2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

Live systems of varying dimension: modeling and stability

Andrii Mironchenko

Abstract— **A major limitation of the classical control theory is the assumption that the state space remains stationary in time. This prevents analyzing and even formalizing the stability and control problems for open multi-agent systems whose agents may enter or leave the network, industrial processes where the sensors or actuators may be exchanged frequently, smart grids, etc. In this work, we propose a framework of live systems that covers a rather general class of systems with a time-varying state space. We argue that input-to-state stability is a proper stability notion for this class of systems, and many of the classic tools and results, such as Lyapunov methods and superposition theorems, can be extended to this setting.**

Keywords: nonlinear systems, modeling, infinite-dimensional systems, input-to-state stability, Lyapunov methods, impulsive systems, multi-agent systems

I. Introduction

Systems theory constitutes a powerful paradigm for the analysis and control of linear and nonlinear systems despite the lack of information about the system, acting disturbances, communication constraints, and many further obstructions on the way to the practical implementation of the controllers. This astonishing progress was achieved under a foundational structural assumption that goes through the whole body of the mathematical systems theory: the state space does not change in time.

However, this assumption is frequently not fulfilled in natural and human-made systems. The number of individuals in the populations of organisms changes in time due to the birth and decay processes. Plants (considered as a system consisting of repeated units) intermittently create new organs such as leaves, flowers, or bracts. Dynamical systems with a variable state space, which we will call *live systems*, appear naturally in control applications. An archetypal problem of this kind is designing the optimized adaptive traffic control system to make the controller viable and scalable even though new roads and cars are entering or leaving the network [15, Section 4.1]. A related problem is the organization of the smart grids that ensure robust and effective generation and transport of energy even though the size and topology of the network change due to the attachment and detachment of the micrograds to the system [4], [5]. Another recent problem of this kind is a consensus of multi-agent systems with new agents entering or leaving the network (open multi-agent systems) [25], [10], [7], [22], [31]. In social networks, the users may enter and leave communities modifying the group behavior [9]. Last but not least, most large-scale industrial processes are live systems, which are updated or modified from time to time: new sensors may be added, certain actuators can be exchanged, components may become malfunctioning, and subparts of the network may be added or excluded from the system, etc. [28].

Most current control designs do not take into account the variable dimension of the state space. Thus they are developed only for a model of a system that is valid for a relatively short period. After a change of the model of a system, one has to develop a new controller from scratch, which is a costly and time-consuming process [28].

Challenges. Although live systems are omnipresent, few works are devoted to their analysis within the control and dynamical systems theory. This is due to several conceptual problems in the modeling of live systems. First, one must go beyond the classical concept of state space. Secondly, the stability concepts need to be revisited. Indeed, if new agents with a magnitude of the state bounded away from zero may enter the network at arbitrarily large times, there is no hope that the reasonably defined state of the system approaches zero (or any other point). Thus, the classical attractivity and asymptotic stability properties may fail to be the proper stability concepts for control systems.

Approaches to live systems in control theory. Some particular questions in the theory of live systems have been addressed in systems theory. The adaptive control theory for systems with unknown parameters [14], [1] allows us to handle the problem of updating the existing sensors or actuators with similar dynamics but distinct parameters. Yet, the problem of adding new sensors or actuators is outside of the scope of adaptive control. Fault-tolerant and reconfigurable control [2], [24] considers the design of controllers that are robust with respect to malfunctions of some of its components (sensors, actuators, observers). However, most of the research is devoted to analyzing some particular types of failures that may occur. Plug-and-play control envisaged in [28] aims to develop self-configuring control methods that will remain viable if new sensors or controllers are added.

If the maximal number of components is known, one can study the dynamics of such a network in the largest possible state space. The entering or leaving of subsystems to/from the network can thus be modeled by switches in the system's structure [29]. In [30], the concept of pseudo-continuous multi-dimensional multi-mode systems has been introduced that can be understood as a switched system with finitely many modes of a distinct dimension. The change of

A. Mironchenko is with Faculty of Computer Science and Mathematics, University of Passau, Innstraße 33, 94032 Passau, Germany. Email: andrii.mironchenko@uni-passau.de,

This research has been supported by the German Research Foundation (DFG), grant MI 1886/2-2.

a dimension is modeled by a switch into the other mode. Both in [29] and [30], the maximal dimension of the system is finite and known in advance.

In some cases, it is possible to overapproximate a large but finite network of a possibly unknown size via an infinite network and transfer the stability results for the infinite overapproximation to any possible subnetwork [23], [17].

In [7], the authors consider open discrete-time multi-agent systems. The authors do not assume any a priori upper bound for the dimension of the network. Thus the dimension of the system may well converge to infinity with the time, though it remains finite at any moment. The asymptotic stability in [7] is studied in weighted norms, given by the usual norms of finite-dimensional vectors divided by a square root of the dimension of the vector. The stability in such weighted norms may be interesting for some applications but can be non-typical for other ones.

Contribution. The main contribution of this paper is to propose a new framework for modeling and stability analysis of live systems. We consider a live system as an impulsive system that changes at impulse times not only the state's value but also the system's configuration. Each configuration of a live system is a control system in a classical sense, characterized by the state set, input set, and the flow map [12]. Our setting is very general, including open multi-agent systems, systems with unknown dimension, switched systems of variable dimension, etc.

