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Do 0-GAS-Guaranteeing Impulse Sequences Preserve ISS or iISS Properties? Not Always

Antonio Russo and Shenyu Liu

***Abstract*— Based on the recent work on determination of classes of impulse/switching signals, uniformly over which a switched or impulsive system is input-to-state stable (ISS) or integral input-to-state stable (iISS), it is conjectured that for a switched system with all ISS or iISS subsystems, as long as the impulse/switching signal makes the system 0-input globally asymptotically stable (0-GAS), then it also ensures ISS and iISS, respectively. This work disproves this conjecture by showing examples where the impulse sequence amplifies the input, causing 0-GAS impulsive system to lose the ISS or iISS properties. As a compensation, we provide sufficient conditions on the impulse sequences which guarantee ISS or iISS. It turns out that these conditions are strictly stronger than those ensuring 0-GAS; the differences between them are in fact necessary conditions for the impulsive system to have the asymptotic gain (AG) property or uniformly bounded energy bounded state (UBEBS) property.**

I. Introduction

Impulsive systems combine differential equations describing the continuous flow, with difference equations characterizing the state jumps triggered by an impulse sequence that specifies when the state jumps occur [1]. Similarly, switched systems are characterized by several subsystems and a piece-wise constant signal, called a switching signal, which governs the switching between these subsystems [2]. Many engineering problems are described in terms of impulsive or switched systems such as power electronic systems [3], robotics [4] and automotive [5].

One important research subject for switched/impulsive systems is stability [6], [7]. For instance, one would expect that the ideal internal stability property (e.g., 0-GAS) infers external stability against perturbations or noise (e.g., ISS or iISS). This is true for linear systems, but unfortunately, not true for general continuous-time nonlinear systems, as it is well-known that both ISS and iISS are stronger than 0-GAS [8], [9]. Similar arguments also hold for switched systems [10], impulsive systems [11], and hybrid systems [12], [13]. To rule out such ill behaviors, we assume that all the subsystems of a switched system are ISS, or the continuous dynamics of an impulsive system is ISS. Although it is known that with this additional assumption, the switched system or impulsive system may still not be ISS under arbitrary switching/impulses, it is of interest to investigate whether 0-GAS-guaranteeing switching signals/impulse sequences preserve the system to be ISS or iISS in this case.

The above question stems from our recent discovery in the study of switched/impulsive systems. The pioneering work [14] shows that a switched system with all stable subsystems is GAS uniformly over all switching signals whose average dwell-time (ADT) is strictly larger than a proposed lower bound. Later in [15], it was proved that the same strict inequality condition also guarantees ISS. In the recent work [16], a generalized version of the previous ADT condition is shown to guarantee ISS or iISS for switched systems with inputs even when the growth/decay rates of the subsystems are nonlinear. On the other hand, in the work [17], some switched nonlinear systems are discovered to be neither ISS or iISS no matter how slowly the systems switch. These systems do not satisfy the ISS or iISS properties because their unforced versions are not GAS under slow switching, consistent with the discovery in [18]. Similar results are also obtained for impulsive systems in [19], [20], [21]. In addition, a broad class of impulse sequences is shown to be sufficient for both ISS and uniform 0-GAS for time-varying nonlinear impulsive systems in [22]. All the aforementioned observations give us the impression that if a condition on the switching signal or impulse sequence is sufficient for 0-GAS, then perhaps it is also sufficient for ISS or iISS.

This paper aims to disprove this conjecture, by first providing examples where an impulse sequence makes an impulsive system 0-GAS but not ISS or iISS. Based on the Lyapunov characterizations on the flow and jumps, we provide sufficient conditions on the impulse sequences which guarantee 0-GAS, ISS or iISS. It turns out that the sufficient condition for 0-GAS is strictly weaker than the conditions for ISS or iISS; the differences between them are in fact necessary conditions for the impulsive system to have the AG property or UBEBS property, consistent with the known results in the literature.

