available signals. Realizable PD-type couplings for PDE synchronization are presented

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in Section III and the abstraction to a class of positive-real infinite dimensional systems is given in Section IV. The special class of finite dimensional positive real systems with a temporal PID coupling is summarized in Section V and numerical studies for both PDE and ODE systems are presented in Section VI. Conclusions follow in Section VII.

II. Problem motivation

To put the proposed concept of *spatial PID coupling in the synchronization of PDEs* into perspective, we consider the dual framework for the problem presented in [11]. There, the goal was to ensure that N filters trying to reconstruct the state of a parabolic PDE using output measurements, were able to exchange valuable information that enhanced consensus. The idea behind this was *not only* to exchange information on the estimated states between the networked filters, *but also* exchange the spatial gradient of the estimated states between the networked filters. The former information exchange constituted the *proportional* coupling of the filters whereas the latter provided the *derivative* coupling. An added feature of these consensus filters was the integral coupling whereby each networked filter exchanged the information of its spatial integral (spatial average) with its communicating neighbors. That provided the *integral* consensus coupling of the distributed filters and hence the definition of spatial PID coupling in the consensus filters considered in [11].

For demonstration purposes, we assume that we have N PDE systems that are tasked with following a *virtual leader* (another PDE) and ensure synchronization.

The virtual leader is described by the parabolic PDE

$$\partial_t x_m(t,\xi) = \mathcal{A}_m x_m(t,\xi) + b(\xi) r(t), \quad 0<\xi<\ell,\ t\in\mathbb{R}^+, \tag{1}$$

where for simplicity Dirichlet boundary conditions are assumed with $x_m(t,0)=x_m(t,\ell)=0$ and with initial conditions $x_m(0,\xi)=x_0(\xi)$, $\xi\in[0,\ell]$. The symmetric spatial operator $\mathcal{A}_m$ is assumed to be the generator of an exponentially stable C_0 semigroup on $L^2(0,\ell)$. For example, it can be given by the elliptic operator

$$\mathcal{A}_m\varphi \triangleq \partial_\xi(\alpha(\xi)\partial_\xi\varphi(\xi)) + \beta(\xi)\partial_\xi\varphi(\xi) - \gamma(\xi)\varphi(\xi),$$

for $\varphi\in H_0^1(0,\ell)$. The function $b(\xi)$ denotes the distribution of the actuating device and $r(t)$ is a reference signal. The spatial functions $\alpha(\xi),\beta(\xi),\gamma(\xi)$ satisfy certain regularity conditions necessary for the unforced system to be well-posed, [12].

The N networked systems are governed by

$$\begin{aligned}&\partial_t x_i(t,\xi) = \mathcal{A} x_i(t,\xi) + b(\xi)u_i(t),\\ &x_i(t,0)=x_i(t,\ell)=0,\ \ x_i(0,\xi)=x_{0i}(\xi)\neq x_0(\xi),\end{aligned} \tag{2}$$

where $u_i(t)$, $i=1,\ldots,N$ denotes the control signal to each system. The spatial operator $\mathcal{A}$ is similar to $\mathcal{A}_m$ but it may not have the same stability properties. To differentiate each of the N networked systems, it is assumed that the initial conditions $x_i(0,\xi)$ may not be identical to each other and not equal to the initial condition $x_0(\xi)$ of the virtual leader.

The goal here is to design controllers $u_i(t)$ for the networked systems in (2) so that (i) each agent $x_i(t,\xi)$ tracks the virtual leader $x_m(t,\xi)$ *and* (ii) each agent agrees with all other agents. Similar to the approach in [8], each controller will consist of two parts in order to address each of the two goals. Defining $x_{ij}(t,\xi)$ be the pairwise differences $x_{ij}(t,\xi)\triangleq x_i(t,\xi)-x_j(t,\xi)$, $i,j=1,\ldots,N$ and the estimation errors $e_i(t,\xi)\triangleq x_i(t,\xi)-x_m(t,\xi)$, $i=1,\ldots,N$, the control objectives are expressed in terms of the norm convergence $\lim_{t\to\infty}|x_{ij}(t,\xi)|_{L^2(0,\ell)}=0$, $i,j=1,\ldots,N$, for *synchronization*, and $\lim_{t\to\infty}|e_i(t,\xi)|_{L^2(0,\ell)}=0$, $i=1,\ldots,N$, for *virtual leader tracking*. The information exchange between the networked systems is described via an appropriate communication topology. An undirected simple graph $G=(\mathcal{V},\mathcal{E})$ is assumed here to describe this communication topology. The nodes of the graph denoted by $\mathcal{V}=\{1,2,\ldots,N\}$ represent the networked PDE systems in (2) with the graph edges $\mathcal{E}\subset\mathcal{V}\times\mathcal{V}$ representing the communication links between them. We denote the set of networked systems that the i^{th} system is communicating with via $\mathbb{N}_i=\{j:(i,j)\in\mathcal{E}\}$.

