2024 IEEE 63rd Conference on Decision and Control (CDC)
December 16-19, 2024. MiCo, Milan, Italy

Multiplier analysis of Lurye systems with power signals

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***Abstract*— Multipliers can be used to guarantee both the Lyapunov stability and input-output stability of Lurye systems with time-invariant memoryless slope-restricted nonlinearities. If a dynamic multiplier is used there is no guarantee the closed-loop system has finite incremental gain. It has been suggested in the literature that without this guarantee such a system may be critically sensitive to time-varying exogenous signals including noise. We show that multipliers guarantee the power gain of the system to be bounded and quantifiable. Furthermore power may be measured about an appropriate steady state bias term, provided the multiplier does not require the nonlinearity to be odd. Hence dynamic multipliers can be used to guarantee Lurye systems have low sensitivity to noise, provided other exogenous systems have constant steady state. We illustrate the analysis with an example where the exogenous signal is a power signal with non-zero mean.**

I. INTRODUCTION

We are concerned with Lurye systems where the nonlinearity is memoryless, time-invariant and slope-restricted. The existence of a suitable OZF (O'Shea-Zames-Falb) multiplier can be used to guarantee finite-gain $\mathcal{L}_2$ stability [1]–[3]. Modern tools can search for suitable multipliers and give an upper bound on the $\mathcal{L}_2$ gain [3]–[6].

It has been argued in the literature both specifically with respect to Lurye systems [7], [8] and more generally [9]–[13] that emphasis should be given to finite incremental gain. In particular if exogenous signals are not in $\mathcal{L}_2$ then desirable properties one might infer for linear systems do not necessarily carry over to nonlinear systems. Zames [9] argues that a definition of closed-loop stability should require both continuity and boundedness, inter alia so that outputs are not "critically sensitive to small changes in inputs — changes such as those caused by noise"; Theorem 3 in [9] gives conditions on the incremental positivity of the loop elements that are sufficient to achieve this. Similarly it was known at the time [14] that the circle criterion could be used to guarantee "the existence of unique steady-state oscillations (= absence of "jump phenomena" and subharmonics) in forced nonlinear feedback systems". It was subsequently established [7] that dynamic multipliers do not, in general, preserve the positivity of nonlinearities. As noted in [7] there is some irony that the definition of stability used by Zames and Falb in [1] does *not* require closed-loop continuity.

Certainly lack of finite incremental gain can lead to undesirable effects. A construction for a system where the input period is not preserved is given in [15] (although the specific example is closed-loop unstable). An example of a Lurye system where small changes in input can lead to significant changes in output is given in [16]. We discuss a further example in this paper. However, we argue that finite incremental gain is not necessary to ensure insensitivity to noise signals. In particular the existence of a suitable OZF multiplier can be used to guarantee a Lurye system is insensitive to noise for a wide class of exogenous signal. This is timely in that OZF multipliers have recently been proposed for the design of control systems with saturation [17], [18].

In this paper we characterise the response of such Lurye systems for two classes of exogenous signals that are not in $\mathcal{L}_2$. Specifically we consider power signals and signals with a constant bias. We observe that finite-gain stability ensures small power noise input leads to small power output when all other exogenous signals are in $\mathcal{L}_2$ (Theorem 3). We define a notion of finite-gain offset stability (Definition 2) and observe similarly that finite-gain offset stability ensures small power noise input leads to small power output when all other exogenous signals are bias signals (Theorem 4). Our main result (Theorem 5) is to show that, provided we do not exploit any oddness of the nonlinearity, the existence of a suitable OZF multiplier guarantees such properties. Corresponding discrete-time results follow immediately and we illustrate the results with a discrete-time example.

II. PRELIMINARIES

A. Signals and systems

Let $\mathcal{L}_2$ be the space of finite energy Lebesgue integrable signals on $[0,\infty)$ with norm

$$\|y\| = \left(\int_0^\infty y(t)^2\,dt\right)^{\frac{1}{2}}. \tag{1}$$

Let $\mathcal{L}_{2e}$ be the corresponding extended space (see for example [2]). The truncation $y_T \in \mathcal{L}_2$ of $y \in \mathcal{L}_{2e}$ is given by

$$y_T(t) = \begin{cases} y(t) & \text{for} \quad 0 \leq t \leq T, \\ 0 & \text{for} \quad T < t. \end{cases} \tag{2}$$

Definition 1. *Let $\mathcal{P} \subset \mathcal{L}_{2e}$ be the space of finite power Lebesgue integrable signals on $[0,\infty)$ with seminorm*

$$\|y\|_P = \left(\limsup_{T\to\infty} \frac{1}{T}\|y_T\|^2\right)^{\frac{1}{2}}. \tag{3}$$

We say y is a power signal if $y \in \mathcal{P}$. Let $\theta \in \mathcal{P}$ be the Heaviside step function given by $\theta(t) = 1$ for all $t > 0$.