We show that live systems retain the essential features of control systems. Furthermore, although the set of all states (*state set*) does not have a linear structure (in particular, the dimension of the state vector may vary with time), it is possible to introduce a kind of pseudonorm on it. We understand an arrival of an agent as an input to the system. In fact, it is not a big difference whether a new agent comes into the network or the state of one of the existing agents is changed in an impulsive way. This motivates us to understand the stability of live systems in the sense of input-to-state stability (ISS) introduced in [26] and later extended to impulsive systems [11], [3]. See also [27], [20] for the survey and [18] for systematic development of the theory.

It turns out that many characterizations for ISS of infinite-dimensional systems in Banach spaces shown in [21] are still valid for live systems, as the continuity of trajectories and the specific properties of Banach spaces are not used in the arguments in [21] (as was already stressed in [16]). Finally, we introduce the concept of ISS Lyapunov functions for live systems based on the corresponding concept from the ISS theory of impulsive systems [11], [3]. As in impulsive systems, having an ISS Lyapunov function, we can prove the ISS of a live system under certain dwell-time conditions imposed on the density of impulses.

Notation. For two sets X, Y, denote by $C(X,Y)$ the linear space of continuous functions, mapping X to Y.

For the formulation of stability properties, the following classes of comparison functions are useful:

$$\begin{aligned}
\mathcal{K} \quad &:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous, strictly increasing and } \gamma(0) = 0\},\\
\mathcal{K}_\infty \quad &:= \{\gamma \in \mathcal{K} \mid \gamma \text{ is unbounded}\},\\
\mathcal{L} \quad &:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous and strictly decreasing with } \lim_{t\to\infty} \gamma(t) = 0\},\\
\mathcal{KL} \quad &:= \{\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \mid \beta \text{ is continuous, } \beta(\cdot,t) \in \mathcal{K},\ \beta(r,\cdot) \in \mathcal{L},\ \forall t \geq 0,\ \forall r > 0\}.
\end{aligned}$$

An up-to-date compendium of results concerning comparison functions can be found in [13] and [18, Appendix].

II. LIVE SYSTEMS

As we strive to develop a unified framework of live systems, we start with a rather general classical concept of a control system, whose variations have been used in control theory at least since the 1960s [12].

Definition 2.1: Let a triple $\Sigma = (X, \mathcal{U}, \phi)$ consist of

(i) A set X called the *state set*.

(ii) An *input set* $\mathcal{U} \subset \{u : \mathbb{R}_+ \to U\}$, where U is a set called *set of input values*.
We assume that the following two axioms hold:
The axiom of shift invariance: for all $u \in \mathcal{U}$ and all $\tau \geq 0$ the time-shifted input $u(\cdot + \tau)$ belongs to $\mathcal{U}$.
The axiom of concatenation: for all $u_1, u_2 \in \mathcal{U}$ and all $t > 0$ the *concatenation of* u_1 *and* u_2 *at time* t, given by

$$u_1 \underset{t}{\diamond} u_2(\tau) := \begin{cases} u_1(\tau), & \text{if } \tau \in [0,t], \\ u_2(\tau - t), & \text{if } \tau > t, \end{cases} \tag{1}$$

belongs to $\mathcal{U}$.

(iii) A map $\phi : D_\phi \to X$, with a domain of definition $D_\phi \subseteq \mathbb{R}_+ \times X \times \mathcal{U}$ (called *transition map*), such that for all $(x,u) \in X \times \mathcal{U}$ it holds that
$$D_\phi \cap \big(\mathbb{R}_+ \times \{(x,u)\}\big) = [0, t_m) \times \{(x,u)\} \subset D_\phi,$$
for a certain $t_m = t_m(x,u) \in (0, +\infty]$.
The corresponding interval $[0, t_m)$ is called the *maximal domain of definition* of $t \mapsto \phi(t,x,u)$.

The triple Σ is called a *(control) system*, if the following properties hold:

(Σ1) *The identity property:* for every $(x,u) \in X \times \mathcal{U}$ it holds that $\phi(0,x,u) = x$.

(Σ2) *Causality:* for every $(t,x,u) \in D_\phi$, for every $\tilde{u} \in \mathcal{U}$, such that $u(s) = \tilde{u}(s)$ for all $s \in [0,t]$ it holds that $[0,t] \times \{(x,\tilde{u})\} \subset D_\phi$ and $\phi(t,x,u) = \phi(t,x,\tilde{u})$.

(Σ3) *The cocycle property:* for all $x \in X$, $u \in \mathcal{U}$, for all $t, h \geq 0$ so that $[0, t+h] \times \{(x,u)\} \subset D_\phi$, we have

$$\phi\big(h, \phi(t,x,u), u(t+\cdot)\big) = \phi(t+h, x, u).$$

If $t_m(x,u) = \infty$ for all $x \in X$ and $u \in \mathcal{U}$, we call Σ *forward complete*.