One is tempted to adopt the consensus PID coupling used in [11] for distributed filters to the synchronization control of the N systems in (2). Following the use of spatial PID consensus in [11], and using duality for the case of synchronization controllers here, the proposed synchronization controllers take the form

$$\begin{aligned}u_i(t) = u_i^t(t)+u_i^s(t) &= -\int_0^\ell \kappa(\xi)x_i(t,\xi)\,\mathrm{d}\xi + r(t)\\ &-\pi(\xi)\sum_{j\in\mathbb{N}_i}^N x_{ij}(t,\xi) - \iota(\xi)\sum_{j\in\mathbb{N}_i}^N\int_0^\ell \iota(\psi)x_{ij}(t,\psi)\,\mathrm{d}\psi\\ &-\delta(\xi)\sum_{j\in\mathbb{N}_i}^N \partial_\xi\Big(\delta(\xi)x_{ij}(t,\xi)\Big),\end{aligned} \tag{3}$$

with

$$u_i^t(t) = -\int_0^\ell \kappa(\xi)x_i(t,\xi)\,\mathrm{d}\xi + r(t),$$

where $\kappa(\xi)$ is the feedback gain kernel needed to ensure that the closed-loop systems can match the state dynamics of the virtual leader. The two components of the control signals u_i^t and u_i^s address the tracking and synchronization objectives, respectively. The proportional, integral and derivative coupling functions which form the synchronization component of the controllers, satisfy $0<\pi_l\le\pi(\xi)\le\pi_u$, $0<\iota_l\le\iota(\xi)\le\iota_u$ and $0\le\delta(\xi)\le\delta_u$, for $0\le\xi\le\ell$.

To achieve the leader-following goal, one must impose a *matching condition* which assumes the existence of a feedback kernel $\kappa(\xi)$ with the property that

$$\mathcal{A}_i\varphi - b(\xi)\int_0^\ell \kappa(\xi)\varphi(\xi)\,\mathrm{d}\xi = \mathcal{A}_m\varphi, \quad \varphi\in H_0^1(0,\ell). \tag{4}$$

It is easily seen that the first part of the control signals

$$u_i^t(t) = -\int_0^\ell \kappa(\xi)x_i(t,\xi)\,\mathrm{d}\xi + r(t), \quad i=1,\ldots,N, \tag{5}$$

ensures tracking since each closed-loop system is given by

$$\begin{aligned}&\partial_t x_i(t,\xi) = \mathcal{A}x_i(t,\xi) - b(\xi)\int_0^\ell \kappa(\xi)x_i(t,\xi)\,\mathrm{d}\xi + b(\xi)r(t)\\ &x_i(t,0)=x_i(t,\ell)=0,\ \ x_i(0,\xi)=x_{0i}(\xi)\neq x_0(\xi).\end{aligned} \tag{6}$$

Using the definition of the state error and (1), (6) with the matching condition (4), the state errors are governed by

$$\partial_t e_i(t,\xi) = \mathcal{A}_m e_i(t,\xi),$$
$$e_i(t,0) = e_i(t,\ell) = 0,\ \ e_i(0,\xi) = x_{0i}(\xi) - x_0(\xi), \tag{7}$$

It can be established that the tracking objective with controller (5) and matching condition (4) is achieved since the operator $\mathcal{A}_m$ in (7) generates an exponentially stable C_0 semigroup. This of course requires the availability of each infinite dimensional state $x_i(t,\xi)$ in order for (5) to be realized. Adding the synchronization component u_i^s requires more conditions.

Using the matching condition (4) along with equations (1), (2) and the spatial PID synchronization control signals (3), the resulting closed-loop errors are now given by

$$\begin{aligned}\partial_t e_i(t,\xi) = \mathcal{A}_m e_i(t,\xi) - b(\xi)\pi(\xi)\sum_{j\in\mathbb{N}_i} e_{ij}(t,\xi)\\ - b(\xi)\iota(\xi)\sum_{j\in\mathbb{N}_i}\int_0^\ell \iota(\psi)e_{ij}(t,\psi)\,\mathrm{d}\psi\\ - b(\xi)\delta(\xi)\sum_{j\in\mathbb{N}_i}\partial_\xi\Big(\delta(\xi)e_{ij}(t,\xi)\Big)\\ e_i(t,0) = e_i(t,\ell) = 0,\ \ e_i(0,\xi) = x_{0i}(\xi) - x_0(\xi).\end{aligned} \tag{8}$$

The additional terms in (8) must be shown to achieve the synchronization objective via the convergence of the pairwise difference $|x_{ij}(t)|_{L^2}$ to zero. However, such controllers pose several challenges as described below.

Realization and implementation challenges for the synchronization controllers (3).

The controllers in (3) require:

1) each agent to access its own state $x_i(t,\xi)$ in order to realize the control component $\int_0^\ell \kappa(\xi)x_i(t,\xi)\,\mathrm{d}\xi$ in (5).
2) each agent to access the states of its communicating neighbors in order to realize the synchronization components that use the pairwise errors $x_{ij}(t,\xi)$, their spatial gradients and their spatial integrals in (8).

The above impose a heavy communication load since the j^{th} agent with $j\in\mathbb{N}_i$ must transmit its own state $x_j(t,\xi)$ to the i^{th} controller. Additionally, since the control signals u_i are assumed scalars, then the synchronization components must be modified. First, the term $-\pi(\xi)\sum_{j\in\mathbb{N}_i} x_{ij}(t,\xi)$ must be modified to result in scalar signal since the input operator associated with the function $b(\xi)$ must take an element in the space $H_0^1(0,\ell)$ and map it to $\mathbb{R}^1$. This can easily be managed by obtaining a scalar signal and thus the above term must be modified to $-\sum_{j\in\mathbb{N}_i}^{N}\int_0^\ell \pi(\xi)x_{ij}(t,\xi)\,\mathrm{d}\xi$. Similarly, the second term $-\sum_{j\in\mathbb{N}_i}\iota(\xi)\int_0^\ell \iota(\xi)x_{ij}(t,\xi)\,\mathrm{d}\xi$ must be simplified to $-\sum_{j\in\mathbb{N}_i}\int_0^\ell \iota(\xi)x_{ij}(t,\xi)\,\mathrm{d}\xi$ to also produce a scalar signal, which then becomes identical to the previous one. The last term $-\sum_{j\in\mathbb{N}_i}\delta(\xi)\partial_\xi\big(\delta(\xi)x_{ij}(t,\xi)\big)$, must be modified to $-\sum_{j\in\mathbb{N}_i}\int_0^\ell \delta(\xi)\partial_\xi\big(x_{ij}(t,\xi)\big)\,\mathrm{d}\xi$ in order to produce a scalar signal. Thus, the spatial PID consensus coupling terms used in [11] and proposed in (3) cannot be used in the synchronization control as presented due to the structural constraints imposed by the input operator. The above three modifications result in essentially two distinct components, namely a P-type and a D-type coupling that involve finite dimensional signals. Still, they require the availability of the differences $x_{ij}(t,\xi)$ and their spatial gradients. These will be detailed in the next section.

