2024 IEEE 63rd Conference on Decision and Control (CDC)
December 16-19, 2024. MiCo, Milan, Italy

# On Component-wise Asymptotic Moment Stability of Continuous-time Markov Jump Linear Systems

Shenyu Liu Penghui Wen

*Abstract*— **In addition to the known stochastic stability properties of asymptotic moment stability and almost sure global asymptotic stability for continous-time Markov jump linear systems, in this work, we propose component-wise asymptotic moment stability and study the relations between these stochastic stability properties. Next, we show that the component-wise 1st and 2nd moments of a Markov jump linear system can be precisely computed by solving linear ordinary differential equations. Consequently, necessary and sufficient conditions for component-wise asymptotic 1st and 2nd moment stability are obtained. Lastly, we test stochastic stability of several numerical examples via our criteria, one of which consists of all unstable flow and all unstable jumps, yet has all the stochastic stability properties aforementioned.**

## I. Introduction

Continuous-time Markov jump linear systems (MJLSs) constitute a specialized category within randomly switched systems, where linear subsystems interact with switching signals generated by continuous-time Markov processes [1]. This modeling framework finds application in various areas, including financial systems, communication networks, networked control systems, biological systems, power systems, etc. (cf. [2] and the references therein). Given the significance of such systems, our focus lies in comprehensively understanding the intricacies of MJLSs, with the aim of leveraging this understanding for applications in identification, control, and stabilization.

Extensive research has been devoted to investigating stochastic stability properties of MJLSs, encompassing aspects such as almost sure stability and moment stability [3], [4], [5]. Similar investigations extend to randomly switched nonlinear systems [6], [7], [8], systems with switching signals generated by a broader class of random processes [9], [10], [11], and switched diffusion, where stochastic differential equations replace ordinary differential equations for subsystem dynamics [12], [13], [14].

The motivation for this work stems from a continuation of our previous research on the stability analysis of randomly impulsive switched systems [11]. A primary objective of this work is to enhance moment estimation, departing from conventional stability analysis methods relying on the assumption of multiple Lyapunov functions [15]. The drawbacks of multiple Lyapunov functions approach are twofold: firstly, it introduces natural inequalities, and the choice of Lyapunov functions significantly impacts the conservatism of stability criteria [16]. Secondly, Lyapunov functions map a multidimensional state to a scalar value, resulting in significant information loss. Notably, this method fails to conclusively determine the stability of switched systems with all unstable modes.

To address these limitations, our proposed alternative involves a direct analysis via the state transition matrix, capturing the precise evolution of each state component. It's worth noting, as stated in [17], that moment stability for multidimensional randomly switched systems is considered challenging and remains unresolved. This is because in the context of multidimensional switched systems with arbitrary switching, the challenges posed by non-commutativity become evident when employing traditional stability analysis methods based on the transition matrix [18, Chapter 2.2]. However, when the switching signals adhere to a Markov process, non-commutativity ceases to be an impediment, allowing us to precisely compute the expected mean and mean square of the state. We also highlight that this approach has been previously employed in [3] to derive moment stability criteria for MJLSs. However, it's important to note that the MJLSs considered in that work do not allow state jumps and impulses. In other words, the stability results proposed in this work are more general, as they allow for state jumps and impulses in the context of MJLSs.

This work introduces the concept of component-wise asymptotic moment stability and explores its relationship with classical asymptotic moment stability and almost sure global asymptotic stability. The analysis reveals that the component-wise 1st and 2nd moments of a MJLS can be accurately computed through linear ordinary differential equations. Consequently, the study establishes necessary and sufficient conditions for component-wise asymptotic 1st and 2nd moment stability. To validate the proposed criteria, stochastic stability is tested on several numerical examples, including one with all unstable flow and jumps, yet exhibiting all the stochastic stability properties discussed earlier.

The rest paper is organized as follows. Section II provides the necessary math notions, introduces MJLSs and defines several stochastic stability properties. Section III then shows the relations between all the stochastic stability properties. Section IV then provide the formulae for component-wise moments and the necessary and sufficient conditions for component-wise asymptotic moment stability. Section V contains several numerical examples, where stochastic stability properties are tested by the proposed criteria. Finally, Section VI concludes the paper.

This work was funded by the National Science Foundation of China under Grant 62203053.

S. Liu and P. Wen are with the School of Automation, Beijing Institute of Technology, 100081, Beijing, China {shenyuliu, penghuiwen}@bit.edu.cn

979-8-3503-1632-2/24/$31.00 ©2024 IEEE

## II. PRELIMINARIES

In this section, we provide the necessary math notions, introduce MJLSs and define several stochastic stability properties.

### A. Math notions

Let $\mathbb{N}, \mathbb{N}_{>0}$ be the sets of non-negative integers and positive integers, respectively, and let $\mathbb{R}, \mathbb{R}_{\geq 0}, \mathbb{R}_{>0}$ be the set of all real numbers, non-negative real numbers and positive real numbers, respectively. Let $I_n$ be the $n$-dimensional identity matrix. If $q$ is a vector, $\mathrm{diag}(q)$ means a diagonal matrix, whose diagonal elements are the elements in $q$; if $A_1, \cdots, A_m$ are matrices, then $\mathrm{diag}(A_1, \cdots, A_m)$ means a block diagonal matrix, the blocks on the diagonal of whom are given by $A_1, \cdots, A_m$. The Kronecker product of two matrices $A, B$ is denoted by $A \otimes B$. The operator $\mathrm{vec}: \mathbb{R}^{n\times n} \to \mathbb{R}^{n^2}$ denotes square matrix vectorization, and $\mathrm{vec}^{-1}$ denotes its inverse map. Let $|\cdot|$ denote the absolute value of a real number, and $\|\cdot\|$ denote the Euclidean norm of a vector. If $X$ is a vector of random variables or a matrix of random variables, then $\mathsf{E}(X)$ denotes the vector or matrix of expectations of each component in $X$. If the left limit of $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ exists everywhere, we denote $\lim_{s\to t^-} x(s) =: x(t^-)$.