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Define the bias $\bar{y} \in \mathbb{R}$ of a signal $y \in \mathcal{P}$ as

$$\bar{y} = \arg\min_{\bar{y}\in\mathbb{R}} \|(y - \bar{y}\theta)\|_P. \quad (4)$$

We say y is an $\mathcal{L}_2$-bias signal with bias $\bar{y}$ if $\bar{y}$ is unique and $y - \bar{y}\theta \in \mathcal{L}_2$.

Remark 1. *The limit superior in (3) does not appear in standard definitions of power (e.g. [19], [20]) but is necessary to ensure $\mathcal{P}$ is a vector space [21], [22]*[1].

A map $H : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ is stable if $u \in \mathcal{L}_2$ implies $Hu \in \mathcal{L}_2$. It is finite-gain stable (FGS) if there is some $h < \infty$ such that $\|Hu\| \leq h\|u\|$ for all $u \in \mathcal{L}_2$. Its gain is the smallest such h.

Remark 2. *Our definition of finite-gain stability carries the assumption that we have zero initial conditions. Non-zero initial conditions can be accommodated provided they can be represented with a nonlinear state-space description that is reachable and uniformly observable [19].*

Definition 2. *Let $H : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$. We say H is offset stable if there is some function $H_0 : \mathbb{R} \to \mathbb{R}$ such that if u is an $\mathcal{L}_2$-bias signal with bias $\bar{u}$ then Hu is an $\mathcal{L}_2$-bias signal with bias $H_0(\bar{u})$. We call H_0 the steady state map of H. Define $H_{\bar{u}} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ as*

$$H_{\bar{u}}u = H(u + \bar{u}\theta) - H_0(\bar{u}). \quad (5)$$

It follows that H is offset stable if $H_{\bar{u}}$ is stable for all $\bar{u} \in \mathbb{R}$. We say H is finite-gain offset stable (FGOS) if there is some $h < \infty$ such that $H_{\bar{u}}$ is FGS with gain less than or equal to h for all $\bar{u} \in \mathbb{R}$. We call the minimum such h the offset gain of H.

B. Lurye systems

We are concerned with the behaviour of the Lurye system (Fig. 1) given by

$$y_1 = \boldsymbol{G}u_1,\ y_2 = \boldsymbol{\phi}u_2,\ u_1 = r_1 - y_2 \text{ and } u_2 = y_1 + r_2. \quad (6)$$

The Lurye system (6) is assumed to be well-posed with $\boldsymbol{G} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ linear time invariant (LTI) causal and stable, and with $\boldsymbol{\phi} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ memoryless and time-invariant. We say such a memoryless, time-invariant $\boldsymbol{\phi}$ is characterised by $N : \mathbb{R} \to \mathbb{R}$ with $(\boldsymbol{\phi}(u))(t) = N(u(t))$. We will use G to denote the transfer function corresponding to $\boldsymbol{G}$. Where appropriate we will consider either $G : j\mathbb{R} \to \mathbb{C}$ (i.e. $G(j\omega)$) or $G : \bar{\mathbb{C}}_+ \to \mathbb{C}$ (i.e. $G(s)$) where $\bar{\mathbb{C}}_+ = \{s \in \mathbb{C} : Re(s) \geq 0\}$.

The nonlinearity $\boldsymbol{\phi}$, characterised by N, is assumed to be monotone in the sense that $N(x_1) \geq N(x_2)$ for all $x_1 \geq x_2$. It is also assumed to be bounded in the sense that there exists a $C \geq 0$ such that $|N(x)| \leq C|x|$ for all $x \in \mathbb{R}$. We say $\boldsymbol{\phi}$ is slope-restricted on $[0, k]$ if $0 \leq (N(x_1) - N(x_2))/(x_1 - x_2) \leq k$ for all $x_1 \neq x_2$. We say $\boldsymbol{\phi}$ is odd if $N(x) = -N(-x)$ for all $x \in \mathbb{R}$.