III. Realizable PD couplings in synchronization controllers

Each of the N networked systems is assumed to obtain process information that is provided by N sensors. Thus, each networked system (2) has an output

$$y_i(t) = \int_0^\ell c(\xi)x_i(t,\xi)\,\mathrm{d}\xi,\ \ i = 1,\ldots,N. \tag{9}$$

In a similar fashion to the state error definition, we define the *output errors* as $\varepsilon_i(t) = y_i(t) - y_m(t)$, $i = 1,\ldots,N$ where

$$y_m(t) = \int_0^\ell c(\xi)x_m(t,\xi)\,\mathrm{d}\xi.$$

The goal is to generate synchronizing controllers, similar to (3), but with the property they can be realized and minimize the communication exchange between the N systems.

To realize implementable synchronizing controllers that impose minimal communications costs, one must make some structural assumptions. These take the form of a matching condition which affords a static output feedback controller to yield a closed-loop operator matching $\mathcal{A}_m$.

Assumption 1 (matching condition): There exists a static gain $g>0$ such that

$$\mathcal{A}\phi - b(\xi)g\int_0^\ell c(\xi)\phi(\xi)\,\mathrm{d}\xi = \mathcal{A}_m\phi,\ \ \phi\in H_0^1(0,\ell). \tag{10}$$

The above assumption requires that each networked system is statically stabilizable. With a simple static output feedback each networked system can match the "closed-loop" operator $\mathcal{A}_m$ of the virtual leader by the feedback $u_i(t) = -gy_i(t)$. Using (9) and (10) it is easily observed that the realization of a feedback controller restricts the feedback kernel to the specific form $\kappa(\xi) = gc(\xi)$ with

$$\int_0^\ell \kappa(\xi)x_i(t,\xi)\,\mathrm{d}\xi = g\int_0^\ell c(\xi)x_i(t,\xi)\,\mathrm{d}\xi = gy_i(t).$$

Assumption 2 (dissipativeness): For all $\phi,\psi\in H_0^1(0,\ell)$, the spatial operator $\mathcal{A}_m$ satisfies the inequality

$$\int_0^\ell \phi(\xi)\left(\mathcal{A}_m\psi(\xi)\right) + \left(\mathcal{A}_m\phi(\xi)\right)\psi(\xi)\,d\xi \le -\lambda_m\int_0^\ell \phi(\xi)\psi(\xi)\,d\xi.$$

Since only a spatial PD-type coupling can be realized for this class of systems, one must make an assumption on the spatial gradient measurements at the sensor location.

Assumption 3 (gradient measurements): The N sensing devices can also provide spatial gradient measurements

$$z_i(t) = \int_0^\ell c(\xi)\tfrac{\partial x_i(t,\xi)}{\partial\xi}\,\mathrm{d}\xi,\quad i = 1,\ldots,N. \tag{11}$$

Now that $y_i(t)$ and $z_i(t)$ are available for the realization of the synchronization components, we can propose a realizable controller with minimal communication loads. The counterpart of (3) will not include fixed gains to be optimized, but rather time-varying gains that can be adjusted *adaptively*. The proposed synchronization controllers are

$$u_i(t) = -gy_i(t) - r(t) - \sum_{j\in\mathbb{N}_i}\left(\pi_{ij}(t)y_{ij}(t) + \delta_{ij}(t)z_{ij}(t)\right), \tag{12}$$

for $i=1,\dots,N$, where the differences of the gradient measurements are defined in analogous manner as $z_{ij}(t)=z_i(t)-z_j(t)$. The edge-dependent Proportional and Derivative *synchronization gains* $\pi_{ij}(t)$ and $\delta_{ij}(t)$, $i=1,\dots,N$, $j\in\mathbb{N}_i$ are adjusted adaptively and vary their magnitude in proportion to the agreement of the differences y_{ij} and z_{ij}, respectively.

Using equations (1), (2) and (12), the closed-loop errors are now governed by (*cf.* (8))

$$\begin{aligned}&\dot{e}_i(t,\xi)=\mathcal{A}_m e_i(t,\xi)-b(\xi)\sum_{j\in\mathbb{N}_i}\left(\pi_{ij}(t)y_{ij}(t)+\delta_{ij}(t)z_{ij}(t)\right)\\&e_i(t,0)=e_i(t,\ell)=0,\;\; e_i(0,\xi)=x_{0i}(\xi)-x_0(\xi).\end{aligned}\tag{13}$$