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Antonio Russo is with Dipartimento di Ingegneria, Università degli Studi della Campania "L. Vanvitelli", 81031 Aversa, Italy antonio.russo1@unicampania.it. Shenyu Liu is with the School of Automation, Beijing Institute of Technology, 100081, Beijing, China shenyuliu@bit.edu.cn

II. Preliminaries and Definitions

In the following, we denote with $\mathbb{R}$ and $\mathbb{R}_{\geqslant 0}$ the sets of reals and nonnegative reals, respectively. Let $\mathcal{T} := \{t_1, t_2, \dots\}$ be a countable set with $0 < t_1 < t_2 < \cdots$, which defines the set of strictly increasing sequences of *impulse instants*. We call $\mathcal{T}$ satisfying the aforementioned conditions an *impulse sequence*. In this manuscript, we denote with $N(s,t)$ the number of impulse instants t_k in the semi-open interval $(s,t]$ with $t > s \geqslant 0$ and, without loss of generality, we define $t_0 = 0$. A regularity assumption on the impulse sequences, adapted from [11], is provided below.

Definition 1: An impulse sequence $\mathcal{T}$ is said to be *uniformly incrementally bounded* (UIB) if there exists a continuous and non-decreasing function $\phi : \mathbb{R}_{\geqslant 0} \longrightarrow \mathbb{R}_{\geqslant 0}$ so that $N(s,t) \leqslant \phi(t-s)$ for all $t > s \geqslant 0$.

Note that the UIB condition requires $\mathcal{T}$ to be non-Zeno; i.e., either $\mathcal{T}$ is finite, or $\lim_{i\to\infty} t_i = \infty$.

With a given impulse sequence $\mathcal{T}$, the general impulsive system with an input is defined as

$$\begin{cases} \dot{x} = f(x(t), w(t)), & t \notin \mathcal{T} , \\ x(t) = g(x(t^-), w(t)), & t \in \mathcal{T} , \end{cases} \tag{1}$$

where $x(t) : \mathbb{R}_{\geqslant 0} \longrightarrow \mathbb{R}^n$ is the state trajectory, $w(t) : \mathbb{R}_{\geqslant 0} \longrightarrow \mathbb{R}^l$ is the input and it is measurable and locally essentially bounded on $\mathbb{R}_{\geqslant 0}$ and bounded on $\mathcal{T}$. The functions $f, g : \mathbb{R}^n \times \mathbb{R}^l \longrightarrow \mathbb{R}^n$ are such that $f(0,0) = 0$ and $g(0,0) = 0$, with f being locally Lipschitz, g being continuous. Moreover, for each $x \in \mathbb{R}^n$ and $\epsilon \geqslant 0$, the set $\{f(x,w) \colon |w| \leqslant \epsilon\}$ is convex.

Because of the dual role of w as a continuous and discrete perturbation, given an impulse sequence $\mathcal{T}$, we define the supremum norm of w on the interval $[0,t]$ as

$$\|w\|_t^\infty := \max\left\{ \operatorname*{ess\,sup}_{s\in[0,t]} |w(s)|, \sup_{s\in[0,t]\cap\mathcal{T}} |w(s)| \right\} , \tag{2}$$

and we denote $\|w\|_\infty^\infty := \lim_{t\longrightarrow\infty} \|w\|_t^\infty$. Similarly, for any $\rho \in \mathcal{K}_\infty$[1], define

$$\|w\|_t^\rho := \max\left\{ \int_0^t \rho(|w(s)|)\mathrm{d}s, \sum_{s\in[0,t]\cap\mathcal{T}} \rho(|w(s)|) \right\} . \tag{3}$$

In the following, some stability definitions modified from the ones in [8], [21] are given. These stability properties will be investigated in this work.

[1]The following classes of comparison functions are considered [23]: a function $\alpha : \mathbb{R}_{\geqslant 0} \longrightarrow \mathbb{R}_{\geqslant 0}$ is of class $\mathcal{K}$ if it is continuous, $\alpha(0) = 0$ and it is strictly increasing; α is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and satisfies $\lim_{r\to\infty} \alpha(r) = \infty$. A function $\alpha : \mathbb{R}_{\geqslant 0} \longrightarrow \mathbb{R}_{\geqslant 0}$ is of class $\mathcal{L}$ if it is continuous, decreasing and $\lim_{r\to\infty} \alpha(r) = 0$. A function $\beta \colon \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \longrightarrow \mathbb{R}_{\geqslant 0}$ is of class $\mathcal{KL}$ if, for each fixed r, $\beta(\cdot, r)$ is of class $\mathcal{K}$ and, for each fixed s, $\beta(s, \cdot)$ is of class $\mathcal{L}$.