[1]We are grateful to Andrey Kharitenko for this observation.

The Lurye system is stable if $r_1, r_2 \in \mathcal{L}_2$ implies $u_1, u_2, y_1, y_2 \in \mathcal{L}_2$. It is FGS if there is some $h < \infty$ such that

$$\|y_i\| \leq h(\|r_1\| + \|r_2\|) \text{ and } \|u_i\| \leq h(\|r_1\| + \|r_2\|), \text{ for } i = 1, 2 \text{ and for all } r_1, r_2 \in \mathcal{L}_2. \quad (7)$$

Remark 3. *Non-zero initial conditions can be accommodated in our definition of finite gain stability for such a Lurye system (c.f. Remark 2). Specifically, since the nonlinearity $\boldsymbol{\phi}$ is Lipschitz and the LTI transfer function G admits a minimal state-space representation, non-zero initial conditions can be accommodated by extending the time line backwards and including some fictitious exogenous signal over this extension. See [19], pp290-291.*

Since $\boldsymbol{G}$ is LTI stable we can set $r_1 = 0$ without loss of generality, and it is sufficient for finite-gain stability that there is some $h < \infty$ with $\|y_2\| \leq h\|r_2\|$. We will denote H as the map from r_2 to y_2 and define the gain of the Lurye system to be the gain of H. Similarly for offset stability and offset gain.

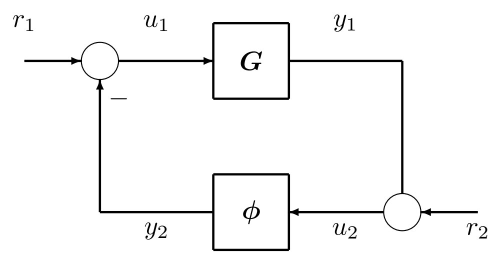

Fig. 1. Lurye system.

C. Multiplier theory

Definition 3 ([1]). *Let $\mathcal{M}$ be the class of transfer functions $M : j\mathbb{R} \to \mathbb{C}$ of systems whose (possibly non-causal) impulse response is given by*

$$m(t) = m_0\delta(t) - h(t) - \sum_{i=1}^{\infty} h_i\delta(t - t_i), \quad (8)$$

with $h(t) \geq 0$ for all $t \in \mathbb{R}$, $h_i \geq 0$ for all i and

$$\|h\|_1 + \sum_{i=1}^{\infty} |h_i| \leq m_0. \quad (9)$$

We say M is an OZF multiplier if $M \in \mathcal{M}$.

Definition 4 ([1]). *Let $\mathcal{M}_{odd}$ be the class of transfer functions $M : j\mathbb{R} \to \mathbb{C}$ of systems whose (possibly non-causal) impulse response is given by (8) with (9). We say M is an OZF multiplier for odd nonlinearities if $M \in \mathcal{M}_{odd}$.*

Definition 5. *Let $M : j\mathbb{R} \to \mathbb{C}$ and let $G : j\mathbb{R} \to \mathbb{C}$. We say M is suitable for G if there exists $\varepsilon > 0$ such that*

$$Re\{M(j\omega)G(j\omega)\} > \varepsilon \text{ for all } \omega \in \mathbb{R}. \quad (10)$$

Theorem 1 ([1], [2]). *If there is an $M \in \mathcal{M}$ ($\mathcal{M}_{odd}$) suitable for G then the Lurye system (6) is FGS for any memoryless* *time-invariant (odd) monotone bounded nonlinearity ϕ. Furthermore, if there is an $M \in \mathcal{M}$ ($\mathcal{M}_{odd}$) suitable for $1/k+G$ then the Lurye system (6) is FGS for any memoryless time-invariant (odd) slope-restricted nonlinearity ϕ in $[0, k]$.*

Remark 4. *We have used a non-strict inequality in (9). While the early literature uses a strict inequality [1], [2], [23] some modern literature (e.g. [24]) uses a non-strict inequality while some (e.g. [3]) retains the strict inequality. The distinction is discussed in [25].*