One must now examine the well-posedness of the above error system and obtain the adaptive laws for the synchronization gains $\pi_{ij}(t),\delta_{ij}(t)$. For the latter, one considers the Lyapunov-like functional

$$V_i=\int_0^\ell e_i^2(t,\xi)\,\mathrm{d}\xi+\sum_{j\in\mathbb{N}_i}\left(\frac{\pi_{ij}^2}{\gamma_i^P}+\frac{\delta_{ij}^2}{\gamma_i^D}\right),\quad i=1,\dots,N,\tag{14}$$

where γ_i^P and γ_i^D are *node-dependent adaptive gains*, [13]. Taking the derivative of (14) along (13) yields

$$\begin{aligned}\dot{V}_i\;&=\;\int_0^\ell e_i(t,\xi)\dot{e}_i(t,\xi)\,\mathrm{d}\xi+2\sum_{j\in\mathbb{N}_i}\left(\frac{\pi_{ij}\dot{\pi}_{ij}}{\gamma_i^P}+\frac{\delta_{ij}\dot{\delta}_{ij}}{\gamma_i^D}\right)\\&=\;\int_0^\ell e_i(t,\xi)\mathcal{A}_m e_i(t,\xi)\,\mathrm{d}\xi+\int_0^\ell \mathcal{A}_m e_i(t,\xi)e_i(t,\xi)\,\mathrm{d}\xi\\&\quad-2\int_0^\ell b(\xi)e_i(t,\xi)\,\mathrm{d}\xi\sum_{j\in\mathbb{N}_i}\left(\pi_{ij}(t)y_{ij}(t)+\delta_{ij}(t)z_{ij}(t)\right)\\&\quad+2\sum_{j\in\mathbb{N}_i}\left(\frac{\pi_{ij}\dot{\pi}_{ij}}{\gamma_i^P}+\frac{\delta_{ij}\dot{\delta}_{ij}}{\gamma_i^D}\right).\end{aligned}$$

The terms

$$-2\int_0^\ell b(\xi)e_i(t,\xi)\,\mathrm{d}\xi\sum_{j\in\mathbb{N}_i}\pi_{ij}(t)y_{ij}(t)+2\sum_{j\in\mathbb{N}_i}\frac{\pi_{ij}\dot{\pi}_{ij}}{\gamma_i^P},$$

and

$$-2\int_0^\ell b(\xi)e_i(t,\xi)\,\mathrm{d}\xi\sum_{j\in\mathbb{N}_i}\delta_{ij}(t)z_{ij}(t)+2\sum_{j\in\mathbb{N}_i}\frac{\delta_{ij}\dot{\delta}_{ij}}{\gamma_i^D},$$

can be simplified when

$$\int_0^\ell b(\xi)e_i(t,\xi)\,\mathrm{d}\xi=\varepsilon_i(t),\quad i=1,\dots,N,$$

is assumed, i.e., when $c(\xi)=b(\xi)$. Then the adaptive laws can be selected as

$$\begin{aligned}\dot{\pi}_{ij}(t)&=\gamma_i^P y_{ij}(t)\varepsilon_i(t)-\pi_{ij}(t)\\\dot{\delta}_{ij}(t)&=\gamma_i^D z_{ij}(t)\varepsilon_i(t)-\delta_{ij}(t)\end{aligned}\qquad i=1,\dots,N, j\in\mathbb{N}_i.\tag{15}$$

The resulting Lyapunov derivative becomes

$$\begin{aligned}\dot{V}_i\;&\leq\;-\lambda_m\int_0^\ell e_i^2(t,\xi)\,\mathrm{d}\xi-2\sum_{j\in\mathbb{N}_i}\frac{\pi_{ij}^2}{\gamma_i^P}-2\sum_{j\in\mathbb{N}_i}\frac{\delta_{ij}^2}{\gamma_i^D}\\&\leq\;-\min\{\lambda_m,2,2\}V_i.\end{aligned}$$

The realization of the adaptive laws (15) relied on the collocation assumption which is stated below.

Assumption 4 (collocated input and output): The control and observations are assumed collocated, which for the current class of PDEs translates to $b(\xi)=c(\xi)$.

Summing over all agents, the collective Lyapunov derivative produces

$$\sum_{i=1}^N\dot{V}_i\leq-\min\{\lambda_m,2,2\}\sum_{i=1}^N V_i.$$

The convergence of e_i to zero in the appropriate norm easily follows. Similarly, due to the architecture of the adaptive laws in (15), the edge-dependent gains converge to zero.

IV. Abstraction to Special Class of Positive Real Infinite Dimensional Systems

In general, to extract adaptive laws for parameters in estimation and control of dynamical systems using Lyapunov-redesign methods, one must require the nominal system be strictly positive real, [14], [15]. This stems from the need to extract the adaptive laws and express them in terms of available signals. The coupling of the input and output operators when they collocated represents the simplest case. However, a more general class of systems relates the input and output operators via the solution to an associated operator Lyapunov equation. A rich class of positive real infinite dimensional systems are the collocated partial differential equations with dissipative operator. Such a class of infinite dimensional systems allows for unbounded input and output operators.

In order to include more PDEs beyond (1) and (2), we express PDEs like (1) as evolution equations in an appropriate Hilbert space. We consider the usual Gelfand space triple with the pivot space H a Hilbert space and the reflexive Banach spaces V that is continuously and densely embedded in H. The conjugate dual of V is denoted by V^* and we have $V\hookrightarrow H\hookrightarrow V^*$ with the embeddings dense and continuous. The class of systems that we study are written as

$$\dot{x}_i(t)=\mathcal{A}_i x_i(t)+\mathcal{B}u_i(t),\quad y_i(t)=\mathcal{B}^*x_i(t),\tag{16}$$

where the state operator $\mathcal{A}\in\mathcal{L}(V,V^*)$ and the input operator $\mathcal{B}\in\mathcal{L}(\mathbb{R}^1,V^*)$. One can easily see that (3) can fit the above framework with

$$\langle\mathcal{B}u,\phi\rangle=\int_0^\ell b(\xi)\phi(\xi)\,\mathrm{d}\xi\, u(t),\quad \phi\in V=H_0^1(0,\ell).$$

Similarly, the abstraction of (1) is

$$\dot{x}_m(t)=\mathcal{A}_m x_m(t)+\mathcal{B}r(t),\quad y_m(t)=\mathcal{B}^*x_m(t),\tag{17}$$

The matching condition takes the form of imposing the existence of a gain $g>0$ such that (*cf.* (10))

$$\mathcal{A}\phi-\mathcal{B}g\mathcal{B}^*\phi=\mathcal{A}_m\phi\tag{18}$$

with $D(\mathcal{A})=D(\mathcal{A}_m)$.