### B. Markov jump linear systems

Let $\mathbb{M} = \{1, \ldots, m\}$. An impulsive switching signal is defined by a pair $(\sigma, \mathcal{T})$, where $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{M}$ is a Càdlàg (right-continuous and left limit exists everywhere) piece-wise constant function and $\mathcal{T} \subset \mathbb{R}_{>0}$ is a countable set satisfying the following two conditions:

1) All discontinuities of $\sigma$ belong to $\mathcal{T}$.
2) For any bounded interval $\mathcal{I}, \mathrm{card}(\mathcal{I} \cap \mathcal{T}) < \infty$,

where $\mathrm{card}(\cdot)$ denotes the cardinality of a set. Note that by our definition, it is allowed to have $\sigma(t^-) = \sigma(t)$ for $t \in \mathcal{T}$. Such time instants are considered as impulse instants.

For any impulsive switching signal $(\sigma, \mathcal{T})$, let $\mathcal{T} = \{t_1, t_2, \cdots\}$ and denote $t_0 = 0$ as a convention. We call the sequence $\sigma(t_0), \sigma(t_1), \cdots$ the *jump chain* and $\Delta t_0, \Delta t_1, \cdots$ where $\Delta t_k := t_{k+1} - t_k$ the *sojourn times* of the impulsive switching signal $(\sigma, \mathcal{T})$. Let $\left(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, \Pr\right)$ be a complete filtered probability space, where $(\mathfrak{F}_t)_{t\geq 0}$ is the natural filtration, satisfying the usual conditions. Let $p_0$ be a probability vector of dimension $m, \Pi$ be a row stochastic matrix[1] of dimension $m$, and $q \in \mathbb{R}^m_{>0}$. We say an impulsive switching signal $(\sigma, \mathcal{T})$ obeys Markov process with data $p_0, \Pi, q$, denoted as $(\sigma, \mathcal{T}) \sim \mathrm{Markov}(p_0, \Pi, q)$, if the following conditions are satisfied on the jump chain and sojourn times:

(a) The jump chain $\sigma(t_0), \sigma(t_1), \cdots$ is a discrete-time-Markov-chain with initial probability vector $p_0$ and probability transition matrix $\Pi$. That is,

$$\Pr(\sigma(t_0) = i) = (p_0)_i,$$
$$\Pr(\sigma(t_k) = j \mid \sigma(t_{k-1}) = i) = \pi_{ij}.$$

(b) For any $k \in \mathbb{N}$, the sojourn time $\Delta t_k$ only depends on the most recent mode $\sigma(t_k)$ and obeys exponential distribution with parameters $q_{\sigma(t_k)}$. That is,

$$\Pr(\Delta t_k \geq t \,|\, \sigma(t_k) = j, \sigma(t_i), \Delta t_i \ \forall i = 0, \ldots k-1) = \Pr(\Delta t_k \geq t \mid \sigma(t_k) = j) = e^{-q_j t}.$$

We remark that we allow $\pi_{ii} > 0$, which is the probability for an incidence of switching back to the same mode. Our definition is consequently a generalized version of Markov process with possible self-jumps.

Let $A_i, B_{ij} \in \mathbb{R}^{n\times n}$ for all $i, j \in \mathbb{M}$. An $n$-dimensional *Markov jump linear system* (abbreviated as MJLS) with $m$ modes is given by

$$\dot{x}(t) = A_i x(t) \qquad \forall t \notin \mathcal{T}, \tag{1a}$$
$$x(t) = B_{ij} x(t^-) \quad \forall t \in \mathcal{T} \text{ s.t } \sigma(t^-) = i, \sigma(t) = j. \tag{1b}$$

where $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is the state trajectory and $(\sigma, \mathcal{T}) \sim \mathrm{Markov}(p_0, \Pi, q)$. Note that contrast to the MJLSs defined in [1], we allow the state to jump linearly at switches. The jump depends on both modes before and after the switch.

### C. Statistical stability properties

**Definition 1** *The MJLS is* asymptotically 1st moment stable *(or* asymptotically mean stable*, abbreviated as AMS), if there exists*[2] $\beta \in \mathcal{KL}$ *such that for any* $x_0 \in \mathbb{R}^n, t \in \mathbb{R}_{\geq 0}$,

$$|\mathsf{E}(v(t))| \leq \beta(\|x_0\|, t), \tag{2}$$

*where* $v(t) = \|x(t)\|$.

**Definition 2** *The MJLS is* asymptotically 2nd moment stable *(or* asymptotically mean square stable*, abbreviated as AM²S), if there exists* $\beta \in \mathcal{KL}$ *such that for any* $x_0 \in \mathbb{R}^n, t \in \mathbb{R}_{\geq 0}$*, the inequality* (2) *holds for* $v(t) = \|x(t)\|^2$.

**Definition 3** *The MJLS is* component-wise asymptotically 1st moment stable *(abbreviated as CAMS), if there exists* $\beta \in \mathcal{KL}$ *such that for any* $x_0 \in \mathbb{R}^n, t \in \mathbb{R}_{\geq 0}$*, the inequality* (2) *holds for* $v(t) = x_i(t)$ *for any* $i = 1, \ldots, n$.

**Definition 4** *The MJLS is* component-wise asymptotically 2nd moment stable *(abbreviated as CAM²S), if there exists* $\beta \in \mathcal{KL}$ *such that for any* $x_0 \in \mathbb{R}^n, t \in \mathbb{R}_{\geq 0}$*, the inequality* (2) *holds for* $v(t) = x_i(t)x_j(t)$ *for any* $i, j = 1, \ldots, n$.