D. Continuity

Definition 6. *[9] The incremental gain of $H : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ is the supremum of $\|(Hx)_T - (Hy)_T\|/\|x_T - y_T\|$ over all $x, y \in \mathcal{L}_{2e}$ and all $T > 0$ for which $\|x_T - y_T\| \neq 0$.*

Finite-gain stability does not guarantee finite incremental gain: if H is FGS and $u_1, u_2 \in \mathcal{L}_2$ the ratio $R = \|Hu_1 - Hu_2\|/\|u_1 - u_2\|$ may be arbitrarily large. In particular, suppose v_1, v_2 are power signals with $v_1 - v_2 \in \mathcal{L}_2$ but $v_1, v_2 \notin \mathcal{L}_2$ and suppose H is FGS but $Hv_1 - Hv_2 \notin \mathcal{L}_2$. Let u_1, u_2 be the truncations $u_1 = (v_1)_T$ and $u_2 = (v_2)_T$. Then $R \to \infty$ as $T \to \infty$.

As noted in the Introduction, dynamic multipliers do not, in general, preserve the positivity of nonlinearities [7]. An example of a Lurye system where multipliers guarantee finite gain stability but where Lipschitz continuity is lost is given in [16].

III. Power analysis

In this section we consider FGS systems. The application to Lurye systems where finite-gain stablilty is guaranteed by the existance of a suitable OZF multiplier is immediate.

Zames [9] argues that "in order to behave properly" a system's "outputs must not be critically sensitive to small changes in inputs - changes such as those caused by noise." Further he argues that the input-output map must be continuous to ensure it is "not critically sensitive to noise." Here we show that if the noise is a power signal then finite-gain stability suffices.

The following is standard, at least for linear systems [20].

Theorem 2. *Suppose $u \in \mathcal{P}$ and $y = Hu$ where H is FGS with gain h. Then $y \in \mathcal{P}$ with $\|y\|_P \leq h\|u\|_P$.*

Proof: We find

$$\begin{aligned}\|y\|_P^2 &= \limsup_{T\to\infty} \frac{1}{T}\|(Hu)_T\|^2,\\ &= \limsup_{T\to\infty} \frac{1}{T}\|(Hu_T)_T\|^2 \text{ since } H \text{ is causal},\\ &\leq \limsup_{T\to\infty} \frac{1}{T}\|Hu_T\|^2,\\ &\leq \limsup_{T\to\infty} \frac{1}{T}h^2\|u_T\|^2,\\ &= h^2\|u\|_P^2. \quad \square \end{aligned} \tag{11}$$

Hence if we define the power gain of H to be

$$h_P = \sup_{u\in\mathcal{P}, \|u\|_P>0} \frac{\|Hu\|_P}{\|u\|_P}, \tag{12}$$

then $h_P \leq h$.

Suppose $u_1, u_2 \in \mathcal{P}$ and H is FGS with gain h. Since $\|\cdot\|_P$ is a seminorm we have the triangle inequalities

$$\begin{aligned}&\|Hu_1 \pm Hu_2\|_P^2\\ &\quad\leq \|Hu_1\|_P^2 + \|Hu_2\|_P^2 + 2\|Hu_1\|_P\|Hu_2\|_P\\ &\quad\leq h^2\left[\|u_1\|_P^2 + \|u_2\|_P^2 + 2\|u_1\|_P\|u_2\|_P\right],\end{aligned} \tag{13}$$

and

$$\begin{aligned}\|H(u_1+u_2)\|_P^2 &\leq h^2\|u_1 + u_2\|^2\\ &\leq h^2\left[\|u_1\|_P^2 + \|u_2\|_P^2 + 2\|u_1\|_P\|u_2\|_P\right].\end{aligned} \tag{14}$$

In particular we may say:

Theorem 3. *Suppose $H : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ is FGS with gain h. Suppose further that $u_1 \in \mathcal{L}_2$ and $u_2 \in \mathcal{P}$. Then*

$$\|H(u_1 + u_2)\|_P \leq h\|u_2\|_P. \tag{15}$$

Proof: The result follows immediately from (14) since $\|u_1\|_P = 0$. $\square$

Hence, if we add small power noise to an $\mathcal{L}_2$ input signal then the output power is small provided the system is FGS, irrespective of the continuity of the input-output map.

IV. Bias signal analysis

Theorem 3 has an immediate counterpart when u_1 is a bias signal and H is FGOS.