Definition 2: The impulsive system (1) is said to be *0-input globally asymptotically stable* (0-GAS) if there exists $\beta \in \mathcal{KL}$ such that, for all $x(0) \in \mathbb{R}^n$ and the input $w(t) \equiv 0$, it holds

$$\forall t \geqslant 0 : \quad |x(t)| \leqslant \beta(|x(0)|, t) . \tag{4}$$

Definition 3: The impulsive system (1) is said to have the *asymptotic gain* (AG) property if there exists a function $\kappa \in \mathcal{K}_\infty$ such that, for all $x(0) \in \mathbb{R}^n$ and all w, it holds

$$\limsup_{t\longrightarrow\infty} |x(t)| \leqslant \kappa(\|w\|_\infty^\infty) . \tag{5}$$

Definition 4: The impulsive system (1) is said to have the *uniformly bounded energy bounded state* (UBEBS) property if there exist functions $\alpha, \rho, \gamma \in \mathcal{K}_\infty$ and $p \geqslant 0$ such that, for all $x(0) \in \mathbb{R}^n$ and all w, it holds

$$\forall t \geqslant 0 : \quad |x(t)| \leqslant \alpha(|x(0)|) + \gamma(\|w\|_t^\rho) + p . \tag{6}$$

Definition 5: The impulsive system (1) is said to be *input-to-state stable* (ISS) if there exist functions $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$ such that, for all $x(0) \in \mathbb{R}^n$ and all w, it holds

$$\forall t \geqslant 0 : \quad |x(t)| \leqslant \beta(|x(0)|, t) + \gamma(\|w\|_t^\infty) . \tag{7}$$

Definition 6: The impulsive system (1) is said to be *integral input-to-state stable* (iISS) if there exist functions $\beta \in \mathcal{KL}$, γ, $\rho \in \mathcal{K}_\infty$ such that, for all $x(0) \in \mathbb{R}^n$ and all w, it holds

$$\forall t \geqslant 0 : \quad |x(t)| \leqslant \beta(|x(0)|, t) + \gamma(\|w\|_t^\rho) . \tag{8}$$

In [11], it is proven that ISS implies iISS if the impulse sequence satisfy the UIB condition defined in Definition 1. We also remark here that all 0-GAS, AG, UBEBS, ISS and iISS defined as above are *non-uniform* with respect to the initial time. The initial time has to be fixed at 0. Such non-uniformity makes the connections between those stability definitions more involved.

Lastly, let us recall the notion of *candidate exponential ISS-Lyapunov function* for impulsive systems, introduced in [19].

Definition 7: A locally Lipschitz function $V \colon \mathbb{R}^n \longrightarrow \mathbb{R}_{\geqslant 0}$ is a candidate exponential ISS-Lyapunov function for (1) with rate coefficients $c, d \in \mathbb{R}$ if there exist $\underline{\alpha}, \overline{\alpha}, \chi \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}(|x|) \leqslant V(x) \leqslant \overline{\alpha}(|x|) \qquad \forall x, \tag{9}$$

$$\nabla V(x) \cdot f(x,w) \leqslant -cV(x) + \chi(|w|) \qquad \forall x \text{ a.e.}, \forall w, \tag{10}$$

$$V(g(x,w)) \leqslant e^{-d} V(x) + \chi(|w|) \quad \forall x, w. \tag{11}$$

III. Motivating examples

In this section, we provide two motivating examples where we show that even when the impulse sequences "neutralize" the destabilizing effects of either the flow or impulses so that the impulsive system is 0-GAS, such impulse sequences still do not guarantee ISS.

A. System with stable flow and destabilizing impulses

Consider the one dimensional impulsive linear system

$$\dot{x}(t) = -x(t) + w(t), \qquad t \notin \mathcal{T}, \tag{12a}$$
$$x(t) = 2x(t^-), \qquad t \in \mathcal{T}. \tag{12b}$$

It can be interpreted that the continuous dynamics (12a) gives an ISS continuous-time system, while the jump (12b) gives an unstable discrete-time system. Intuitively, such an impulsive system is ISS if the impulse sequence is sufficiently infrequent.