**Definition 5** *The MJLS is* almost surely globally asymptotically stable *(abbreviated as GAS a.s.), if for any* $x_0 \in \mathbb{R}^n$,

$$\Pr\left(\lim_{t\to\infty} \|x(t)\| = 0\right) = 1.$$

[1] A probability vector of dimension $m \in \mathbb{N}_{>0}$ is a column vector $v = [v_i] \in \mathbb{R}^m_{\geq 0}$ such that $\sum_{i=1}^m v_i = 1$. A row stochastic matrix of dimension $m \in \mathbb{N}_{>0}$ is a matrix $Q = [q_{ij}] \in \mathbb{R}^{m\times m}_{\geq 0}$ such that $\sum_{j=1}^m q_{ij} = 1$ for all $i = 1, \ldots, m$.

[2] A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ if it is continuous and strictly increasing, $\alpha(0) = 0$ and $\alpha(s) > 0$ for all $s > 0$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each $t \in \mathbb{R}_{\geq 0}$, $\beta(s, \cdot)$ is continuous and decreasing, and $\lim_{t\to\infty} \beta(s, t) = 0$ for each fixed $s \in \mathbb{R}_{\geq 0}$.

## III. Relations between different stability definitions

The relations between the five stochastic stability properties are summarized in the following theorem.

**Theorem III.1** *For the MJLS* (1)*, the implications depicted by the arrows in Figure 1 hold, and any implications not depicted by arrows fail.*

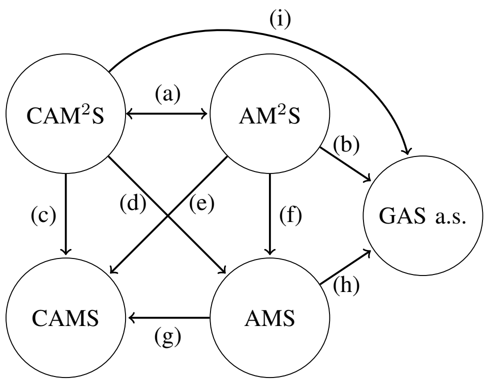



Fig. 1. The relations between stability properties

Except for the bidirectional implication $(a)$, we define $\neg(k)$ as the converse of an implication denoted by an arrow $(k)$. The truth of implications $(b), (f), (h)$, and the failure of implications $\neg(b), \neg(f), \neg(h)$ are well-known results [4]. In this work, the equivalence of $(a)$ is proven by Lemma III.2. The implication $\neg(d)$ is then refuted by combining $(a)$ and $\neg(f)$, while the implication $(g)$ is a straightforward result of the fact that $-\|x\| \leq x_i \leq \|x\|$ for any component $x_i$ in the vector $x$. The implications $(c), (d), (e), (f)$ are consequences of the other true implications. The remaining false implications, namely, $\neg(c), \neg(e), \neg(g), \neg(i)$, and the hidden implications between CAMS and GAS a.s., will be demonstrated by examples in Section V. It's noteworthy that, to disprove, for instance, $\neg(c)$, we must find a MJLS that is CAM$^2$S but not CAMS. This involves establishing necessary and sufficient conditions for component-wise asymptotic moment stability, a topic discussed in the next section.

**Lemma III.2** *CAM$^2$S and AM$^2$S are equivalent.*

*Proof:* To show CAM$^2$S implies AM$^2$S, we use the relation that $\mathsf{E}(\|x\|^2) = \mathsf{E}(\sum_{i=1}^n x_i^2) = \sum_{i=1}^n \mathsf{E}(x_i^2)$. To show the other direction, we use the relation that $|\mathsf{E}(x_i x_j)| \leq \mathsf{E}(|x_i x_j|) \leq \frac{1}{2}\mathsf{E}(x_i^2 + x_j^2) \leq \mathsf{E}(\|x\|^2)$. ■

## IV. Necessary and sufficient conditions for CAMS and CAM$^2$S

Our necessary and sufficient conditions for component-wise asymptotic moment stability is a natural consequence of the following theorem, which provides exact formulae for the component-wise 1st and 2nd moments of an MJLS.

**Theorem IV.1** *Consider an $m$-mode MJLS* (1) *where $(\sigma, \mathcal{T}) \sim \text{Markov}(p_0, \Pi, q)$. For any initial state $x(0) = x_0 \in \mathbb{R}^n$, it holds that*

$$\mathsf{E}(x(t)) = U(t)(p_0 \otimes x_0), \tag{3}$$

$$\mathsf{E}\big(x(t)x(t)^\top\big) = \text{vec}^{-1}\left(V(t)\left(p_0 \otimes \text{vec}(x_0 x_0^\top)\right)\right), \tag{4}$$

*where $U(t) \in \mathbb{R}^{n \times mn}$ is the solution to the matrix-valued linear ordinary differential equation*

$$\frac{d}{dt}U(t) = U(t)(\bar{A}_1 - \bar{Q}_1 + \bar{B}_1\bar{Q}_1), \quad U(0) = 1_m^\top \otimes I_n, \tag{5}$$

*and $V(t) \in \mathbb{R}^{n^2 \times mn^2}$ is the solution to the matrix-valued linear ordinary differential equation*

$$\frac{d}{dt}V(t) = V(t)(\bar{A}_2 - \bar{Q}_2 + \bar{B}_2\bar{Q}_2), \quad V(0) = 1_m^\top \otimes I_{n^2}, \tag{6}$$