Theorem 4. *Suppose $H : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ is FGOS with steady state map H_0 and offset gain h. Suppose further that u_1 is a bias signal with bias $\bar{u}_1$ and $u_2 \in \mathcal{P}$. Then*

$$\|H(u_1 + u_2) - H_0(\bar{u}_1)\|_P \leq h\|u_2\|_P. \tag{16}$$

Proof: The result follows from Theorem 3 since

$$\|H(u_1 + u_2) - H_0(\bar{u}_1)\|_P = \|H_{\bar{u}}(u_1 - \bar{u}_1\theta + u_2)\|_P, \tag{17}$$

where $H_{\bar{u}}$ is given by Definition 2. $\square$

Hence, if we add small power noise to a input bias signal then the output power (measured about the noise-free output bias) is small provided the system is FGOS, irrespective of the continuity of the input-output map.

The application to a Lurye system is immediate provided we can show the system is FGOS. It turns out that the existence of a suitable OZF multiplier $M \in \mathcal{M}$ guarantees this, but the existence of a suitable $M \in \mathcal{M}_{\text{odd}}$ does not.

Theorem 5. *If there is an $M \in \mathcal{M}$ suitable for G then the Lurye system (6) is FGOS for any memoryless time-invariant monotone bounded nonlinearity ϕ. If there is an $M \in \mathcal{M}$ suitable for $1/k + G$ then the Lurye system (6) is FGOS for any memoryless time-invariant slope-restricted nonlinearity ϕ in $[0, k]$.*

Proof: Without loss of generality suppose $r_1 = 0$ and let H be the map from r_2 to y_2. Let r_2 be a bias signal with bias $\bar{r}_2$. Suppose first that r_2 is constant $r_2 = \bar{r}_2\theta$ and G is a fixed gain $G = \bar{G}$. The monotonicity of ϕ ensures there is a unique fixed solution $y_1 = \bar{y}_1\theta$ [2]. This defines our candidate bias function $H_o(\bar{r}_2) = \bar{y}_1$. The input to the nonlinearity is $u_2 = \bar{u}_2\theta$ where $\bar{u}_2 = \bar{r}_2 + \bar{y}_1$ and its output is $y_2 = \bar{y}_2\theta$ where $\bar{y}_2 = N(\bar{u}_2)$ and the nonlinearity ϕ is characterised by $N : \mathbb{R} \to \mathbb{R}$. We can define a memoryless, time-invariant, bounded and monotone nonlinearity $\phi_{\bar{r}_2}$ characterised by $N_{\bar{r}_2}$ where

$$N_{\bar{r}_2}(x) = N(x + \bar{u}_2) - \bar{y}_2 \text{ for all } x \in \mathbb{R}. \tag{18}$$

This in turn defines a normalized Lurye system with linear element G and nonlinear element $\phi_{\bar{r}_2}$. Denote $H_{\bar{r}_2}$ as the map from $r_2 - \bar{r}_2\theta$ to $y_1 - \bar{y}_1\theta$. If $M \in \mathcal{M}$ is suitable for G then $H_{\bar{r}_2}$ is FGS. It follows that H is FGOS.

If ϕ is in addition slope-restricted on $[0, k]$ then $\phi_{\bar{r}_2}$ is also slope-restricted on $[0, k]$. □

Remark 5. *There is no corresponding result when $M \in \mathcal{M}_{odd} - \mathcal{M}$. In this case $\phi_{\bar{r}_2}$ need not be odd even if ϕ is odd. Hence if $M \in \mathcal{M}_{odd} - \mathcal{M}$ is suitable for G there is no guarantee that $H_{\bar{r}_2}$ is stable.*

Remark 6. *Suppose M is suitable for G (or for $1/k + G$) and can be used to ensure the $\mathcal{L}_2$ gain of H is bounded above by h. It follows from the proof of Theorem 5 that the offset gain of H is also bounded above by h.*

V. Example

A. Discrete time preliminaries

Although our development has been for continuous-time systems, we illustrate with a discrete-time example. Discrete-time counterparts to our results follow immediately, with the spaces ℓ of real-valued sequences $h : \mathbb{Z}^+ \to \mathbb{R}$ and ℓ_2 of square-summable sequences $h : \mathbb{Z}^+ \to \mathbb{R}$ taking the places of $\mathcal{L}_{2e}$ and $\mathcal{L}_2$ respectively. Similarly ℓ_2 gain takes the place of $\mathcal{L}_2$ gain.