The associated error equations are described by the following evolution equations

$$\begin{aligned}&\dot{e}_i(t)=\mathcal{A}_m e_i(t)-\mathcal{B}\sum_{j\in\mathbb{N}_i}\pi_{ij}(t)\left(y_i(t)-y_j(t)\right)\\&-\mathcal{B}\sum_{j\in\mathbb{N}_i}\delta_{ij}(t)\left(z_i(t)-z_j(t)\right),\quad e_i(0)=x_{0i}-x_0.\end{aligned}\tag{19}$$

Lemma 1: Consider the infinite dimensional systems (16) with collocated unbounded input and output operators and with the leader system given by (17). Assume that the virtual leader operator satisfies the dissipativity identity $\mathcal{A}_m+\mathcal{A}_m^*<-\mu I$ with $r\in L^2(0,\infty;\mathbb{R}^1)$, then the adaptive laws of the synchronization gains are given by

$$\begin{aligned}\dot{\pi}_{ij}(t) &= \gamma_i^P \varepsilon_i(t) y_{ij}(t) - \pi_{ij}(t),\\ \dot{\delta}_{ij}(t) &= \gamma_i^D \varepsilon_i(t) z_{ij}(t) - \delta_{ij}(t).\end{aligned} \tag{20}$$

The proposed controllers given by

$$u_i = -g y_i + r - \sum_{j\in\mathbb{N}_i} \pi_{ij} y_{ij} + \delta_{ij} z_{ij}$$

result in well-posed systems with error dynamics (19) and

$$\lim_{t\to\infty} |e_i(t)| = 0, \quad \lim_{t\to\infty} |x_i(t) - x_j(t)| = 0, \quad \forall i,j = 1,\dots,N.$$

Proof: The extraction of adaptive laws is made possible via Lyapunov-redesign methods. Consider the functionals

$$V_i = \langle e_i, e_i\rangle_H + \sum_{j\in\mathbb{N}_i} \left(\frac{\pi_{ij}^2}{\gamma_i^P} + \frac{\delta_{ij}^2}{\gamma_i^D} \right), \; i = 1,\dots,N. \tag{21}$$

Taking the time derivative along the error dynamics, we have

$$\begin{aligned}\dot{V}_i &= \langle e_i, \mathcal{A}_m e_i\rangle + \langle \mathcal{A}_m e_i, e_i\rangle - 2\langle e_i, \mathcal{B} \sum_{j\in\mathbb{N}_i} \pi_i(t) y_{ij}\rangle\\ &\quad -2\langle e_i, \mathcal{B}\sum_{j\in\mathbb{N}_i} \delta_{ij}(t) z_i\rangle + 2\sum_{j\in\mathbb{N}_i} \frac{\pi_i \dot{\pi}_{ij}}{\gamma_i^P} + \frac{\delta_{ij}\dot{\delta}_{ij}}{\gamma_i^D}\\ &= \langle e_i, (\mathcal{A}_m + \mathcal{A}_m^*) e_i\rangle - 2\sum_{j\in\mathbb{N}_i} \frac{\pi_{ij}^2}{\gamma_i^P} - 2\sum_{j\in\mathbb{N}_i} \frac{\delta_{ij}^2}{\gamma_i^D}\\ &\leq -\min\{\lambda_m, 2\} V_i\end{aligned} \tag{22}$$

The convergence of the pairwise differences and of the state errors e_i to zero in the H norms easily follows. The error systems (19) along with the leader dynamics in (17) and the adaptive laws (20) result in a well-posed aggregate system. The aggregate system falls into the general abstract system given in [16], see also [17]. ■

V. Case of finite dimensional SPR systems

The finite dimensional systems are described by

$$\dot{x}_i(t) = A x_i(t) + B u_i(t), \;\; y_i(t) = C x_i(t), \;\; x_i(0) = x_{i0}, \tag{23}$$

with $x_i \in \mathbb{R}^n$. The three spaces collapse into $V = H = V^* = \mathbb{R}^n$ and the system has $q \geq 1$ inputs and outputs with B an $n\times q$ and C a $q\times n$ matrices. The virtual leader is given by

$$\dot{x}_m(t) = A_m x_m(t) + B r(t), \;\; y_m(t) = C x_m(t), \;\; x_m(0) = x_0 \tag{24}$$

where the matrix triple (A_m, B, C) satisfies Lur'e equations

$$A_m^T P + P A_m = -Q, \quad B^T P = C.$$

The collocated assumption is removed for the more general case of $B^T P = C$ which produces a strictly positive real transfer function from r to y_m. The condition of static stabilizability is also assumed and which requires the existence of a $q\times q$ gain matrix G such that $A - BGC = A_m$.