*with matrices*

$$\begin{aligned}
\bar{A}_1 &= \text{diag}(A_1, \cdots, A_m),\\
\bar{A}_2 &= \text{diag}(A_1 \otimes I_n + I_n \otimes A_1, \cdots, A_m \otimes I_n + I_n \otimes A_m),\\
\bar{B}_1 &= \begin{bmatrix} \pi_{11}B_{11} & \cdots & \pi_{m1}B_{m1} \\ \vdots & \ddots & \vdots \\ \pi_{1m}B_{1m} & \cdots & \pi_{mm}B_{mm} \end{bmatrix},\\
\bar{B}_2 &= \begin{bmatrix} \pi_{11}B_{11} \otimes B_{11} & \cdots & \pi_{m1}B_{m1} \otimes B_{m1} \\ \vdots & \ddots & \vdots \\ \pi_{1m}B_{1m} \otimes B_{1m} & \cdots & \pi_{mm}B_{mm} \otimes B_{mm} \end{bmatrix},\\
\bar{Q}_1 &= \text{diag}(q) \otimes I_n,\\
\bar{Q}_2 &= \text{diag}(q) \otimes I_{n^2}
\end{aligned}$$

*Proof:* Let $\Phi(t,s) \in \mathbb{R}^{n \times n}$ be the random state transition matrix from time $s$ to $t$. We have $\mathsf{E}(x(t)|\sigma(0) = i) = \mathsf{E}(\Phi(t,0)x_0|\sigma(0) = i) = \mathsf{E}(\Phi(t,0)|\sigma(0) = i)x_0$. Thus in order to find $\mathsf{E}(x(t))$, we first compute $\mathsf{E}(\Phi(t,0)|\sigma(0) = i)$.

Condition on the first switch instance $t_1 = s$ and first mode-to-go $\sigma(t_1) = j$. If $s > t$, then

$$\mathsf{E}(\Phi(t,0)|\sigma(0) = i, t_1 = s > t, \sigma(t_1) = j) = e^{A_i t}.$$

Otherwise when $s \leq t$, then

$$\begin{aligned}
&\mathsf{E}(\Phi(t,0)|\sigma(0) = i, t_1 = s \leq t, \sigma(t_1) = j)\\
&\quad = \mathsf{E}(\Phi(t,s)|\sigma(s) = j)B_{ij}e^{A_i s}\\
&\quad = \mathsf{E}(\Phi(t-s,0)|\sigma(0) = j)B_{ij}e^{A_i s},
\end{aligned}$$

where we have used the property that Markov process is memoryless and time-invariant. Hence, we conclude that

$$\begin{aligned}
&\mathsf{E}(\Phi(t,0)|\sigma(0) = i)\\
&= \sum_{j \in \mathbb{M}} \pi_{ij} \int_0^\infty q_i e^{-q_i s} \mathsf{E}(\Phi(t,0)|\sigma(0) = i, t_1 = s, \sigma(t_1) = j) ds\\
&= \int_t^\infty q_i e^{-q_i s} e^{A_i t} ds\\
&\quad + \sum_{j \in \mathbb{M}} \pi_{ij} \int_0^t q_i e^{-q_i s} \mathsf{E}(\Phi(t-s,0)|\sigma(0) = j) B_{ij} e^{A_i s} ds\\
&= e^{(A_i - q_i I_n)t}
\end{aligned}$$

$$+ q_i\sum_{j\in\mathbb{M}}\pi_{ij}\int_0^t \mathsf{E}(\Phi(t-s,0)|\sigma(0)=j)B_{ij}e^{(A_i-q_iI_n)s}ds, \quad (7)$$

where we have used the fact that $A_i$ is commutative with $I_n$ so that $e^{-q_is}e^{A_is} = e^{(A_i-q_iI_n)s}$. By defining $U_k(\tau) := \mathsf{E}(\Phi(\tau,0)|\sigma(0)=k) \in \mathbb{R}^{n\times n}$ for any $\tau \in \mathbb{R}_{\geq 0}, k \in \mathbb{M}$, we can re-write (7) as

$$U_i(t)=e^{(A_i-q_iI_n)t}+q_i\sum_{j\in\mathbb{M}}\pi_{ij}\int_0^t U_j(t-s)B_{ij}e^{(A_i-q_iI_n)s}ds. \quad (8)$$

Differentiate (8) with respect to $t$, we have

$$\frac{d}{dt}U_i(t)=e^{(A_i-q_iI_n)t}(A_i-q_iI_n)+q_i\sum_{j\in\mathbb{M}}\pi_{ij}\Big(B_{ij}e^{(A_i-q_iI_n)t} + \int_0^t \frac{\partial}{\partial t}U_j(t-s)B_{ij}e^{(A_i-q_iI_n)s}ds\Big), \quad (9)$$

where we have used the fact that $U_j(0)=\mathsf{E}(\Phi(0,0)|\sigma(0)=j)=I_n$, and the property that $\frac{d}{dt}e^{Mt}=Me^{Mt}=e^{Mt}M$. Now, note that $\frac{\partial}{\partial t}U_j(t-s)=-\frac{\partial}{\partial s}U_j(t-s)$. Meanwhile, it follows from integration by parts that

$$-\int_0^t \frac{\partial}{\partial s}U_j(t-s)B_{ij}e^{(A_i-q_iI_n)s}ds=U_j(t)B_{ij}-B_{ij}e^{A_i-q_iI_n}t + \int_0^t U_j(t-s)B_{ij}e^{(A_i-q_iI_n)s}(A_i-q_iI_n)ds. \quad (10)$$