Specifically, let $\mathbb{D}$ denote the unit circle. Respective counterparts to Definitions 3, 4 and 5 are as follows.

Definition 7. *Let $\mathcal{M}^d$ be the class of discrete-time transfer functions $M : \mathbb{D} \to \mathbb{C}$ of systems whose (possibly non-causal) impulse response is given by*

$$m(t) = m_0\delta(t) - \sum_{i\neq 0} h_i\delta(t - i), \tag{19}$$

with $h_i \geq 0$ for all i and

$$\sum_{i\neq 0} |h_i| \leq m_0. \tag{20}$$

We say M is a discrete-time OZF multiplier if $M \in \mathcal{M}^d$.

Definition 8. *Let $\mathcal{M}^d_{odd}$ be the class of discrete-time transfer functions $M : \mathbb{D} \to \mathbb{C}$ of systems whose (possibly non-causal) impulse response is given by (19) with (20). We say M is a discrete-time OZF multiplier for odd nonlinearities if $M \in \mathcal{M}^d_{odd}$.*

Definition 9. *Let $M : \mathbb{D} \to \mathbb{R}$ and let $G : \mathbb{D} \to \mathbb{C}$. We say M is suitable for G if*

$$\text{Re}\left\{M(e^{j\omega})G(e^{j\omega})\right\} > 0 \text{ for all } \omega \in [0, 2\pi). \tag{21}$$

The counterpart to Theorem 1 is direct and standard [26], [27]. Direct counterparts to Theorems 2-5 follow immediately.

Remark 7. *Once again we have used a non-strict inequality in the definition of the OZF multipliers (20); see Remark 4. The original literature [26], [27] includes time-varying multipliers but this is unnecessary [28].*

B. The example

Consider the Lurye system (6) where ϕ is the saturation characterised by $N : \mathbb{R} \to \mathbb{R}$ with $N(x) = x$ when $|x| < 1$ and $N(x) = x/|x|$ when $|x| \geq 1$, and $\boldsymbol{G}$ has the discrete-time transfer function

$$G(z) = g\frac{2z + 0.92}{z(z - 0.5)}, \tag{22}$$

and where the gain g takes one of three values: $g = 0.6$, $g = 0.8$ or $g = 1$. NB we have considered this Lurye system previously with $g = 1$ [29], [30].

Following standard analysis (e.g. [3]), we may say that if $M \in \mathcal{M}^d_{odd}$ is suitable for G and

$$\begin{aligned} &2\text{Re}[M(e^{j\omega})(1 + G(e^{j\omega}))]\gamma^2 \\ &\quad - (|G(e^{j\omega})|^2 + |M(e^{j\omega})|^2)\gamma - 2\text{Re}[M(e^{j\omega})] > 0, \end{aligned} \tag{23}$$

for all $\omega \in [0, 2\pi)$ and for some $\gamma \geq 1$ then the l_2 gain from r_2 to u_2 is bounded above by γ. If $M \in \mathcal{M}^d \subset \mathcal{M}^d_{odd}$ then the offset gain is also bounded above by γ (see Remark 4).

- When $g = 0.6$ finite incremental gain of the Lurye system may be established via the circle criterion. Specifically $\text{Re}\left[1 + G(e^{j\omega})\right] > 0$ for all $\omega \in [0, 2\pi)$. Although dynamic multipliers are not required to establish ℓ_2 stability, they may still be useful to find reduced upper bounds on the ℓ_2 gain, and hence on the offset gain. In this case the circle criterion alone establishes an upper bound $h \leq 16.3156$ but the multiplier $M(e^{j\omega}) = 1 - 0.66e^{-j\omega}$ establishes an upper bound $h \leq 4.1795$.
- When $g = 0.8$ the circle criterion can no longer be applied, but there are multipliers $M \in \mathcal{M}^d$ suitable for G. It follows from the discrete-time counterpart of Theorem 5 that the Lurye system is FGOS. For example the multiplier $M = 1 - 0.85e^{-j\omega}$ is suitable for G and establishes an upper bound $h \leq 12.8983$ on the ℓ_2 gain, and hence on the offset gain.
- When $g = 1$ we find $\angle[1 + gG(e^{2\pi j/3})] = -\pi + \text{atan}\frac{31\sqrt{3}}{48} < -\frac{2\pi}{3}$. It follows from the phase limitations at single frequencies [31] that there is no $M \in \mathcal{M}^d$ suitable for G. Nevertheless there are multipliers $M \in \mathcal{M}^d_{odd}$ suitable for G. It follows that the Lurye system is FGS, but not necessarily FGOS in this case. For example the multiplier $M = 1 + 0.9e^{j\omega}$ is suitable for G and establishes an upper bound $h \leq 31.332$ on the ℓ_2 gain.