Definition 2.1 comprises the most basic and essential properties of control systems. Notably, X and U are merely sets, and we do not require any further structure (linearity, topology, norm, metric, etc.) from these sets. This is a typical feature in the early references [12].

Fix any set $\hat{S}$, which we call *configuration set*. We call its elements *configurations*. With each configuration $Q \in \hat{S}$, we associate a control system $\Sigma_Q := (X_Q, \mathscr{U}, \phi_Q)$. To avoid the notational complications, we assume that the set of inputs is the same for all systems Σ_Q, $Q \in \hat{S}$.

Remark 2.2: In this work, for any $Q \in \hat{S}$, we identify any element $y \in X_Q$ with a labeled pair (Q, y). However, we drop the labels to simplify the notation.

We will define a live system Σ as a system induced by the family $(\Sigma_Q)_{Q\in\hat{S}}$, whose configuration may change with time. We denote the configuration of Σ at time t by $I(t) \in \hat{S}$.

Assumption 2.3: The changes of the state space or the impulsive changes of the system's state occur at certain time instants given by the increasing infinite sequence $\mathscr{T} := (t_k)_{k\in\mathbb{N}}$ without accumulation points. We call $\mathscr{T}$ *impulse time sequence*.

Furthermore, the configuration $I(\cdot)$ remains constant between the impulse times, that is $I(t) = I(t_k)$ for $t \in [t_k, t_{k+1})$.

The transitions between configurations are given by the map $q : \hat{S} \times \mathbb{R}_+ \to \hat{S}$. The change of the configuration of the system Σ at impulse times t_k is given by

$$I(t_k) := q(I(t_k^-), t_k), \quad k \in \mathbb{N}. \tag{2}$$

As the configuration changes, a system trajectory with an initial condition in X_Q leaves X_Q and jumps to the state set of the system in another configuration. This motivates us to consider the following set as the state set for the live system composed of $(\Sigma_Q)_{Q\in\hat{S}}$:

$$X := \bigcup_{Q\subset\hat{S}} X_Q. \tag{3}$$

In view of Remark 2.2, each element of X "knows" its configuration. Thus, the union in (3) is disjoint in the sense that for any $x \in X$ there is a unique $Q \in \hat{S}$ such that $x \in X_Q$.

From now on, we assume that $\mathscr{U}$ is the space of piecewise continuous functions from $\mathbb{R}_+$ to U (with this assumption, the values of inputs are well-defined at the impulse times).

We fix a sequence of impulse times $\mathscr{T} = (t_k)_{k=1}^{\infty}$, and construct the flow map ϕ of a live system Σ.

Pick any initial configuration $I_0 \in \hat{S}$, any initial condition $x \in X_{I_0}$, any input $u \in \mathscr{U}$. As long as the system stays in the initial configuration, we define the dynamics of Σ by

$$\phi(t,x,u) := \phi_{I_0}(t,x,u), \quad t \in [0, \min\{t_{m,I_0}(x,u), t_1\}),$$

where $t_{m,I_0}(x,u)$ is the maximal existence time of $\phi_{I_0}(\cdot,x,u) \subset \Sigma_{I_0}$. If $t_{m,I_0}(x,u) < t_1$, then $t_m(x,u) := t_{m,I_0}(x,u)$, and $\phi(\cdot,x,u)$ is constructed. Otherwise, define

$$\phi(t_1,x,u) := g(\phi(t_1^-,x,u), u(t_1^-), t_k), \tag{4}$$

where $g : X \times U \times \mathbb{R}_+ \to X$ describes the jump of an element from the state set. Now the system Σ is in a new configuration $I(t_1)$, and its evolution is governed by the flow $\phi_{I(t_1)}$.

$$\phi(t,x,u) := \phi_{I(t_1)}\big(t - t_1, \phi(t_1,x,u), u(\cdot + t_1)\big),$$

for $t \in [t_1, t_1 + \min\{t_{m,I(t_1)}(\phi(t_1,x,u), u(\cdot+t_1)), t_2 - t_1\})$.

Note that the map ϕ is also dependent on the sequence $\mathscr{T}$. However, we do not express it in our notation explicitly.

Doing this procedure repeatedly, we obtain the flow map

$$\phi : D_\phi \to X, \quad D_\phi \subseteq \mathbb{R}_+ \times X \times \mathscr{U},$$

such that $\forall (x,u) \in X \times \mathscr{U}$ $\exists t_m = t_m(x,u) \in (0, +\infty]$:

$$D_\phi \cap \big(\mathbb{R}_+ \times \{(x,u)\}\big) = [0, t_m) \times \{(x,u)\} \subset D_\phi.$$

We immediately see by the construction of the flow ϕ that

Proposition 2.4: For each impulse time sequence $\mathscr{T}$, the corresponding triple $\Sigma^{\mathscr{T}} := (X, \mathscr{U}, \phi)$ is a control system.

Remark 2.5: If the map q in (2) describing the change of configuration, depends on the state of a system, we need to go beyond the paradigm of impulsive systems, and consider live systems as hybrid systems (see [8]) changing their configuration at impulse times.

To introduce the stability concepts of live systems, we need to measure distances. From now on, we assume that all X_Q are normed vector spaces endowed with the norm $\|\cdot\|_{X_Q}$.