By plugging (10) into (9), we conclude

$$\begin{aligned}\frac{d}{dt}U_i(t) &= e^{(A_i-q_iI_n)t}(A_i-q_iI_n)\\ &= \left(e^{(A_i-q_iI_n)t}+q_i\sum_{j\in\mathbb{M}}\pi_{ij}\int_0^t U_j(t-s)B_{ij}e^{(A_i-q_iI_n)s}ds\right)\\ &\quad\times(A_i-q_iI_n)+q_i\sum_{j\in\mathbb{M}}\pi_{ij}U_j(t)B_{ij}\\ &= U_i(t)(A_i-q_iI_n)+q_i\sum_{j\in\mathbb{M}}\pi_{ij}U_j(t)B_{ij},\end{aligned}$$

where we have used (8) on the right-hand side for the last equality. By defining $U(t)=\begin{bmatrix}U_1(t) & \cdots & U_m(t)\end{bmatrix}$, we recover the matrix-valued differential equation (5). Finally,

$$\begin{aligned}\mathsf{E}(x(t)) &= \mathsf{E}(\Phi(t,0))x_0\\ &= \left(\sum_{i\in\mathbb{M}}\mathsf{E}(\Phi(t,0)|\sigma(0)=i)(p_0)_i\right)x_0\\ &= \begin{bmatrix}U_1(t) & \cdots & U_m(t)\end{bmatrix}\begin{bmatrix}(p_0)_1x_0\\ \vdots\\ (p_0)_mx_0\end{bmatrix},\end{aligned}$$

and we recover the expression (3).

To find $\mathsf{E}\big(x(t)x(t)^\top\big)$, we first compute $\mathsf{E}\big(x(t)x(t)^\top|\sigma(0)=i\big)=\mathsf{E}\big(\Phi(t,0)x_0x_0^\top\Phi(t,0)^\top|\sigma(0)=i\big)$. Apply similar analysis for the derivation of (7), we conclude that

$$\begin{aligned}&\mathsf{E}\big(\Phi(t,0)x_0x_0^\top\Phi(t,0)^\top|\sigma(0)=i\big)\\ &\quad= e^{-q_it}e^{A_it}x_0x_0^\top e^{A_i^\top t}+q_i\sum_{j\in\mathbb{M}}\pi_{ij}\int_0^t e^{-q_is}\mathsf{E}^*_{ij}(s)ds. \quad (11)\end{aligned}$$

with $\mathsf{E}^*_{ij}(s) := \mathsf{E}\big(\Phi(t-s,0)B_{ij}e^{A_is}x_0x_0^\top e^{A_i^\top s}B_{ij}^\top\Phi(t-s,0)^\top|\sigma(0)=j\big)$. Now, vectorize both sides of (11) and use the identity that $\operatorname{vec}(Mxx^\top M^\top)=(M\otimes M)\operatorname{vec}(xx^\top)$, we conclude that

$$\begin{aligned}&\mathsf{E}\big(\Phi(t,0)\otimes\Phi(t,0)|\sigma(0)=i\big)\operatorname{vec}(x_0x_0^\top)\\ &= e^{-q_it}\left(e^{A_it}\otimes e^{A_it}\right)\operatorname{vec}(x_0x_0^\top)\\ &\quad+q_i\sum_{j\in\mathbb{M}}\pi_{ij}\int_0^t e^{-q_is}\mathsf{E}\big(\Phi(t-s,0)\otimes\Phi(t-s,0)|\sigma(0)=j\big)\\ &\quad\times(B_{ij}\otimes B_{ij})\left(e^{A_is}\otimes e^{A_is}\right)\operatorname{vec}(x_0x_0^\top)ds.\end{aligned}$$

Since this equation holds for any $x_0\in\mathbb{R}^n$, by defining $V_k(\tau) := \mathsf{E}\big(\Phi(\tau,0)\otimes\Phi(\tau,0)|\sigma(0)=k\big)$, we conclude that

$$\begin{aligned}&V_i(t)=e^{-q_it}\left(e^{A_it}\otimes e^{A_it}\right)\\ &+q_i\sum_{j\in\mathbb{M}}\pi_{ij}\int_0^t e^{-q_is}V_j(t-s)\,(B_{ij}\otimes B_{ij})\left(e^{A_is}\otimes e^{A_is}\right)ds. \quad (12)\end{aligned}$$

Similar to the previous analysis, we aim to take time derivative of $V_i(t)$. First, note the identity that

$$\begin{aligned}\frac{d}{dt}\big(e^{A_it}\otimes e^{A_it}\big) &= \big(e^{A_it}A_i\big)\otimes e^{A_it}+e^{A_it}\otimes\big(e^{A_it}A_i\big)\\ &= \big(e^{A_it}\otimes e^{A_it}\big)(A_i\otimes I_n+I_n\otimes A_i). \quad (13)\end{aligned}$$

Meanwhile, integration by parts gives

$$\begin{aligned}&-\int_0^t e^{-q_is}\left(\frac{\partial}{\partial s}V_j(t-s)\right)(B_{ij}\otimes B_{ij})\left(e^{A_is}\otimes e^{A_is}\right)ds\\ &= -e^{-q_is}V_j(t-s)\,(B_{ij}\otimes B_{ij})\left(e^{A_is}\otimes e^{A_is}\right)|_0^t\\ &\quad+\int_0^t\left(\frac{d}{ds}e^{-q_is}\right)V_j(t-s)(B_{ij}\otimes B_{ij})\left(e^{A_is}\otimes e^{A_is}\right)ds\\ &\quad+\int_0^t e^{-q_is}V_j(t-s)\,(B_{ij}\otimes B_{ij})\,\frac{d}{ds}\left(e^{A_is}\otimes e^{A_is}\right)ds\\ &= -e^{-q_it}\,(B_{ij}\otimes B_{ij})\left(e^{A_it}\otimes e^{A_it}\right)+V_j(t)\,(B_{ij}\otimes B_{ij})\\ &\quad+\int_0^t e^{-q_is}V_j(t-s)\,(B_{ij}\otimes B_{ij})\left(e^{A_is}\otimes e^{A_is}\right)\\ &\quad\times(A_i\otimes I_n+I_n\otimes A_i-q_iI_{n^2})\,ds,\end{aligned}$$