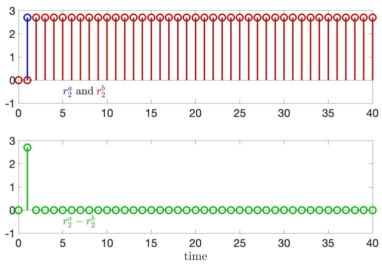

Fig. 2. Deterministic step signals r_2^a and r_2^b. Both are bias signals, with r_2^b a delayed version of r_2^a. Their difference $r_2^a - r_2^b \in \ell_2$.

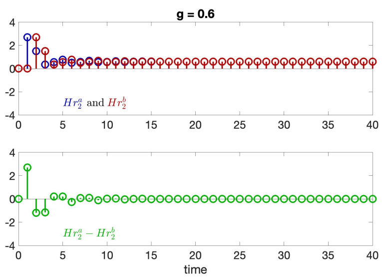

Fig. 3. Respective responses Hr_2^a and Hr_2^b to deterministic step signals r_2^a and r_2^b when the gain is $g = 0.6$. The circle criterion guarantees finite incremental gain and hence the difference $Hr_2^a - Hr_2^b \in \ell_2$.

C. Deterministic step response

Let r_2 be either the step response r_2^a or the delayed step response r_2^b given by

$$r_2^a(t) = \begin{cases} 0 & \text{when} \quad t = 0, \\ 2.7 & \text{when} \quad t \geq 1, \end{cases}$$
$$r_2^b(t) = \begin{cases} 0 & \text{when} \quad t = 0, 1, \\ 2.7 & \text{when} \quad t \geq 2. \end{cases} \tag{24}$$

Although $r_2^a \notin \ell_2$ and $r_2^b \notin \ell_2$ we have $r_2^a - r_2^b \in \ell_2$ (see Fig. 2).

When $g = 0.6$ (Fig. 3) the closed-loop system is incrementally stable so the difference $Hr_2^a - Hr_2^b \in \ell_2$. When $g = 0.8$ (Fig. 4) the results of this paper show that the closed-loop system is FGOS so once again $Hr_2^a - Hr_2^b \in \ell_2$. By contrast, when $g = 1$ (Fig. 5) the outputs Hr_2^a and Hr_2^b oscillate in steady state, and hence the difference $Hr_2^a - Hr_2^b$ also oscillates (since Hr_2^b is a delayed version of Hr_2^a); i.e. the difference $Hr_2^a - Hr_2^b \notin \ell_2$ is a power signal. If we consider the response to the truncated signals $(r_2^a)_T, (r_2^b)_T, \in \ell_2$ then we find that when $g = 1$ we obtain

$$\|H(r_2^a)_T - H(r_2^b)_T\| / \|(r_2^a)_T - (r_2^b)_T\| \rightarrow \infty \text{ as } T \rightarrow \infty. \tag{25}$$

Remark 8. *Since r_2 is a power signal, Theorem 2 can be applied to all three cases. However for the case $g = 1$, since the closed-loop system is not FGOS, the power must be normalized around 0 rather than around the steady state values.*

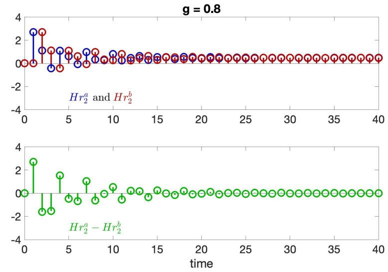

Fig. 4. Respective responses Hr_2^a and Hr_2^b to deterministic step signals r_2^a and r_2^b when the gain is $g = 0.8$. There is a suitable multiplier $M \in \mathcal{M}^d$ for G. It follows that H is FGOS and the difference $Hr_2^a - Hr_2^b \in \ell_2$.