Definition 2.6: We introduce the map $\|\cdot\|_X : X \to \mathbb{R}_+$ as

$$\|x\|_X := \|x\|_{X_Q}, \quad x \in X_Q,\ Q \in \hat{S}.$$

We call this map a *pseudonorm* on X.

As X does not possess a linear structure, there is no triangle property for the map $\|\cdot\|_X$. However, it is not hard to check the following:

Proposition 2.7: The map $\|\cdot\|_X$ satisfies the following properties:

- $\|x\|_X \geq 0$ for all $x \in X$.
- $\|x\|_X = 0$ iff x is a zero element in X_Q for some $Q \in \hat{S}$.
- For each $x \in X$ and any $\lambda \in \mathbb{R}$ it holds that $\lambda x \in X$, and

$$\|\lambda x\|_X = |\lambda| \|x\|_X. \tag{5}$$

III. Stability of live systems and its characterization

As new agents with a state of magnitude bounded away from zero may enter the system at arbitrarily large times, a system cannot be stable or attractive in the classical sense. At the same time, as we view the arrival of new subsystems as the inputs to our system, it is reasonable to use the concept of input-to-state stability to study the stability of such systems. Since X is endowed with a map $\|\cdot\|_X$ that has many properties of a norm, we can define ISS as usual:

Definition 3.1: For a given sequence $\mathscr{T}$ of impulse times, we call a system $\Sigma^{\mathscr{T}} := (X, \mathscr{U}, \phi)$ *input-to-state stable (ISS)* if it is forward complete and there exist $\beta \in \mathscr{KL}$, $\gamma \in \mathscr{K}_\infty$, such that for any $x \in X$, all $u \in \mathscr{U}$, and all $t \geq 0$ it holds that

$$\|\phi(t,x,u)\|_X \leq \beta(\|x\|_X, t) + \gamma(\|u\|_{\mathscr{U}}). \tag{6}$$

A live system Σ is called *uniformly ISS* over a given set $\mathscr{S}$ of admissible sequences of impulse times if $\Sigma^{\mathscr{T}}$ is ISS for every $\mathscr{T} \in \mathscr{S}$, with β and γ independent of the choice of the sequence from the class $\mathscr{S}$.

It turns out that many properties of "classic" ISS infinite-dimensional systems (as defined in [20]) can be transferred to the general live systems.

Definition 3.2: For a given sequence $\mathscr{T}$ of impulse times, a forward complete system $\Sigma^{\mathscr{T}} := (X, \mathscr{U}, \phi)$ has a *convergent input - uniformly convergent state (CIUCS) property*, if for each $u \in \mathscr{U}$ such that $\lim_{t\to\infty} \|u(t+\cdot)\|_{\mathscr{U}} = 0$, and for any $r > 0$, it holds that

$$\lim_{t\to\infty} \sup_{\|x\|_X \le r} \|\phi(t,x,u)\|_X = 0.$$

Proposition 3.3: Every input-to-state stable live system (for a fixed impulse time sequence) has the CIUCS property.

For a live system whose only inputs are due to the entering new agents, Propositon 3.3 means that if the magnitude of the incoming agents tends to zero as time goes to infinity, then the system's state converges to zero.

IV. CHARACTERIZATIONS OF ISS

In this section, assuming that the impulse time sequence $\mathscr{T}$ is fixed (known a priori), we state a criterion for ISS of live systems in terms of weaker properties introduced next.

Denote $B_{r,\mathscr{U}} := \{u \in \mathscr{U} : \|u\|_{\mathscr{U}} \le r\}$ and $B_r := \{x \in X : \|x\|_X \le r\}$.

Definition 4.1: $\Sigma^{\mathscr{T}} = (X, \mathscr{U}, \phi)$ *has bounded reachability sets (BRS)*, if for any $C > 0$ and any $\tau > 0$ it holds that

$$\sup\big\{\|\phi(t,x,u)\|_X : \|x\|_X \le C,\ \|u\|_{\mathscr{U}} \le C,\ t \in [0,\tau]\big\} < \infty.$$

Definition 4.2: Consider a live system $\Sigma^{\mathscr{T}} = (X, \mathscr{U}, \phi)$ with equilibrium point $0 \in X$. We say that ϕ *is continuous at the equilibrium* if for every $\varepsilon > 0$ and for any $h > 0$ there exists a $\delta = \delta(\varepsilon, h) > 0$, so that $[0,h] \times B_\delta \times B_{\delta,\mathscr{U}} \subset D_\phi$, and

$$t \in [0,h],\ \|x\|_X \le \delta,\ \|u\|_{\mathscr{U}} \le \delta\ \Rightarrow\ \|\phi(t,x,u)\|_X \le \varepsilon. \quad (7)$$

In this case, we will also say that $\Sigma^{\mathscr{T}}$ has the *CEP property*.