In this case, one can consider a *temporal PID* coupling for synchronization. The counterpart of (3), which implements a temporal PID coupling using available output signals is

$$\begin{aligned}u_i(t) &= -G y_i(t) + r(t) - \sum_{j\in\mathbb{N}_i} \pi_{ij}(t) y_{ij}(t)\\ &\quad - \sum_{j\in\mathbb{N}_i} \iota_{ij}(t) \int_0^t y_{ij}(\tau)\,d\tau - \sum_{j\in\mathbb{N}_i} \delta_{ij}(t) \dot{y}_{ij}(t),\end{aligned} \tag{25}$$

and which results in the closed-loop networked systems

$$\begin{aligned}\dot{x}_i(t) &= A_m x_i(t) + B r(t) - B\sum_{j\in\mathbb{N}_i} \pi_{ij}(t) y_{ij}(t)\\ &\quad -B\sum_{j\in\mathbb{N}_i} \iota_{ij}(t) \int_0^t y_{ij}(\tau)\,d\tau - B\sum_{j\in\mathbb{N}_i} \delta_{ij}(t)\dot{y}_{ij}(t),\\ x_i(0) &= x_{i0}.\end{aligned} \tag{26}$$

Using the identity for the output errors

$$\varepsilon_{ij} = \varepsilon_i - \varepsilon_j = (y_i - y_m) + (y_m - y_j) = C e_{ij} = y_{ij},$$

then the state errors $e_i = x_i - x_m$ are governed by

$$\begin{aligned}\dot{e}_i(t) &= A_m e_i - B\sum_{j\in\mathbb{N}_i} \pi_{ij}(t) \varepsilon_{ij}(t)\\ &\quad -B\sum_{j\in\mathbb{N}_i} \iota_{ij}(t) \int_0^t \varepsilon_{ij}(\tau)\,d\tau - B\sum_{j\in\mathbb{N}_i} \delta_{ij}(t)\dot{\varepsilon}_{ij}(t).\end{aligned} \tag{27}$$

To analyze the stability of (27) and extract the adaptive laws, we consider the following Lyapunov-like function

$$V_i = e_i^T P e_i + \sum_{j\in\mathbb{N}_i} \left(\tfrac{1}{\gamma_i^P} \pi_{ij}^2(t) + \tfrac{1}{\gamma_i^I} \iota_{ij}^2(t) + \tfrac{1}{\gamma_i^D} \delta_{ij}^2(t) \right), \tag{28}$$

where $\gamma_i^P, \gamma_i^I, \gamma_i^D$ are the node-dependent adaptive gains.

Lemma 2: Given the finite dimensional networked systems (23) with the SPR virtual leader (24) and the matching condition satisfied, then the proposed PID synchronization controllers with adaptations

$$\begin{aligned}\dot{\pi}_{ij}(t) &= \gamma_i^P \varepsilon_i^T(t) \varepsilon_{ij}(t) - \pi_{ij}(t),\\ \dot{\iota}_{ij}(t) &= \gamma_i^I \varepsilon_i^T(t) \int_0^t \varepsilon_{ij}(\tau)\,d\tau - \iota_{ij}(t), \qquad i = 1,\dots,N, \; j \in \mathbb{N}_i,\\ \dot{\delta}_{ij}(t) &= \gamma_i^D \varepsilon_i^T(t) \dot{\varepsilon}_{ij}(t)(t) - \delta_{ij}(t),\end{aligned} \tag{29}$$

produce stable closed-loop systems with $\lim_{t\to\infty} |e_i| = 0$, and the adaptive gains converging to zero asymptotically.

Proof: The derivative of the Lyapunov functions (28) along the trajectories of the error systems (27) becomes

$$\begin{aligned}\dot{V}_i &= e_i^T P \dot{e}_i + \dot{e}_i^T P e_i + 2\sum_{j\in\mathbb{N}_i} \left(\tfrac{\pi_{ij}\dot{\pi}_{ij}}{\gamma_i^P} + \tfrac{\iota_{ij}\dot{\iota}_{ij}}{\gamma_i^I} + \tfrac{\delta_{ij}\dot{\delta}_{ij}}{\gamma_i^D} \right)\\ &= e_i^T \left(P A_m + A_m^T P \right) e_i - 2 e_i^T P B \sum_{j\in\mathbb{N}_i} \pi_{ij}\varepsilon_{ij}\\ &\quad -2 e_i^T P B \sum_{j\in\mathbb{N}_i} \iota_{ij} \int_0^t \varepsilon_{ij}(\tau)\,d\tau - 2 e_i^T P B \sum_{j\in\mathbb{N}_i} \delta_{ij}\dot{\varepsilon}_{ij}\\ &\quad +2\sum_{j\in\mathbb{N}_i} \left(\tfrac{\pi_{ij}\dot{\pi}_{ij}}{\gamma_i^P} + \tfrac{\iota_{ij}\dot{\iota}_{ij}}{\gamma_i^I} + \tfrac{\delta_{ij}\dot{\delta}_{ij}}{\gamma_i^D} \right)\\ &= -e_i^T Q e_i + 2\sum_{j\in\mathbb{N}_i} \pi_{ij} \left(\tfrac{\dot{\pi}_{ij}}{\gamma_i^P} - \varepsilon_i^T \varepsilon_{ij} \right)\\ &\quad +2\sum_{j\in\mathbb{N}_i} \iota_{ij} \left(\tfrac{\dot{\iota}_{ij}}{\gamma_i^I} - \varepsilon_i^T \int_0^t \varepsilon_{ij}(\tau)\,d\tau \right) + 2\sum_{j\in\mathbb{N}_i} \delta_{ij} \left(\tfrac{\dot{\delta}_{ij}}{\gamma_i^D} - \varepsilon_i^T \dot{\varepsilon}_{ij} \right)\end{aligned}$$

Substitution of the adaptive laws results in

$$\begin{aligned}\dot{V}_i &\leq -\tfrac{\lambda_{min}(Q)}{\lambda_{max}(P)} e_i^T P e_i - 2\sum_{j\in\mathbb{N}_i} \left(\tfrac{\pi_{ij}^2}{\gamma_i^P} + \tfrac{\iota_{ij}^2}{\gamma_i^I} + \tfrac{\delta_{ij}^2}{\gamma_i^D} \right)\\ &\leq -\min\left\{ \tfrac{\lambda_{min}(Q)}{\lambda_{max}(P)}, 2 \right\} V_i\end{aligned}$$

Collectively, the Lyapunov functionals satisfy

$$\sum_{i=1}^N \dot{V}_i \leq -\min\left\{ \tfrac{\lambda_{min}(Q)}{\lambda_{max}(P)}, 2 \right\} \sum_{i=1}^N V_i.$$

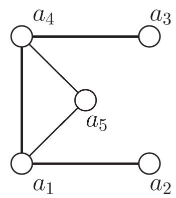

Fig. 1. Communication topology.