Denote $t_0 = 0$ and let us consider an impulse sequence $\mathcal{T} = \{t_1, t_2, \cdots\}$, described as follows:

$$t_i = t_{i-1} + \ln 2 + \ln(i+1) - \ln(i), i = 1, 2, \cdots \tag{13}$$

We show that the impulsive system (12) with the impulse sequence (13) is 0-GAS but not ISS. For the sake of clear presentation, define $\varepsilon(i) := \ln(i+1) - \ln i$. For proving 0-GAS, let $w(t) = 0$ for all $t \geqslant 0$. In this case, it follows from (12) that for $t_i \in \mathcal{T}$,

$$x(t_i) = 2x(t_i^-) = 2e^{-(t_i - t_{i-1})}x(t_{i-1}) = e^{-\varepsilon(i)}x(t_{i-1}) .$$

Hence, iterating continuous and discontinuous dynamics, it holds that

$$x(t_i) = e^{-\sum_{j=1}^{i}\varepsilon(j)}x(0) = \frac{1}{i+1}x(0) ,$$

where the second equality comes from telescopic sum. It is easy to see that $|x(t)| \leqslant |x(t_i)|$ for all $t \in [t_i, t_{i+1})$. Since the sequence $x(t_i)$ converges to 0 as t_i approaches infinity, the system is 0-GAS.

To disprove ISS, let $w(t) = 1$ for all $t \geqslant 0$. In this case, again it can be concluded from (12) that

$$\begin{aligned} x(t_i) = 2x(t_i^-) &= 2\left(e^{-(t_i - t_{i-1})}x(t_{i-1}) + \int_{t_{i-1}}^{t_i} e^{\tau - t_i}d\tau\right) \\ &= e^{-\varepsilon(i)}x(t_{i-1}) + 2 - e^{-\varepsilon(i)} . \end{aligned}$$

Denoting $y_i := x(t_i) - 1$, we have $y_i = e^{-\varepsilon(i)}y_{i-1} + 1$, from which it can be recursively computed that

$$\begin{aligned} y_i &= e^{-\sum_{j=1}^{i}\varepsilon(j)}y_0 + 1 + \sum_{k=2}^{i} e^{-\sum_{j=k}^{i}\varepsilon(j)} \\ &= \frac{1}{i+1}y_0 + 1 + \sum_{k=2}^{i}\frac{k}{i+1} . \end{aligned}$$

Thus $x(t_i) = \frac{1}{i+1}x(0) + \frac{i(i+5)}{2(i+1)}$, which diverges as i approaches to infinity. Therefore, the system is not ISS. Fig. 1 shows both state trajectories starting from initial state $x(0) = 10$, when there is no input or with constant input $w(t) = 1$.

We can in fact show that the impulsive system (12) with impulse sequence (13) is iISS. This is because

$$\begin{aligned} |x(t_i)| &\leqslant 2\left|e^{-(t_i - t_{i-1})}x(t_{i-1}) + \int_{t_{i-1}}^{t_i} e^{\tau - t_i}w(\tau)d\tau\right| \\ &\leqslant e^{-\varepsilon(i)}|x(t_{i-1})| + 2\int_{t_{i-1}}^{t_i}|w(\tau)|d\tau . \end{aligned}$$

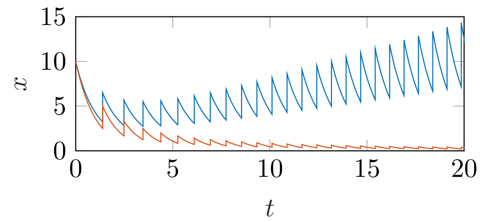

Fig. 1. State trajectories of the first example system. Red curve: no input; blue curve: constant input $w(t) = 1$.