where we have used the identity (13) and the fact that $V_j(0)=\mathsf{E}\big(\Phi(0,0)\otimes\Phi(0,0)|\sigma(0)=j\big)=I_{n^2}$. Hence, we obtain

$$\begin{aligned}\frac{d}{dt}V_i(t) &= e^{-q_it}\big(e^{A_it}\otimes e^{A_it}\big)\big(A_i\otimes I_n+I_n\otimes A_i-q_iI_{n^2}\big)\\ &\quad+q_i\sum_{j\in\mathbb{M}}\pi_{ij}\Big(V_j(t)\,(B_{ij}\otimes B_{ij})\\ &\quad+\int_0^t e^{-q_is}V_j(t-s)\,(B_{ij}\otimes B_{ij})\left(e^{A_is}\otimes e^{A_is}\right)\\ &\quad\times(A_i\otimes I_n+I_n\otimes A_i-q_iI_{n^2})\,ds\Big)\\ &= V_i(t)\big(A_i\otimes I_n+I_n\otimes A_i-q_iI_{n^2}\big)\\ &\quad+q_i\sum_{j\in\mathbb{M}}\pi_{ij}V_j(t)\,(B_{ij}\otimes B_{ij})\end{aligned}$$

where we have used (12) on the right-hand side for the last equality. By defining $V(t)=\begin{bmatrix}V_1(t) & \cdots & V_m(t)\end{bmatrix}$, we recover the matrix-valued differential equation (6). Finally,

$$\begin{aligned}&\operatorname{vec}\left(\mathsf{E}\big(x(t)x(t)^\top\big)\right)=\mathsf{E}\big(\Phi(t,0)\otimes\Phi(t,0)\big)\operatorname{vec}(x_0x_0^\top)\\ &= \left(\sum_{i\in\mathbb{M}}\mathsf{E}\big(\Phi(t,0)\otimes\Phi(t,0)|\sigma(0)=i\big)(p_0)_i\right)\operatorname{vec}(x_0x_0^\top)\end{aligned}$$

$$= \begin{bmatrix} V_1(t) & \cdots & V_m(t) \end{bmatrix} \begin{bmatrix} (p_0)_1 \operatorname{vec}(x_0 x_0^\top) \\ \vdots \\ (p_0)_m \operatorname{vec}(x_0 x_0^\top) \end{bmatrix}$$

and we recover the expression (4). ■

Unlike the upper bounds provided by [11] for the moment, the formulae (3) and (4) are exact for the component-wise moment. This advantage stems from our departure from the multiple Lyapunov function approach in our analysis. It has also come to the authors' attention that similar formulae for the 1st and 2nd moment appear in [3] and [1, Chapter 3]. However, it is crucial to note that the MJLSs considered in those works do not account for state jumps at switches and do not allow impulses. Therefore, our results are more general and applicable in a broader context.

Based on Theorem IV.1, we immediately have the following corollaries.

**Corollary IV.2** *The MJLS (1) is CAMS (resp. CAM$^2$S) if and only if the matrix $\Xi_1 := \bar{A}_1 - \bar{Q}_1 + \bar{B}_1\bar{Q}_1$ (resp. $\Xi_2 := \bar{A}_2 - \bar{Q}_2 + \bar{B}_2\bar{Q}_2$) is Hurwitz.*

**Corollary IV.3** *The MJLS (1) is CAMS (resp. CAM$^2$S) if and only if it is CAMS (resp. CAM$^2$S) with exponential decay rates (i.e., $\beta(s,t) = \alpha(s)e^{-\lambda t}$ for some $\alpha \in \mathcal{K}, \lambda > 0$.)*

Corollary IV.2 provides a necessary and sufficient condition for CAMS and CAM$^2$S. Corollary IV.3, on the other hand, suggests that component-wise *asymptotic* moment stability is the same as component-wise *exponential* moment stability for MJLS.

## V. Examples

In this section, we test stochastic stability of four numerical examples using our stability criteria. All examples have only 2 modes and we assume that the switching signal obeys the same Markov process with data $p_0 = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, $\Pi = \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$, $q = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$. The first three examples are used to disprove some implications in Theorem III.1 and we further assume that the state does not jump at switches; i.e., $B_{i,j} = I_n$ for all $i, j \in \{1, 2\}$. The last example consists of all unstable flow and all unstable jumps, yet we show that the system has all the five stochastic stability properties.

### A. GAS a.s. but not CAMS example

Consider a one dimensional example with $A_1 = 0.25, A_2 = -0.3$. Intuitively, as the expected total activation time of mode 1 and mode 2 converge on the long run, and because the state decays at a faster rate in mode 2 compared with the growth rate in mode 1, it is expected that asymptotically the state should converge to 0. This is consistent with the conclusion drawn by [4, Section 4] that this system is GAS a.s.. On the other hand, it is not difficult to compute that $\Xi_1 = \left[\begin{smallmatrix}-0.75&1\\1&-1.3\end{smallmatrix}\right]$, which is not Hurwitz. Hence, by Corollary IV.2, this system is not CAMS. The 1st and 2nd moments, both theoretically computed and statistically estimated, are plotted in Figure 2. As a result, this example disproves the implication from GAS.a.s. to CAMS in Figure 1.

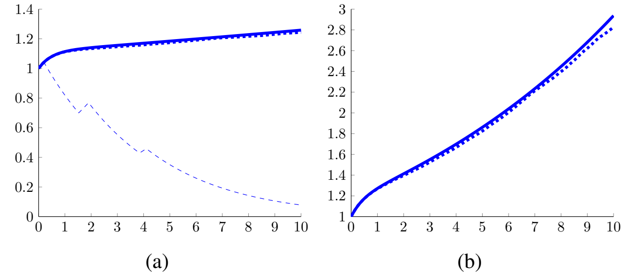



Fig. 2. 1st moment plot (left) and 2nd moment plot (right) for the first example. Solid curves are the computed values by (3), (4). Dotted curves are the statistical averages from 5000 Monte Carlo simulations. The thinner dashed curve is a sample trajectory of $x(t)$.