Definition 4.3: A system $\Sigma^{\mathscr{T}} = (X, \mathscr{U}, \phi)$ is called *uniformly locally stable (ULS)*, if there exist $\sigma \in \mathscr{K}_\infty$, $\gamma \in \mathscr{K}_\infty \cup \{0\}$ and $r > 0$ such that for all $x \in \overline{B_r}$ and all $u \in \overline{B_{r,\mathscr{U}}}$ the trajectory $\phi(\cdot,x,u)$ is defined on $\mathbb{R}_+$, and

$$\|\phi(t,x,u)\|_X \le \sigma(\|x\|_X) + \gamma(\|u\|_{\mathscr{U}}) \quad \forall t \ge 0. \quad (8)$$

Definition 4.4: A forward complete system $\Sigma^{\mathscr{T}} = (X, \mathscr{U}, \phi)$ has the *bounded input uniform asymptotic gain (bUAG) property*, if there exists a $\gamma \in \mathscr{K}_\infty \cup \{0\}$ such that for all $\varepsilon, r > 0$ there is a $\tau = \tau(\varepsilon, r) < \infty$ such that

$$\|u\|_{\mathscr{U}} \le r,\ \ x \in B_r,\ \ t \ge \tau \ \ \Rightarrow\ \ \|\phi(t,x,u)\|_X \le \varepsilon + \gamma(\|u\|_{\mathscr{U}}).$$

Definition 4.5: We say that $\Sigma^{\mathscr{T}} = (X, \mathscr{U}, \phi)$ has the *uniform limit property on bounded sets (bULIM)*, if there exists $\gamma \in \mathscr{K}_\infty \cup \{0\}$ so that for every $\varepsilon > 0$ and for every $r > 0$ there is a $\tau = \tau(\varepsilon, r)$ such that

$$\|x\|_X \le r\ \wedge\ \|u\|_{\mathscr{U}} \le r \quad \Rightarrow \quad \exists t \le \tau:\ \|\phi(t,x,u)\|_X \le \varepsilon + \gamma(\|u\|_{\mathscr{U}}). \quad (9)$$

It turns out that the ISS superposition theorem from [21] can be transferred to general live systems without significant changes in the formulation and the proof.

Theorem 4.6 (ISS superposition theorem): Let the impulsive time sequence $\mathscr{T}$ be fixed, and $\Sigma^{\mathscr{T}} = (X, \mathscr{U}, \phi)$ be a forward complete live control system. The following statements are equivalent:

(i) $\Sigma^{\mathscr{T}}$ is ISS.
(ii) $\Sigma^{\mathscr{T}}$ is bUAG, CEP and BRS.
(iii) $\Sigma^{\mathscr{T}}$ is bULIM, ULS and BRS.

V. SPECIAL CLASSES OF LIVE SYSTEMS

To demonstrate the generality of our formalism, let us consider two special scenarios.

A. Open multi-agent systems

Open multi-agent systems (OMAS) are multi-agent systems with a time-varying number of agents, which may grow to infinity with time. Let S be a (finite or infinite) set, which we call *index set*, that labels all possible components of the network. Let for each $i \in S$ a Euclidean space X_i endowed with the norm $\|\cdot\|_{X_i}$ be the state space of the i-th agent.

As a configuration set, we take a certain subset $\hat{S}$ of all finite nonempty subsets of S. Let $Q \in \hat{S}$. In what follows, we denote by $(x_j)_{j\in Q}$ the vector consisting of $x_j \in X_j$ for $j \in Q$. Define the vector space

$$X_Q := \bigtimes_{j\in Q} X_j, \quad (10)$$

and endow it with a certain norm $\|\cdot\|_{X_Q}$ making X_Q a (real) normed linear space.

We define the state set X for the OMAS Σ by (3).

As before, we denote for each time instant t the configuration of the system at time t by $I(t) \in \hat{S}$. The state space of the system at time t is thus given by $X_{I(t)} \in X$. Also, we take $\mathscr{T}$ as in Assumption 2.3.

Let for all $t \in \mathbb{R}_+ \backslash \mathscr{T}$ the dynamics of the system be given by the following equations:

$$\Sigma: \quad \dot{x}_i(t) = f_i(x(t), u_i(t)), \quad i \in I(t), \quad t > 0. \quad (11)$$

Here $f_i : X \times \mathbb{R}^{m_i} \to X_i$ is a map that is defined on the whole state set. We assume that $u_i \in L_\infty(\mathbb{R}_+, \mathbb{R}^{m_i})$, for all $i \in S$, and define the total input to the system Σ as $u := (u_i)$, with $\|u\|_{\mathscr{U}} := \sup_{i\in S} \|u_i\|_\infty$. For the simplicity of notation, we also assume that all the signals u_i are right-continuous.

As for any $k \in \mathbb{N}$ and any $t \in (t_k, t_{k+1})$, it holds that $x(t) \in X_{I(t_k)}$, the dynamics of the system Σ can be equivalently represented for $t \in (t_k, t_{k+1})$ by

$$\dot{x}(t) = f_{I(t_k)}(x(t), u(t)) := (f_i(x(t), u_i(t)))_{i\in I(t_k)}. \quad (12)$$

Now $f_{I(t_k)}$ acts on $X_{I(t_k)} \times \mathscr{U}$, and thus (12) is a "usual" ODE representing the dynamics of the overall system on (t_k, t_{k+1}). We denote the system Σ in the configuration $I(t)$ by $\Sigma_{I(t)}$. The state space of this configuration is $X_{I(t)}$. We understand the solutions of (12) in the sense of Caratheodory.

