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# On the Reachability of Stable Linear Systems with Quantized Input Alphabet

Donglei Fan

***Abstract*— In this paper, we study reachability of linear systems with a finitely quantized input alphabet. We give motivations for a new notion of reachability for this class of systems, and propose a notion of local asymptotic reachability. We then derive a sufficient condition and a necessary condition for the proposed local asymptotic reachability, and present an algorithmic procedure to verify these conditions.**

***Index Terms*— Quantized systems, reachability, linear systems.**

## I. Introduction

In this paper, we study discrete-time, linear time invariant systems where the input takes values in a finitely quantized set. Such systems naturally arise in applications where the underlying controlled plant is continuous while the controller is digitally quantized. We are interested in the reachability of this class of systems. Conventionally, a system is reachable if for every state in its state space, the system can be driven to this state from the origin under some control sequence in finite time. As we shall see, conventional reachability does not generalize well for systems with quantized inputs, motivating us to study and propose new notions of reachability for this class of systems.

Reachability of quantized systems has been studied [1], [2], [3], [8]. In [1], the author studies linear systems with integer inputs and shows that reachability is characterized by whether a linear combination of the eigenvalues of the $A$ matrix is an integer. In [2], the authors study reachability of quantized control systems and present conditions such that the reachable set is dense for a class of driftless linear systems. They also show results on the reachability of nonlinear chained form systems. In [3], the authors derive conditions regarding the existence of a finite input set such that the reachable set is dense. Specifically, they show that given a controllable pair $(A, B)$, there exist a finite control set such that the reachable set is dense in a compact set if and only if $A$ is invertible. In [8], the authors study reachability of hybrid systems in a stochastic setting. They first set up a stochastic automaton model of a hybrid system, and then show that reachability of the model implies reachability of the hybrid system.

More generally, linear systems with quantized input has been studied extensively. For instance in [4], the author studies the stabilization of linear systems when quantized state information is available. The author shows that for any quantized state feedback control law, for an unstable linear system, the set of all initial conditions whose closed-loop trajectories tend to the origin has measure zero. The author then studies a new notion of stabilization.

In this paper we continue our previous study on systems over finite alphabets [6] [7]. We are interested in investigating what is fundamentally lost in terms of reachability when the control input is constrained to a finite set. We briefly point out some of the distinctions between the current work and the existing results: We study systems with any given finite input set, while the results in [3] focus on the existence of a quantized input set ensuring reachability. In [2], the reachability of driftless systems is studied, while in this paper we study systems with Schur stable $A$ matrix. In [1] the $A$ matrix is assumed to be diagonalizable and the input is assumed to be integers, while no such assumptions are made here. Our main contributions are as follows.

1) We give the motivations of, and propose a new notion of local asymptotic reachability for linear systems with quantized input set.
2) We present both a sufficient condition and a necessary condition for the proposed local asymptotic reachability. If the sufficient condition is satisfied, then the reachable set is dense in some neighborhood of the origin, while if the necessary condition is satisfied the reachable set is nowhere dense in the state space.
3) We provide an algorithm to facilitate the verification of the proposed conditions.

*Notation:* We use $\mathbb{N}$ to denote the non-negative integers, $\mathbb{Z}_+$ to denote the positive integers, $\mathbb{R}$ to denote the real numbers, and $\mathbb{C}$ to denote the complex numbers. We use $\mathcal{A}^{\mathbb{N}}$ to denote the collection of infinite sequences over set $\mathcal{A}$, that is $\mathcal{A}^{\mathbb{N}} = \{f : \mathbb{N} \to \mathcal{A}\}$. For $\mathbf{a} \in \mathcal{A}^{\mathbb{N}}$, we use $a_t$ to denote its $t^{th}$ component. We use $\{a_t\}_{t\in\mathcal{I}}$ to denote the subsequence over index set $\mathcal{I} \subset \mathbb{N}$. For $v \in \mathbb{R}^n$, we use $\|v\|$ to denote its Euclidean norm, and $\|v\|_\infty$ to denote its infinity norm. We use $B_r(v) = \{x \in \mathbb{R}^n | \|x - v\| < r\}$ to denote the open ball centered at $v$ with radius $r$. A neighborhood of $v$ is any set containing an open ball about $v$. For a square matrix $A$, we use $\|A\|$ to denote the induced 2-norm, and $\|A\|_\infty$ to denote the induced infinity norm. We use $\rho(A)$ to denote the spectral radius of $A$, and we say that $A$ is Schur stable if $\rho(A) < 1$. For set $\mathcal{A}$ in $\mathbb{R}^n$, we use $|\mathcal{A}|$ to denote the cardinality of $\mathcal{A}$, $int(\mathcal{A})$ to denote the interior of $\mathcal{A}$, $cl(\mathcal{A})$ to denote the closure of $\mathcal{A}$, and $conv(\mathcal{A})$ to denote the convex hull of $\mathcal{A}$. For sets $\mathcal{S}, \mathcal{R} \subset \mathbb{R}^n$, and $A \in \mathbb{R}^{n\times n}$, $A\mathcal{S}$ denotes the set $\{z \in \mathbb{R}^n | z = Ax, \text{ for some } x \in \mathcal{S}\}$, $\mathcal{S} + \mathcal{R}$ denotes the set $\{x + r | x \in \mathcal{S}, r \in \mathcal{R}\}$, and $\mathcal{S} \setminus \mathcal{R}$ denotes the set $\{x | x \in \mathcal{S} \text{ and } x \notin \mathcal{R}\}$. We use $d(v, \mathcal{S}) = \inf\{\|v - \alpha\| | \alpha \in$