Recursively, we have

$$\begin{aligned} |x(t_i)| &\leqslant e^{-\sum_{j=1}^{i}\varepsilon(j)}|x(0)| + 2\int_0^{t_1}|w(\tau)|d\tau \\ &\quad + 2\sum_{k=2}^{i}\left(e^{-\sum_{j=k}^{i}\varepsilon(j)}\int_{t_{i-1}}^{t_i}|w(\tau)|d\tau\right) \\ &\leqslant \frac{1}{i+1}|x(0)| + 2\int_0^{t_i}|w(\tau)|d\tau . \end{aligned}$$

B. System with unstable flow and stabilizing impulses

We then consider another one dimensional impulsive linear system

$$\dot{x}(t) = x(t), \qquad t \notin \mathcal{T} , \tag{14a}$$
$$x(t) = \frac{1}{2}x(t^-) + w(t), \qquad t \in \mathcal{T} . \tag{14b}$$

Contrary to the system (12), the continuous dynamics (12a) gives an unstable continuous-time system, while the jump (12b) gives an ISS discrete-time system. Intuitively, such an impulsive system is ISS if the impulse sequence is sufficiently frequent.

Pick arbitrary $\epsilon \in (0, \ln 2)$. We consider an impulse sequence $\mathcal{T} = \{t_1, t_2, \cdots\}$ described as follows:

$$t_1 = 1, \tag{15a}$$
$$t_{i+1} - t_i = \begin{cases} j(\ln 2 - \epsilon), & i = \frac{j(j+1)}{2}, j = 1, 2, \cdots , \\ \epsilon, & \text{otherwise.} \end{cases} \tag{15b}$$

In other words, starting from t_1, the impulse sequence consists of infinitely many cycles. The j-th cycle consists of firstly j impulses separated by flow of ϵ unit of time, and then a continuous flow of $j(\ln 2 - \epsilon)$ unit of time. We show that the impulsive system (14) with the impulse sequence (15) is 0-GAS but not iISS (and hence not ISS).

To show 0-GAS, let $w(t) = 0$ for all $t \geqslant 0$. For any $i = \frac{j(j+1)}{2} + 1, j = 1, 2, \cdots$, t_i is the end of the j-th cycle and we have $t_i = \frac{(j-1)j}{2}\epsilon + \frac{j(j+1)}{2}(\ln 2 - \epsilon) + 1$. Hence it is not difficult to see that

$$x(t_i^-) = e^{t_i}2^{-i}x(0)$$

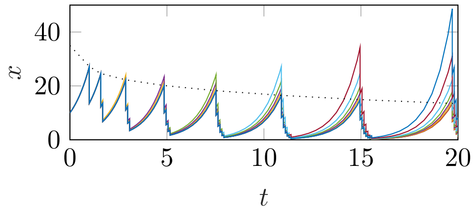

Fig. 2. State trajectories of system (14). Different colors represent unit impulsive inputs at different times $t_i, i = \frac{j(j+1)}{2}, j = 1, 2, \cdots, 7$. The dotted curve represents an upper bound for the unforced trajectory, which is convergent.

$$= e^{\frac{(j-1)j}{2}\epsilon + \frac{j(j+1)}{2}(\ln 2 - \epsilon) + 1} 2^{-\frac{j(j+1)}{2}} x(0) = e^{1-j} x(0) \,,$$

which uniformly converges to 0 as j goes to infinity. Because the flow from time t_{i-1} to t_i is unstable, we have $|x(t)| \leqslant |x(t_i^-)|$ for all $t \in [t_{i-1}, t_i)$. Meanwhile, it can also be verified that since $\epsilon < \ln 2$, we always have $|x(t)| \leqslant |x(t_i^-)|$ for all $t \in [t_i, t_i^*)$ where $i^* = \frac{(j+1)(j+2)}{2}$. Hence the sequence of $|x(t_i^-)|$ serves as the local maxima for $|x(t)|$ and because it is bounded from above by $e^{1-j}|x(0)|$, the system is 0-GAS.