Convergence of the estimation errors to zero and of the adaptive gains immediately follows. ■

VI. Numerical Examples

A. Example 1: Infinite dimensional collocated systems

We consider the networked systems examined in [8] and fit the proposed framework of collocated dissipative systems. The PDEs with boundary control and observation are

$$\dot{x}_i(t,\xi) = \alpha x_i''(t,\xi), \quad x_i(0,\xi) = x_{i0}(\xi), \quad \alpha > 0, \; i = 1,\dots,5.$$
$$x_i(t,0) = 0, \quad x_i'(t,1) - kx_i(t,1) = u(t), \; k > 0.$$

The connectivity of the five systems is described by the undirected graph in Figure 1. The five systems have collocated input and output unbounded operator with $\mathcal{B} = \mathcal{C}^* = \delta(\xi - 1)$. The state operator is defined as $\mathcal{A}\phi = \alpha\Delta\phi$ and the physical parameters were $\alpha = 0.1$ and $\ell = 0.1$. The virtual leader operator is $\mathcal{A}_m\phi = \alpha\Delta\phi - 0.1b(\xi)\int_\Omega c(\xi)\phi(\xi)\,\mathrm{d}\xi$. Assuming for simplicity $r(t) = 0$ in (1) along with $x_m(0,\xi) = 0$, it reduces the tracking problem into a regulation problem and thus requiring only the synchronization of the five systems with a prescribed regulation dictated by the spectrum of $\mathcal{A}_m$.

To examine the effects of the spatial PD synchronizing controller in enhancing regulation, we consider the spatial distribution of the mean state $x_{mean}(t,\xi) = \sum_{i=1}^{5} x_i(t,\xi)$ at the instances $t = 2$ and $t = 4$s. Figure 2 depicts the mean state and it is easily observed that the absence of any form of synchronization coupling lacks in performance. The adaptive PD, the adaptive proportional and the adaptive derivative synchronization coupling shows significant improvement over the case of no synchronization coupling in the control signal. Table I summarizes the same results for $|E(t)|_H^2 = \sum_{i=1}^{5} |x_i(t,\xi) - x_m(t,\xi)|_H^2$ and $|\Delta(t)|_H^2 = \sum_{i=1}^{N} |x_i(t,\xi) - x_{mean}(t,\xi)|_H^2$. The positive effects of adaptive weights on both $|E(t)|_H$ and $|\Delta(t)|_H$ are immediately seen.

gains	$\int_0^\infty \lvert E(t)\rvert^2\,\mathrm{d}t$	$\int_0^\infty \lvert\Delta(t)\rvert^2\,\mathrm{d}t$	$\int_0^\infty \lvert E(t)\rvert^2 + \lvert\Delta(t)\rvert^2\,\mathrm{d}t$
α_{ij}, δ_{ij}	52.81	34.18	86.99
α_{ij}	54.23	35.06	89.29
δ_{ij}	52.81	34.20	87.02
no	79.51	51.24	130.75

TABLE I
CUMULATIVE EFFECT OF GAINS α_{ij} AND δ_{ij} ON $E(t)$ AND $\Delta(t)$.

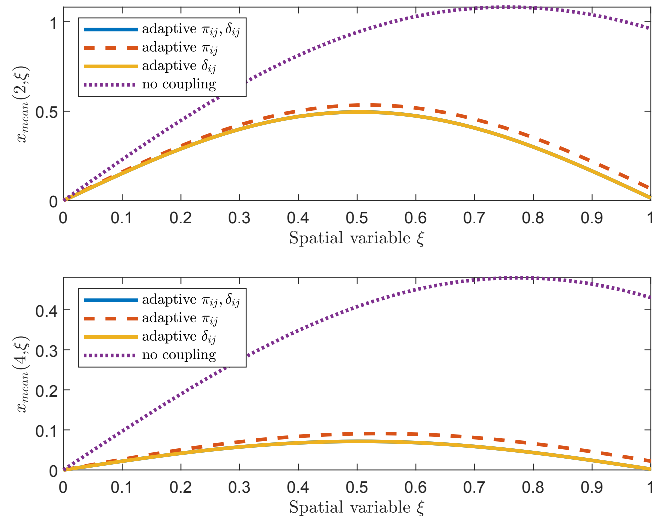

Fig. 2. Spatial distribution of mean state at $t = 2$ and $t = 4$s.