Assumption 5.1: We suppose that Σ_Q is well-posed for any configuration Q. That is, for each initial condition $x_0 \in X_{I(t_k)}$ and any $u \in \mathscr{U}$, there exists a unique maximal solution of (12) with $x(t_k) = x_0$ on $[t_k, \tau)$ for some $\tau \in (t_k, t_{k+1})$. We denote this solution as $\hat{\phi}(\cdot, t_k, x_0, u)$.

At impulse times, the configuration may change as some subsystems may leave the system, and others may enter the system. We denote by $D(t_k) \subset I(t_k^-)$ the set of indices of all subsystems that are leaving at time t_k, and by $B(t_k)$ the indices of subsystems that enter the system at time t_k.

Assumption 5.2: We assume that

$$B(t_k) \cap I(t_k^-) = \emptyset,$$

that is, only subsystems which were not a part of the network immediately prior to time t_k can join the system.

Thus, at impulse times t_k, $k \in \mathbb{N}$, we have that

$$I(t_k) = \big(I(t_k^-) \cup B(t_k)\big) \backslash D(t_k).$$

We assume that $I(t_k) \in \hat{S}$ for all $k \in \mathbb{N}$, which means that the configuration of the system remains admissible for all times.

At impulse time t_k, the subsystems with indices belonging to $B(t_k)$ are entering the network Σ. We treat their initial conditions as an input to the system:

$$x_i(t_k) = u_i(t_k), \quad i \in B(t_k). \tag{13}$$

The states that remain in the network may jump instantaneously at time t_k:

$$x_i(t_k) = g_i(x(t_k^-), u(t_k^-)), \quad i \in I(t_k^-) \backslash D(t_k). \tag{14}$$

We treat the initial states of new subsystems entering or leaving the network as an (impulsive) input to the system, similar to instantaneous changes in the states of subsystems within the network.

OMAS, as defined previously, constitute an important special case of live systems. On the other hand, this subclass is still quite general. In particular, if there is only one agent in the network, and thus the configuration does not change with time, we obtain a classical impulsive system. Other special cases of OMAS are systems with an unknown initial configuration and switched systems.

Remark 5.3: *(System with an unknown initial configuration).* All our previous considerations are also valid for the empty time sequence $\mathscr{T}$. In this case, the system remains in the initial configuration for any initial condition and at the whole interval of existence. However, the initial configuration of the system is not fixed, and thus such a live system Σ can be interpreted as a "classical" system with an unknown initial configuration. Furthermore, Σ is ISS if and only if each configuration Σ_Q, $Q \subset S$ is ISS, and the corresponding functions β_Q and γ_Q can be chosen uniformly in Q. If S is a finite set, then clearly, Σ is ISS iff each Σ_Q is ISS.

VI. Lyapunov methods

The concept of ISS Lyapunov functions can be naturally extended to live systems:

Definition 6.1: A continuous map $V : X \to \mathbb{R}_+$ is called an *ISS Lyapunov function* for Σ if $\exists\ \psi_1, \psi_2 \in \mathscr{K}_\infty$, such that

$$\psi_1(\|x\|_X) \le V(x) \le \psi_2(\|x\|_X), \quad x \in X \tag{15}$$

and $\exists \chi \in \mathscr{K}_\infty$, $\alpha \in \mathscr{P}$ and continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ with $\varphi(x) = 0 \Leftrightarrow x = 0$, such that $\forall x \in X$, $\forall \xi \in U$ it holds

$$V(x) \ge \chi(\|\xi\|_U) \Rightarrow \begin{cases} \dot{V}_u(x) \le -\varphi(V(x)) \\ V(g(x,\xi)) \le \alpha(V(x)), \end{cases} \tag{16}$$

for all $u \in \mathscr{U}$ with $u(0) = \xi$. Here g is given by (4) (and we assume it to be independent on time).

For a given input $u \in \mathscr{U}$, the Lie derivative is defined by

$$\dot{V}_u(x) = \overline{\lim_{t \to +0}} \frac{1}{t}\big(V(\phi(t,x,u)) - V(x)\big). \tag{17}$$

If in addition

$$\varphi(s) = cs \text{ and } \alpha(s) = e^{-d}s \tag{18}$$

for some $c, d \in \mathbb{R}$, then V is called *exponential ISS Lyapunov function with rate coefficients c, d.*

For any given impulse time sequence, the configuration of Σ does not change for small enough times. Thus $\dot{V}_u(x)$ does not depend on the sequence of impulse times.

For a given sequence of impulse times, denote by $N(t,s)$ the number of jumps within the interval $(s,t]$.

Theorem 6.2: Let V be an exponential ISS Lyapunov function for Σ with corresponding coefficients $c \in \mathbb{R}$, $d \neq 0$. For arbitrary function $h : \mathbb{R}_+ \to (0, \infty)$, for which there exists $g \in \mathscr{L}$: $h(x) \le g(x)$ for all $x \in \mathbb{R}_+$ consider the class $\mathscr{S}[h]$ of impulse time-sequences, satisfying the generalized average dwell-time (gADT) condition:

$$-dN(t,s) - c(t-s) \le \ln h(t-s), \quad \forall t \ge s \ge t_0. \tag{19}$$

Then the system Σ is uniformly ISS over $\mathscr{S}[h]$.