### B. CAMS but not CAM$^2$S example

Now we alter the dynamics of mode 1 of the example in Section V-A so that $A_1 = 0.2$. Because the divergence rate becomes slower, by a similar argument as for the previous example we again conclude GAS a.s.. On the other hand, we have $\Xi_1 = \left[\begin{smallmatrix}-0.8&1\\1&-1.3\end{smallmatrix}\right]$ which is Hurwitz, and $\Xi_2 = \left[\begin{smallmatrix}-0.6&1\\1&-1.6\end{smallmatrix}\right]$ which is not Hurwitz. Hence, by Corollary IV.2, this system is CAMS but not CAM$^2$S. Similar to the previous example, the 1st and 2nd moments are plotted in Figure 3. As a result, this example disproves the implications $\neg(c), \neg(i)$ in Figure 1. With the equivalence $(a)$, we further disprove the implication $\neg(e)$.

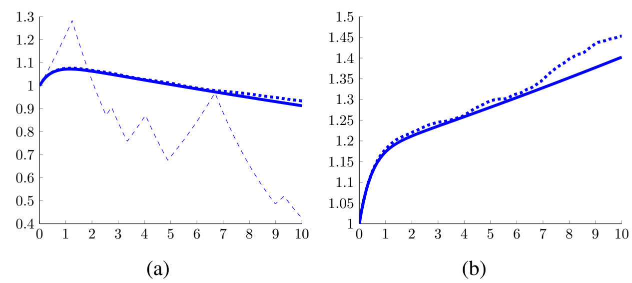


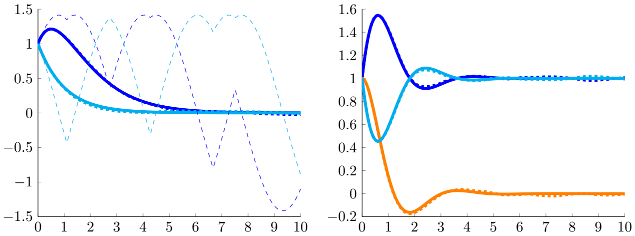

Fig. 3. 1st moment plot (left) and 2nd moment plot (right) for the 2nd example. Same explanation as for Figure 2.

### C. CAMS but not AMS example

Consider a two dimensional example with $A_1 = \left[\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right]$, $A_2 = \left[\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right]$. It can be intuitively understood that on the phase plane, the state $x(t)$ rotates counter clockwise in mode 1 and clockwise in mode 2. Therefore, $\|x(t)\|$ is constant and the system is not AMS, nor GAS a.s.. However, it can be computed that

$$\Xi_1 = \begin{bmatrix} -1 & -1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

which is Hurwitz. Hence, by Corollary IV.2, this system is CAMS. Intuitively, as $t$ increases, the state $x(t)$ will tend to be "uniformly" distributed on the unit circle of radius $\|x(0)\|$ on the phase plane. Thus, while $\|x(t)\|$ remains constant, component-wisely $|\mathsf{E}(x_i)|$ converges to 0. The component-wise 1st and 2nd moments are plotted in Figure 4. As a result, this example disproves the implication $\neg(g)$ and the implication from CAMS to GAS a.s. in Figure 1.



(a) (b)

Fig. 4. 1st moment plot (left) with blue, cyan curves meaning $x_1, x_2$ respectively, and 2nd moment plot (right) with blue, cyan and orange curves meaning $x_1^2, x_2^2, x_1x_2$ respectively, for the 3rd example. Same explanation as for Figure 2.

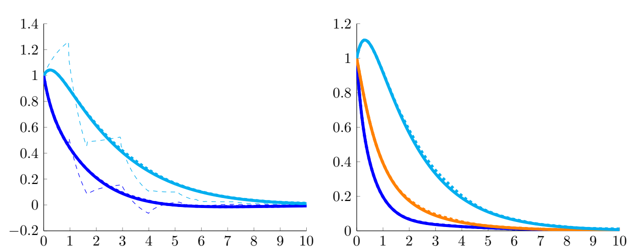
### D. CAMS example with all unstable flow and jumps

Motivated by the example in [19], we consider a two dimensional example with $A_1 = \begin{bmatrix} -1.9 & 0.6 \\ 0.6 & -0.1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0.1 & -0.9 \\ 0.1 & -1.4 \end{bmatrix}$, $B_{11} = B_{12} = \begin{bmatrix} 1.1 & 0 \\ 0 & 0.9 \end{bmatrix}$, $B_{21} = B_{22} = \begin{bmatrix} 0.9 & 0 \\ 0 & 1.1 \end{bmatrix}$. The flow of the system is all unstable, since both $A_i$ matrices have eigenvalues with positive real parts. Meanwhile, all jumps of the system are unstable as well, since all $B_{ij}$ matrices have eigenvalues with magnitude larger than 1. Nevertheless, it can be computed that

$$\Xi_1 = \begin{bmatrix} -2.9 & 0.6 & 1.1 & 0 \\ 0.6 & -1.1 & 0 & 0.9 \\ 0.9 & 0 & -0.9 & 0.1 \\ 0 & 1.1 & -0.9 & -2.4 \end{bmatrix},$$

$$\Xi_2 = \begin{bmatrix} -4.8 & 0.6 & 0.6 & 0 & 1.21 & 0 & 0 & 0 \\ 0.6 & -3 & 0 & 0.6 & 0 & 0.99 & 0 & 0 \\ 0.6 & 0 & -3 & 0.6 & 0 & 0 & 0.99 & 0 \\ 0 & 0.6 & 0.6 & -1.2 & 0 & 0 & 0 & 0.81 \\ 0.81 & 0 & 0 & 0 & -0.8 & 0.1 & 0.1 & 0 \\ 0 & 0.99 & 0 & 0 & -0.9 & -2.3 & 0 & 0.1 \\ 0 & 0 & 0.99 & 0 & -0.9 & 0 & -2.3 & 0.1 \\ 0 & 0 & 0 & 1.21 & 0 & -0.9 & -0.9 & -3.8 \end{bmatrix}.$$