We now show that the system is not iISS. Suppose (8) holds with some $\gamma, \rho \in \mathcal{K}_\infty$. Consider an input $w(t)$ such that $w(t_{i^*}) = 1$ where $i^* = \frac{j^*(j^*+1)}{2}$, j^* is an integer such that $j^* > \ln_2 \gamma \circ \rho(1)$, and $w(t) = 0$ for all $t \neq t_{i^*}$. This is a unit impulsive input and clearly $\|w\|_t^\chi = \chi(1)$ for all $t \geqslant t_{i^*}$. Now because the initial state is 0, $x(t) = 0$ for all $t \in [0, t_{i^*})$ and (8) implies

$$|x(t)| \leqslant \gamma \circ \rho(1) \quad \forall t \geqslant t_{i^*}. \tag{16}$$

Meanwhile, it follows from the dynamics (14) and the fact that there is a continuous flow of $j^* \ln 2$ unit time from t_{i^*} to t_{i^*+1} that

$$x(t_{i^*+1}^-) = e^{j^* \ln 2} x(t_{i^*}) = e^{j^* \ln 2} w(t_{i^*}) = 2^{j^*} > \gamma \circ \rho(1).$$

which is a contradiction to (16). Hence the system is not iISS. The state trajectories, starting from initial state $x(0) = 10$ and subjected to unit impulses at different impulse instants, are plotted in Fig. 2.

IV. Stability-guaranteeing sufficient conditions on the impulse sequences

As reflected by the examples in Section III, ISS, iISS are stronger stability properties than 0-GAS for impulsive systems. We would like to study the relations between 0-GAS, ISS and iISS, as well as seek distinct sufficient conditions on the impulse sequences which lead to these stability properties. The first lemma identifies the gaps between 0-GAS and ISS/iISS for impulsive systems.

Lemma 4.1: Consider the general impulsive system (1) with an UIB impulse sequence, then the following two statements hold

1) System (1) is ISS if and only if it is 0-GAS and has the AG property,
2) System (1) is iISS if and only if it is 0-GAS and has the UBEBS property.

Proof: The first statement can be proven by appealing to [12, Theorem 3.1] and considering impulsive systems as a special type of hybrid systems. The second statement is proven by [11, Theorem 3.1]. ■

In the presence of candidate ISS-Lyapunov functions as defined in Definition 7, sufficient conditions on the impulse sequences that guarantee the 0-GAS and the AG properties are given.

Theorem 4.2: Let V be a candidate exponential ISS-Lyapunov function for the impulsive system (1), with $c, d \in \mathbb{R}$. Given an impulse sequence $\mathcal{T} = \{t_1, t_2, \cdots\}$, if there exists a function $\theta \in \mathcal{L}$ such that

$$ct + dN(0,t) \geqslant -\ln \theta(t) \quad \forall t \geqslant 0 \,, \tag{17}$$

then the impulsive system is 0-GAS. On the other hand, if the following conditions hold

$$\lim_{t \longrightarrow \infty} ct + dN(0,t) = \infty \,, \tag{18}$$

$$\limsup_{t \longrightarrow \infty} e^{-ct - dN(0,t)} \sum_{j=1}^{N(0,t)} e^{ct_j + dj} < \infty \,, \tag{19}$$

then the impulsive system has the AG property.

Due to space limitation, we omit the proof of Theorem 4.2. Note that $\ln \theta(0) - \ln \theta(t) \in \mathcal{K}_\infty$, thus, the condition (17) can also be interpreted as that the function

$$h_1(t) := ct + dN(0,t) \tag{20}$$

is bounded below by a shifted $\mathcal{K}_\infty$ function. Following the definition (20), the condition (19) is equivalent to $\limsup_{t \longrightarrow \infty} h_2(t) < \infty$, where

$$h_2(t) := e^{-h_1(t)} \sum_{j=1}^{N(0,t)} e^{h_1(t_j)} \,. \tag{21}$$

Also note that the condition (18) holds when (17) is satisfied. Because of Lemma 4.1, we have the following immediate result.

Corollary 4.3: Let V be a candidate exponential ISS-Lyapunov function for the impulsive system (1), with $c, d \in \mathbb{R}$. The impulsive system is ISS if the impulse sequence satisfies the UIB condition and there exist $\theta \in \mathcal{L}$ and a constant $K > 0$ such that (17) holds and

$$\sum_{j=1}^{N(0,t)} e^{ct_j - dj} \leqslant \frac{K}{\theta(t)} \quad \forall t \geqslant 0. \tag{22}$$

The next theorem contains conditions which guarantees UBEBS and iISS properties.