B. Example 2: Finite dimensional SPR systems

We consider the following SPR system

$$\dot{x}_i = \begin{bmatrix} -3.25 & -3.25 \\ 1.00 & 0 \end{bmatrix} x_i + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_i, \; y_i = \begin{bmatrix} 1 & 1 \end{bmatrix} x_i$$

The virtual leader has

$$A_m = \begin{bmatrix} -4 & -4 \\ 1.00 & 0 \end{bmatrix},$$

and with the static gain $G = 0.75$ one has $A_m = A - BGC$. The triple (A_m, B, C) satisfies Lur'e equations with $B^T P = C$

$$P = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \qquad Q = \begin{bmatrix} 7 & 6 \\ 2 & 4 \end{bmatrix}.$$

A total of $N = 5$ agents are used and the same connectivity from Example 1 is used. The laws (29) were used to implement the PID consensus coupling of the 5 systems. To create an initial mismatch of the 5 systems amongst themselves and with $x_m(t)$, the initial conditions were

$$x_1(0) = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \quad x_2(0) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \quad x_3(0) = \begin{bmatrix} 6 \\ -1 \end{bmatrix},$$
$$x_4(0) = \begin{bmatrix} 7 \\ 3 \end{bmatrix}, \quad x_2(0) = \begin{bmatrix} -4 \\ 3 \end{bmatrix}, \quad x_m(0) = \begin{bmatrix} 20 \\ 15 \end{bmatrix}.$$

The reference signal in this case was $r(t) = \sin(10\pi t)$.

The norm of the difference x_m and the mean state $x_{mean}(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t)$ is depicted in Figure 3. This depicts the performance of the mean state from the virtual leader thus representing the virtual leader tracking in the mean. The case of an adaptive PID consensus coupling, an adaptive P consensus coupling and no consensus coupling are depicted. As expected the consensus coupling significantly improves the virtual leader tracking. Assessing synchronization is captured by the aggregate deviation from the mean

$$\begin{bmatrix} x_1(t) - x_{mean}(t) \\ \vdots \\ x_N(t) - x_{mean}(t) \end{bmatrix}$$

and depicted in Figure 4. Both the adaptive PID and adaptive consensus coupling show an improvement over the no consensus coupling case. The $L_2(0,5;\mathbb{R}^2)$ norm for the error from the mean $|x_m(t) - x_{mean}(t)|^2$ plus the aggregate

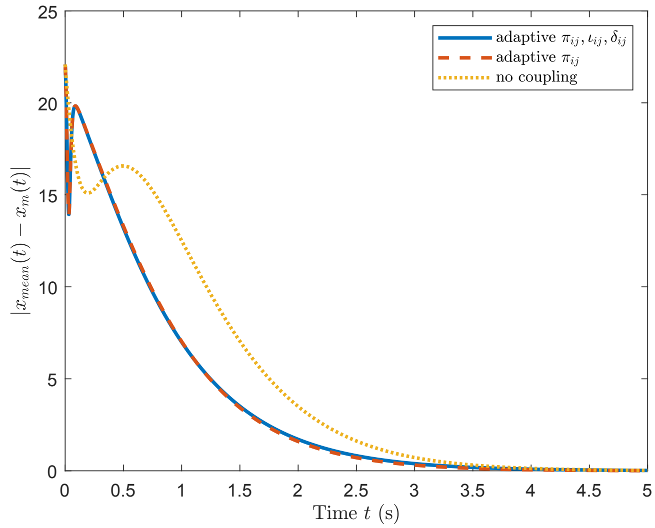

Fig. 3. Evolution of error between virtual leader and mean state of the $N = 5$ networked systems.

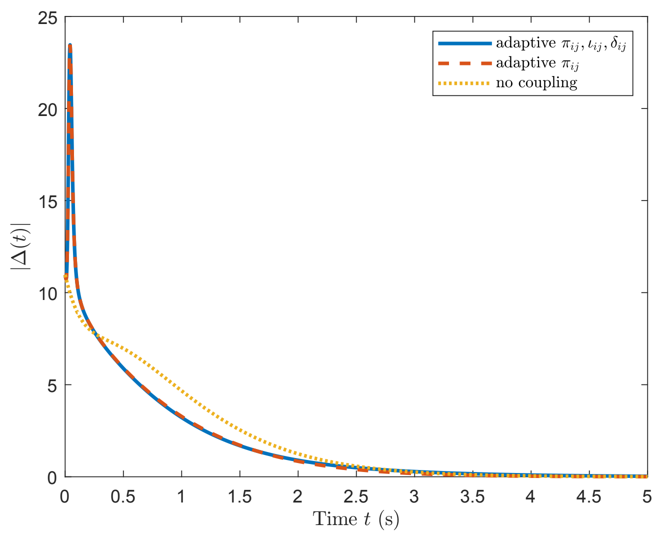

Fig. 4. Evolution of aggregate deviation from the mean of the $N = 5$ networked systems.

deviation from the mean $\sum_{i=1}^{5} |x_i(t) - x_{mean}(t)|^2$ is tabulated in Table II providing similar conclusions to the PDE case.

VII. Conclusions

The use of output signal derivatives, spatial or temporal, when used in the synchronization control of networked PDE systems, provides performance enhancement. For networked PDE systems, a spatial proportional and spatial derivative components of the consensus protocols used for synchronization and leader following, provides spatial synchronization and enhances leader following. For networked finite dimensional systems that are strictly positive real, an added feature of the consensus protocol involves the integral component, thus arriving at a PID-type components in the consensus protocols. In both cases, the synchronization gains were optimized via adaptations which utilized available signals.

	adaptive PID coupling	adaptive P coupling	no coupling
norm	16.5763	16.6163	19.3497

TABLE II
$L_2(0,5;\mathbb{R}^2)$ norm.

Both networked PDE systems with collocated input-output operators and networked ODE strictly positive real systems utilized the spatial PD and temporal PID couplings for synchronization enhancement were presented and demonstrated the beneficial effects of the additional adaptive modifications of the synchronization controllers.

An immediate extension involves the application of higher order spatial couplings in PDEs of first and second order in time form and with spatial operators involving more than two spatial derivatives. Such an extension should involve both in-domain and boundary observation and actuation.

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