D. Fan is currently with NVIDIA Semiconductor Technologies (Beijing), Co., Ltd. (email: dongleifan4@126.com)

$\mathcal{S}\}$ to denote the distance between the point $v$ and the set $\mathcal{S}$, and $d(\mathcal{S}, \mathcal{R}) = \inf\{\|\alpha - \beta\| | \alpha \in \mathcal{S}, \beta \in \mathcal{R}\}$ to denote the distance between sets $\mathcal{S}$ and $\mathcal{R}$. For $w \in \mathbb{C}^n$, we use $w^*$ to denote its complex conjugate.

## II. Proposed Notion of Reachability

### A. Systems of Interest and Motivation

In this manuscript, we consider linear systems with quantized control input

$$x_{t+1} = Ax_t + Bu_t, \tag{1}$$

where $t \in \mathbb{N}$ is the discrete time index, $x_t \in \mathbb{R}^n$ is the state, and $u_t \in \mathcal{U} \subset \mathbb{R}^m$, $|\mathcal{U}| < \infty$, is the input. Set $\mathcal{U}$ is a finitely quantized input alphabet. Matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are given.

Next, we recall the definition of reachability: A state $x$ is *reachable* in time $T \in \mathbb{N}$ if there exists $\{u_t\}_{t=0}^{T-1}$ such that when $x_0 = 0$, $x_T = x$, where $x_T = \sum_{t=0}^{T-1} A^{T-1-t}Bu_t$. A system is *reachable* if every state $x \in \mathbb{R}^n$ is reachable. For reachability of system (1), we first make the following observation.

*Observation*: No system (1) is reachable.

The reasoning for this observation is that the state space $\mathbb{R}^n$ is uncountable, while the reachable states of system (1) is countable due to $|\mathcal{U}| < \infty$.

Although system (1) is not reachable in the "conventional" sense, the characteristics of the reachable set of different systems (1) could be rather distinct. Specifically, for some system (1), as shown in the following example, the collection of reachable states is *dense*.

*Example* 1. f Consider a scalar system (1) with parameters $A = 1/2, B = 1$, and $\mathcal{U} = \{0, 1\}$. It is relatively straightforward to show that every state $x \in [0, 1]$ could be approached by an input sequence $\sum_{t=0}^{T-1}(1/2)^{T-1-t}u_t$ arbitrarily closely. On the other hand, if instead $A = 1/4$, then there are "many" states, say within $(1/3, 2/3)$, that can not be approached closely by any input sequence. As we will see in the following sections, the reachable set of the system $A = 1/4, B = 1, \mathcal{U} = \{0, 1\}$ is actually *nowhere dense*.

### B. Definition of Local Asymptotic Reachability

To better describe the characteristics of the reachable set of system (1), we propose the following notions of reachability. We begin with introducing a weaker notion of a state being reachable.

*Definition* 1. Given system (1), a state $x$ is *asymptotically reachable* if for any $\epsilon > 0$, there is $T \in \mathbb{N}$ and a sequence $\{u_t\}_{t=0}^{T-1}$, $u_t \in \mathcal{U}$ such that when $x_0 = 0$, $\|x_T - x\| < \epsilon$ under this input sequence.

Next we propose a notion of reachability for system (1). In this manuscript, we mainly constrain our attention to systems with Schur stable $A$ matrix. In this case, since the input set $\mathcal{U}$ is finite and consequently bounded, the reachable set is bounded as well. Therefore we consider reachability around the origin and propose the following notion.

*Definition* 2. A system (1) is *locally asymptotically reachable* if every state in some neighborhood of the origin is asymptotically reachable.

Essentially, local asymptotic reachability means that every state in some open ball around the origin can be approached by an input sequence arbitrarily closely.

Next, we intend to characterize conditions for local asymptotic reachability. Particularly, we consider the following problem of interest.

*Question:* Given a system (1), under what conditions of $A$, $B$, and $\mathcal{U}$ such that it is locally asymptotically reachable ?

## III. Conditions for Local Asymptotic Reachability

In this section, we propose both a sufficient condition and a necessary condition for local asymptotic reachability of system (1).