Both of them are Hurwitz. Hence, by Corollary IV.2, this system is both CAMS and CAM$^2$S. The component-wise 1st and 2nd moments are plotted in Figure 4. In addition, by Theorem III.1, this system is also AMS, AM$^2$S and GAS a.s.. In particular, GAS a.s.. means that despite all unstable flow and jumps, each solution will eventually converge to 0 with probability 1.



(a) (b)

Fig. 5. 1st moment plot (left) and 2nd moment plot (right) for the 2nd example. Same explanation as for Figure 4.

## VI. Conclusion

In this work, we introduced and studied component-wise asymptotic moment stability for continuous-time MJLSs. we found exact formulae for the component-wise 1st and 2nd moment, from which we obtain necessary and sufficient conditions for the system to be component-wise asymptotic moment stable. In the numerical simulation, we show that our stability criteria can even be used to conclude stochastic stability of a system with all unstable flow and jumps. In the future work, we can extend our analysis to randomly impulsive switched systems whose impulsive switching signals are generated by semi-Markov processes, and to randomly switched diffusion.

## References


[1] O. L. do Valle Costa, M. D. Fragoso, and M. G. Todorov, *Continuous-time Markov jump linear systems*. Springer Science & Business Media, 2012.

[2] A. R. Teel, A. Subbaraman, and A. Sferlazza, “Stability analysis for stochastic hybrid systems: A survey,” *Automatica*, vol. 50, no. 10, pp. 2435–2456, 2014.

[3] M. Mariton, “Almost sure and moments stability of jump linear systems,” *Systems & Control Letters*, vol. 11, no. 5, pp. 393–397, 1988.

[4] X. Feng, K. Loparo, Y. Ji, and H. Chizeck, “Stochastic stability properties of jump linear systems,” *IEEE Transactions on Automatic Control*, vol. 37, no. 1, pp. 38–53, 1992.

[5] S. Cong, “A result on almost sure stability of linear continuous-time Markovian switching systems,” *IEEE Transactions on Automatic Control*, vol. 63, no. 7, pp. 2226–2233, 2018.

[6] X. Mao, “Stability of stochastic differential equations with Markovian switching,” *Stochastic Processes and their Applications*, vol. 79, pp. 45–67, 1999.

[7] D. Chatterjee and D. Liberzon, “On stability of randomly switched nonlinear systems,” *IEEE Transactions on Automatic Control*, vol. 52, no. 12, pp. 2390–2394, 2007.

[8] J. Leth, H. Schioler, M. Gholami, and V. Cocquempot, “Stochastic stability of markovianly switched systems,” *IEEE Transactions on Automatic Control*, vol. 58, no. 8, pp. 2048–2054, 2013.

[9] H. Schioler, M. Simonsen, and J. Leth, “Stochastic stability of systems with semi-Markovian switching,” *Automatica*, vol. 50, no. 11, pp. 2961–2964, 2014.

[10] X. Wu, Y. Tang, J. Cao, and X. Mao, “Stability analysis for continuous-time switched systems with stochastic switching signals,” *IEEE Transactions on Automatic Control*, vol. 63, no. 9, pp. 3083–3090, 2018.

[11] S. Liu and P. Wen, “Nonconservative stability criteria for semi-markovian impulsive switched systems,” *SIAM Journal on Control and Optimization*, vol. 62, no. 3, pp. 1783–1808, 2024.

[12] H. Schioler, J. Leth, M. Simonsen, and A. R. Khan, “Stochastic stability of diffusions with semi-Markovian switching,” in *Proceedings of the 54th IEEE Conf. on Decision and Control*, Dec 2015, pp. 1870–1877.

[13] X. Wu, P. Shi, Y. Tang, S. Mao, and F. Qian, “Stability analysis of semi-Markov jump stochastic nonlinear systems,” *IEEE Transactions on Automatic Control*, vol. 67, no. 4, pp. 2084–2091, 2021.

[14] X. Zong, F. Wu, G. Yin, and Z. Jin, “Almost sure and $p$th-moment stability and stabilization of regime-switching jump diffusion systems,” *SIAM Journal on Control and Optimization*, vol. 52, no. 4, pp. 2595–2622, 2014.

[15] R. Decarlo, M. Branicky, S. Pettersson, and B. Lennartson, “Perspectives and results on the stability and stabilizability of hybrid systems,” *Proceedings of the IEEE*, vol. 88, no. 7, pp. 1069–1082, 2000.

[16] S. Liu, S. Martinez, and J. Cortes, “Average dwell-time minimization of switched systems via sequential convex programming,” *IEEE Control Systems Letters*, vol. 6, pp. 1076–1081, 2022.

[17] G. G. Yin and C. Zhu, *Hybrid Switching Diffusions: Properties and Applications*. New York: Springer, 2010.

[18] D. Liberzon, *Switching in Systems and Control*. Boston, MA: Birkhäuser, 2003.

[19] W. Xiang and J. Xiao, “Stabilization of switched continuous-time systems with all modes unstable via dwell time switching,” *Automatica*, vol. 50, no. 3, pp. 940–945, 2014.