A. Illustrative example

Consider an infinite number of identical exponentially stable subsystems given by

$$\dot{x}_i = Ax_i, \tag{20}$$

where $i \in \mathbb{N}$, $A \in \mathbb{R}^{s \times s}$ is a Hurwitz matrix, for a certain $s \in \mathbb{N}$.

Thus, $S := \mathbb{N}$, $X_i := \mathbb{R}^s$ for all $i \in S$, and we endow each X_i with Euclidean norm. Now we set for each $Q \subset S$

$$\|x\|_{X_Q} := \Big(\sum_{j \in Q} |x_j|^2\Big)^{1/2}, \quad x \in X_Q.$$

Consider an impulse time sequence $\mathscr{T} := (t_k)_{k \in \mathbb{N}}$, and assume that at each time t_k one new agent of the form (20) is entering the system. The initial state of the newly added agent at time t_k is $u(t_k)$.

We are going to study the ISS of this system.

As A is Hurwitz, there is a positive matrix $P = P^T \in \mathbb{R}^{s \times s}$ such that the Lyapunov inequality holds:

$$PA + A^T P \le -I.$$

For any state $x \in X$, let $Q = Q(x) \subset S$ be such that $x \in X_Q$. Define the ISS Lyapunov function candidate as

$$V(x) := \sum_{j \in Q(x)} x_j^T P x_j, \quad x \in X. \tag{21}$$

Clearly, the following sandwich estimates hold:

$$\begin{aligned} \lambda_{\min}(P)\|x\|_X^2 &= \lambda_{\min}(P)\|x\|_{X_Q}^2 \\ &\le V(x) \le \|P\|\|x\|_{X_Q}^2 = \|P\|\|x\|_X^2, \end{aligned}$$

where $\lambda_{\min}(P) > 0$ is the minimal eigenvalue of P.

Take any $Q \subset S$ and any $x \in X_Q$. Computing the Lie derivative of V along the trajectories, we see that

$$\begin{aligned} \dot{V}(x) &= \sum_{j \in Q} x_j^T (PA + A^T P) x_j \\ &\le -\sum_{j \in Q} x_j^T x_j = -\|x\|_X^2 \le -\frac{1}{\|P\|} V(x). \end{aligned} \tag{22}$$

Consider the impulsive dynamics. At each time t_k, a new subsystem enters a system Σ, with the initial state $u(t_k)$, and thus the new state is

$$x(t_k) = g(x(t_k^-), u(t_k)) := (x(t_k^-), u(t_k)).$$

Thus, the dynamics of V at impulse times is

$$V((x,u)) = V(x) + u^T P u \le V(x) + \|P\||u|^2.$$

This implies that

$$\|P\||u|^2 \le \varepsilon V(x) \quad \Rightarrow \quad V((x,u)) \le (1+\varepsilon)V(x). \qquad (23)$$

Taking $c := \|P\|^{-1}$, $d := -\ln(1+\varepsilon)$, and any h as in the formulation of Theorem 6.2, Theorem 6.2 ensures UISS of our system over the class of impulse time sequences $\mathscr{S}[h]$ satisfying (19).

VII. Discussion

The framework of live systems allows us to formulate and analyze new approaches for control.

One of the applications of live systems theory is the development of *plug-and-play control* methods that allow the controllers to work properly even though some parts of the system may detach from the system or become unfunctional or, conversely, some new parts/agents may join the network. In particular, one could consider the synchronization of open multi-agent systems, distributed control, and observation over networks with varying topology, etc.

In plug-and-play control, a system can *react to the changes* of the system configuration. A complementary problem is controlling a system by *inducing the changes* of the system configuration. Examples are adding drones to the drones flock, deployment of new military units to the battlefield, adding predators to counteract parasites, etc. One can, however, go one step further and consider *self-governing systems*, which redesign themselves by adding new elements, sensors, actuators, estimators, etc. As an example, consider optimal allocation models, which are the most advanced class of mathematical models for the modeling of life histories of living organisms; see [19] and the references therein. In a basic setting of these models, one considers a plant consisting of a fixed number of compartments and assumes that the plant can control itself to maximize fitness. These models neglect the modularity, which makes them less exact and leads to false predictions [6]. By employing live systems theory, one could allow a plant to control not only the allocation of energy to existing compartments but also to control which modules have to be produced (brackets, flowers, leaves, etc.), at which time, and in which order.

References

[1] K. J. Åström and B. Wittenmark. *Adaptive Control*. Dover, 2008.

[2] M. Blanke, M. Kinnaert, J. Lunze, M. Staroswiecki, and J. Schröder. *Diagnosis and Fault-tolerant Control*, volume 2. Springer, 2006.

[3] S. Dashkovskiy and A. Mironchenko. Input-to-state stability of nonlinear impulsive systems. *SIAM Journal on Control and Optimization*, 51(3):1962–1987, 2013.