Theorem 4.4: Let V be a candidate exponential ISS Lyapunov function for the impulsive system (1), with $c, d \in \mathbb{R}$. The impulsive system is UBEBS if the function h_1 defined in (20) is uniformly bounded to a smooth $\mathcal{K}$

function. That is, if there exist a smooth function $\gamma \in \mathcal{K}$ and a constant $T \geqslant 0$ such that

$$|ct + dN(0,t) - \gamma(t)| \leqslant T \quad \forall t \geqslant 0. \tag{23}$$

Moreover, if $\gamma \in \mathcal{K}_\infty$ and $\mathcal{T}$ satisfies the UIB condition, then the impulsive system is iISS.

The proof of Theorem 4.4 is also omitted.

V. Discussion

In this section, we first revisit the examples studied in Section III, by applying the 0-GAS, ISS and iISS criteria developed in Section IV. We then compare our stability results with some known results in the literature.

A. Revisiting the examples

As discussed in Section IV, it is sufficient to visualize the functions h_1, h_2 defined in (20), (21) in order to conclude whether the system is 0-GAS or has the AG or UBEBS properties.

For the system (12), it is not difficult to see that $V(x) = |x|$ is a candidate ISS Lyapunov function with $c = 1, d = -\ln 2$. With the particular impulse sequence (13), the plots of functions h_1, h_2 are shown in Fig. 3. Note that although $h_1(t)$ grows slowly in the order of $\ln(t)$, it is unbounded and hence can be bounded from below by a shifted $\mathcal{K}_\infty$ function, implying that the system (12) is 0-GAS via Theorem 4.2. Also, the fluctuation of $h_1(t)$ is also uniformly bounded, implying that the system also has the UBEBS property via Theorem 4.4 and hence it is iISS, consistent with the observations in Section III-A. On the other hand, the function h_2 is unbounded. This is expected, as otherwise the system would have the AG property by Theorem 4.2 and hence it would be ISS, contradicting the previous observation.

Regarding system (14), $V(x) = |x|$ is a candidate ISS Lyapunov function with $c = -1, d = \ln 2$. With the particular impulse sequence (15), the plots of functions h_1, h_2 are shown in Fig. 4. Again, $h_1(t)$ grows slowly but it is unbounded and hence can be bounded from below by a shifted $\mathcal{K}_\infty$ function, implying that the system (12) is 0-GAS via Theorem 4.2. However, the fluctuation of $h_1(t)$ becomes larger and unbounded when t becomes larger, implying that $h_1(t)$ is not uniformly bounded to any class $\mathcal{K}$ function. Thus, Theorem 4.4 is inconclusive and indeed, this system is not iISS as discussed in Section III-B. Meanwhile, $h_2(t)$ is also unbounded. In fact, as iISS is necessary for ISS, the system (14) is not ISS.

B. Comparison to the literature

We first observe that [21, Section 3.5] contains an example where 0-GUAS and UBEBS do not imply iISS. The impulse sequence in this example, however, does

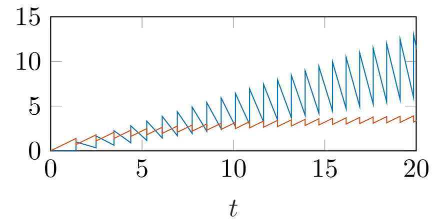

Fig. 3. Plots of $h_1(t)$ (red) and $h_2(t)$ (blue) for the system (12).

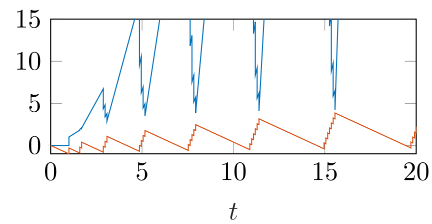

Fig. 4. Plots of $h_1(t)$ (red) and $h_2(t)$ (blue) for the system (14).

not satisfy the UIB condition. Consequently, it is not a contradiction to our Lemma 4.1.