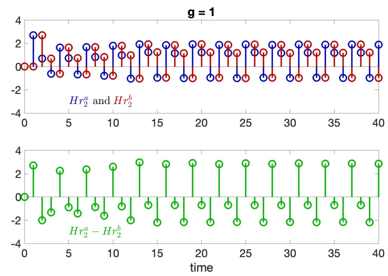

Fig. 5. Respective responses Hr_2^a and Hr_2^b to deterministic step signals r_2^a and r_2^b when the gain is $g = 1$. There is no suitable multiplier $M \in \mathcal{M}^d$ for G. Both responses oscillate: their difference $Hr_2^a - Hr_2^b$ is a power signal but $Hr_2^a - Hr_2^b \notin \ell_2$.

D. Step response with noise

Now suppose r_2 is a pulse signal taking values 0 or 2.7 with period 400. In addition r_1 is pseudo-random Gaussian white noise with mean 0 and variance 10^{-5}. Fig. 6 shows the signals r_1 and r_2 over three periods of r_2 while Fig. 7 shows the corresponding signal u_2 for the three cases $g = 0.6$, $g = 0.8$ and $g = 1$. While there are quantitative differences between all responses, there is a qualitative difference when $g = 1$ and when r_2 is high. This is exactly what me might expect, as the discrete-time counterpart of Theorem 3 can be applied to all three cases when r_2 is low, but the discrete-time counterpart of Theorem 4 only applies to the cases $g = 0.6$ and $g = 0.8$. For the first two periods with $g = 1$ an oscillatory response is evoked when r_2 is high, but not for the third period. In this case the response is indeed "critically sensitive to small changes in inputs" [9].

VI. Discussion

We have shown that the existence of a suitable OZF multiplier in $\mathcal{M}$ (but not $\mathcal{M}_{\text{odd}} - \mathcal{M}$) ensures a Lurye system is FGOS. This in turn ensures that if the exogenous signal has small power measured around some bias then the output also has small power measured around a uniquely determined bias.

The example of [16] shows that the result cannot be extended (in general) to exogenous signals that have small power measured around some periodic signal. From a practical point of view it would be interesting to know if there is an extension to signals that are sufficiently slow moving.

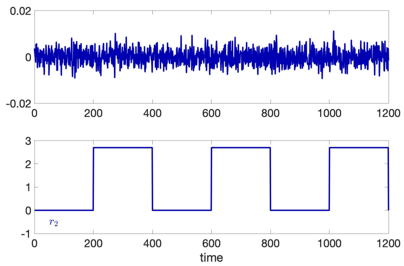

Fig. 6. Exogenous signals r_1 and r_2 for the step response with noise. Both are power signals.

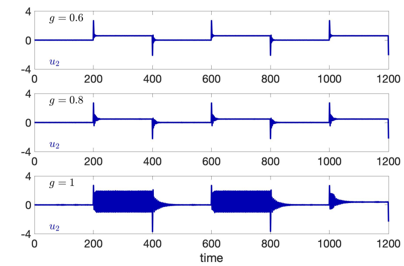

Fig. 7. Response u_2 for the cases $g = 0.6$, $g = 0.8$ and $g = 1$. There is a qualitative difference for the case $g = 1$ in the time intervals $[200, 400]$ and $[600, 800]$ (when r_2 is high), indicating that the response is critically sensitive to noise in this case.

It is worth noting that Theorem 4 does not exclude discontinuity. The example in [16] shows Lipschitz discontinuity when the exogenous signal is $r_2(t) = \sin(2t)$. For $T > 0$ sufficiently big the discontinuity will still occur if the input is $r_2(t) = \sin(2t)$ for $t < T$ and $r_2(t) = \sin(2T)$ for $t \geq T$. For this example the presence of small power noise may significantly affect the transient behaviour, even if the output power is guaranteed small.

Finally we note that both the example of [16] and our example with $g = 1$ have high $\mathcal{L}_2$ or ℓ_2 gain h. A lower bound is given by the peak sensitivity when the nonlinearity with a gain corresponding to its maximum slope. For the example in [16] this lower bound is $\|(1 + G)^{-1}\|_\infty$ with $G(s) = \frac{909}{(s^2+0.1s+1)(s+100)}$, viz $h \geq 311.35$. For our example this lower bound is $\|(1+G)^{-1}\|_\infty$ with G given by (22) and $g = 1$, viz $h \geq 28.58$. We know of no such example when the gain h is small.

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