[4] F. Dörfler, J. W. Simpson-Porco, and F. Bullo. Plug-and-play control and optimization in microgrids. In *Proc. of 53rd IEEE Conference on Decision and Control*, pages 211–216. IEEE, 2014.

[5] X. Fang, S. Misra, G. Xue, and D. Yang. Smart grid-the new and improved power grid: A survey. *IEEE Communications Surveys & Tutorials*, 14(4):944–980, 2011.

[6] G. Fox. Annual plant life histories and the paradigm of resource allocation. *Evolutionary Ecology*, 6(6):482–499, 1992.

[7] M. Franceschelli and P. Frasca. Stability of open multiagent systems and applications to dynamic consensus. *IEEE Transactions on Automatic Control*, 66(5):2326–2331, 2020.

[8] R. Goebel, R. Sanfelice, and A. R. Teel. *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. Princeton University Press, 2012.

[9] S. Grauwin and P. Jensen. Opinion group formation and dynamics: Structures that last from nonlasting entities. *Physical Review E*, 85(6):066113, 2012.

[10] J. M. Hendrickx and S. Martin. Open multi-agent systems: Gossiping with deterministic arrivals and departures. In *2016 54th Annual Allerton Conference on Communication, Control, and Computing (Allerton)*, pages 1094–1101. IEEE, 2016.

[11] J. P. Hespanha, D. Liberzon, and A. R. Teel. Lyapunov conditions for input-to-state stability of impulsive systems. *Automatica*, 44(11):2735–2744, 2008.

[12] R. E. Kalman, P. L. Falb, and M. A. Arbib. *Topics in Mathematical System Theory*. McGraw-Hill New York, 1969.

[13] C. M. Kellett. A compendium of comparison function results. *Mathematics of Control, Signals, and Systems*, 26(3):339–374, 2014.

[14] M. Krstic, I. Kanellakopoulos, and P. V. Kokotovic. *Nonlinear and Adaptive Control Design*. Wiley, 1995.

[15] F. Lamnabhi-Lagarrigue, A. Annaswamy, S. Engell, A. Isaksson, P. Khargonekar, R. M. Murray, H. Nijmeijer, T. Samad, D. Tilbury, and P. Van den Hof. Systems & control for the future of humanity, research agenda: Current and future roles, impact and grand challenges. *Annual Reviews in Control*, 43:1–64, 2017.

[16] J. L. Mancilla-Aguilar, J. E. Rojas-Ruiz, and H. Haimovich. Characterization of integral input-to-state stability for nonlinear time-varying systems of infinite dimension. *SIAM Journal on Control and Optimization*, 61(4):1979–2003, 2023.

[17] A. Mironchenko. Non-uniform ISS small-gain theorem for infinite networks. *IMA Journal of Mathematical Control and Information*, 38(4):1029–1045, 2021.

[18] A. Mironchenko. *Input-to-State Stability: Theory and Applications*. Springer Nature, 2023.

[19] A. Mironchenko and J. Kozłowski. Optimal allocation patterns and optimal seed mass of a perennial plant. *Journal of Theoretical Biology*, 354:12–24, 2014.

[20] A. Mironchenko and C. Prieur. Input-to-state stability of infinite-dimensional systems: Recent results and open questions. *SIAM Review*, 62(3):529–614, 2020.

[21] A. Mironchenko and F. Wirth. Characterizations of input-to-state stability for infinite-dimensional systems. *IEEE Transactions on Automatic Control*, 63(6):1602–1617, 2018.

[22] C. Monnoyer de Galland and J. M. Hendrickx. Fundamental performance limitations for average consensus in open multi-agent systems. *IEEE Transactions on Automatic Control*, 68(2):646–659, 2022.

[23] N. Noroozi, A. Mironchenko, and F. Wirth. A relaxed small-gain theorem for discrete-time infinite networks. *Automatica*, 142:110363, 2022.

[24] R. J. Patton. Fault-tolerant control: the 1997 situation. *IFAC Proceedings Volumes*, 30(18):1029–1051, 1997.

[25] I. Pinyol and J. Sabater-Mir. Computational trust and reputation models for open multi-agent systems: A review. *Artificial Intelligence Review*, 40(1):1–25, 2013.

[26] E. D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Transactions on Automatic Control*, 34(4):435–443, 1989.

[27] E. D. Sontag. Input to state stability: Basic concepts and results. In *Nonlinear and Optimal Control Theory*, chapter 3, pages 163–220. Springer, Heidelberg, 2008.

[28] J. Stoustrup. Plug & play control: Control technology towards new challenges. *European Journal of Control*, 15(3-4):311–330, 2009.

[29] V. S. Varma, I. Morărescu, and D. Nešić. Open multi-agent systems with discrete states and stochastic interactions. *IEEE Control Systems Letters*, 2(3):375–380, 2018.

[30] E. I. Verriest. Pseudo-continuous multi-dimensional multi-mode systems: Behavior, structure and optimal control. *Discrete Event Dynamic Systems*, 22:27–59, 2012.

[31] R. Vizuete. *Contributions to open multi-agent systems: consensus, optimization and epidemics*. PhD thesis, Université Paris-Saclay, 2022.