We then compare our results to the work [19], which states that if there is a candidate exponential ISS Lyapunov function for the impulsive system (1) with rate coefficients $c, d \in \mathbb{R}, d \neq 0$, then the system is uniformly ISS over $\mathcal{S}[\mu, \lambda]$ for any $\mu, \lambda > 0$, where $\mathcal{S}[\mu, \lambda]$ is a class of impulse sequences satisfying

$$-dN(s,t) - (c - \lambda)(t - s) \leqslant \mu \quad \forall t \geqslant s \geqslant 0 \tag{24}$$

Note that by replacing $s = 0$, a necessary condition for (24) to hold is $ct + dN(0,t) \geqslant \lambda t - \mu$, where the right-hand side is increasing and affine in t and hence strictly stronger than (17), whose right-hand side can be any shifted $\mathcal{K}_\infty$ function. In other words, Theorem 4.2 is less conservative for the determination of 0-GAS impulsive systems. While matching the particular impulse sequences (13), (15) with the condition (24), subjected to the particular c, d values determined for the candidate ISS Lyapunov function $V(x) = |x|$, it turns out that (24) fails for any $\lambda > 0$, except for $\lambda = 0$. In other words, the results in [19] are inapplicable for the stability analysis of our examples.

We then compare our stability results to the work [24]. Given a candidate ISS-Lyapunov function $V : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geqslant 0}$ in the implicit form such that (9) holds for some $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$, and

$$V(x) \geqslant \chi'(|w|) \Rightarrow \begin{cases} \nabla V(x) \cdot f(x,w) \leqslant -\varphi(V(x)), \\ V(g(x,w)) \leqslant \psi(V(x)), \end{cases} \tag{25}$$

holds for some $\chi' \in \mathcal{K}_\infty$, positive definite functions φ and ψ, if there exists a positive constant ρ (which, for both examples, are $\frac{1}{\ln 2}$) such that

$$\lim_{t \longrightarrow \infty} \frac{N(t,s)}{t - s} = \rho \tag{26}$$

holds uniformly with respect to $s \in [0, \infty)$ and there exists $\delta > 0$ such that

$$\int_a^{\psi(a)} \frac{\mathrm{d}s}{\varphi(s)} \leqslant \frac{1}{\rho} - \delta \tag{27}$$

holds for all $a > 0$, then the impulsive system is ISS. When (10) and (11) hold with $c > 0$, the condition (25) is satisfied with $\chi' = \frac{1}{\epsilon}\chi$ for any $\epsilon \in (0, c)$, and $\varphi(s) = (c - \epsilon)s$, $\psi(s) = (e^{-d} + \epsilon)s$. Substituting the numerical values of c, d for the first example,

$$\int_a^{\psi(a)} \frac{\mathrm{d}s}{\varphi(s)} = \int_a^{(2+\epsilon)a} \frac{\mathrm{d}s}{(1-\epsilon)s} = \frac{\ln(2+\epsilon)}{1-\epsilon} > \ln 2 = \frac{1}{\rho},$$

which is a contradiction to (27) for any $\delta \geqslant 0$. Therefore, the result in [24] is inconclusive for the first example in this paper. Indeed, the system (12) with the impulse sequence (13) is not ISS as studied in Section III-A.

Similar problems are also observed for the second example when compared to the work [25], where the case of unstable flow and stabilizing impulses is studied. Briefly speaking, the main stability result in [25] when applied to the second example requires the average number of impulses per unit time to be uniformly strictly bounded below by $\ln 2$, which is not true for the impulse sequence (15). Observe that for an impulsive sequence which has periodic impulses at period of $\ln 2$, both the unforced systems (12) and (14) will have periodic solutions and hence are not stable. Thus, although the two examples are shown to be 0-GAS in our work, in the view of (26) with $\rho = \frac{1}{\ln 2}$, both impulse sequences (13) and (15) are considered *near-critical*, in the sense that the frequency of impulses is close to causing instability. Such impulsive sequences do not affect the internal stability of impulsive systems; however, near critical impulsive sequences may amplify the input, causing a 0-GAS impulsive system to be not ISS or iISS.

VI. Conclusions

In this article, we have provided sufficient conditions on the impulse sequences which preserve 0-GAS, ISS or iISS property for the impulsive systems. We have shown that not all 0-GAS impulse sequences also ensure ISS or iISS. Compared to the existing literature, our proposed sufficient condition for 0-GAS is less conservative and hence it is applicable to a larger class of impulse sequences. Due to space constraints, we have only investigated impulsive systems in this work. In the subsequent work, we will also study the same problem for switched systems with possibly unstable subsystems.

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