### A. Statement of Conditions

We begin with some relevant definitions and notations. Given a system (1), we use $\mathcal{A}$ to denote the set of all states reachable from the origin,

$$\mathcal{A} = \{\alpha \in \mathbb{R}^n : \alpha = \sum_{\tau=0}^{t} A^{t-\tau}Bu_\tau, u_\tau \in \mathcal{U}, t \in \mathbb{N}\}, \tag{2}$$

and use $\mathcal{S}$ to denote the convex hull of the closure of the reachable set $\mathcal{A}$,

$$\mathcal{S} = conv(cl(\mathcal{A})). \tag{3}$$

Next, we propose a sufficient condition, Theorem 1, and a necessary condition, Theorem 2, for local asymptotic reachability.

*Theorem* 1. Given system (1), assume $\rho(A) < 1$, for every $u \in \mathcal{U}$, $-u \in \mathcal{U}$, and there is $(\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n) \in \mathcal{U}^n$ such that $rank([B\bar{u}_1 \; AB\bar{u}_2 \; \cdots \; A^{n-1}B\bar{u}_n]) = n$. Recall $\mathcal{S}$ defined in (3). If $\mathcal{S} \subseteq A\mathcal{S} + B\mathcal{U}$, then every state $x \in \mathcal{S}$ is asymptotically reachable, and the system is locally asymptotically reachable.

*Theorem* 2. Given system (1), assume $\rho(A) < 1$, $A^{-1}$ exists, $0 \in \mathcal{U}$, and there is $(\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n) \in \mathcal{U}^n$ such that $rank([B\bar{u}_1 \; AB\bar{u}_2 \; \cdots \; A^{n-1}B\bar{u}_n]) = n$. Let $\tilde{\mathcal{U}} = \mathcal{U} \cup \{u_1 - u_2 : u_1 \in \mathcal{U}, u_2 \in \mathcal{U}\}$, $\tilde{\mathcal{A}} = \{\alpha \in \mathbb{R}^n : \alpha = \sum_{\tau=0}^{t} A^{t-\tau}Bu_\tau, u_\tau \in \tilde{\mathcal{U}}, t \in \mathbb{N}\}$, and $\tilde{\mathcal{S}} = conv(cl(\tilde{\mathcal{A}}))$. If $d(A\tilde{\mathcal{S}}, B(\tilde{\mathcal{U}} \setminus \{0\}) + A\tilde{\mathcal{S}}) > 0$, then the reachable set $\mathcal{A}$ (2) is *nowhere dense* in $\mathbb{R}^n$, and the system is not locally asymptotically reachable.

Intuitively, we consider the smallest convex set $\mathcal{S}$ covering the reachable states $\mathcal{A}$ and their limit points. If the convex set $\mathcal{S}$ is contained within its own transition according to the linear dynamics under all possible control inputs, then for every state in $\mathcal{S}$, we could find, recursively, a control sequence to approach this state arbitrarily closely. Consequently, the system is locally asymptotically reachable. On the other hand, if the transitions of an augmented convex set $\tilde{\mathcal{S}}$ under zero input and nonzero control input are of some positive distance apart, than there are open balls in the state space that are not reachable by any control input sequence. We could further show that these unreachable open balls occur arbitrarily closely to the origin, and the reachable set is nowhere dense.

To better illustrate the proposed conditions, we present a computational procedure to verify these conditions in Section IV, and provide examples to demonstrate both conditions in Section V.

### B. Derivation of Main Results

We first establish the following observation that will be instrumental in deriving the conditions for local asymptotic reachability.

*Lemma* 1. Given system (1), assume for every $u \in \mathcal{U}$, $-u \in \mathcal{U}$, and there is $(\bar{u}_1, \bar{u}_2, \dots, \bar{u}_n) \in \mathcal{U}^n$ such that $rank([B\bar{u}_1 \ AB\bar{u}_2 \ \cdots \ A^{n-1}B\bar{u}_n]) = n$. Let $\mathcal{S} = conv(cl(\mathcal{A}))$ where $\mathcal{A}$ is defined in (2), then there is $r > 0$ such that $B_r(0) \subseteq \mathcal{S}$.

*Proof:* Write $M := [B\bar{u}_1 \ AB\bar{u}_2 \ \cdots \ A^{n-1}B\bar{u}_n]$, then $M^{-1}$ exist. Let $r = 1/\|M^{-1}\|_\infty$. Then for every $x \in B_r(0) = \{x \in \mathbb{R}^n | \|x\| < r\}$, we can show that $\|x\|_\infty < r$. Next, for every $x \in B_r(0)$, let $c = M^{-1}x$ where $c \in \mathbb{R}^n$. Then $\|c\|_\infty < 1$. Write $c = [c_1 \ c_2 \dots \ c_n]^T$, and note $x = Mc$, then $x = \sum_{i=1}^n c_i A^{i-1} B\bar{u}_i$. Since $\|c\|_\infty < 1$, $|c_i| < 1$ for every $1 \le i \le n$. Consequently, for every $i \in \{1, \dots, n\}$, $c_i \in conv(\{-1, 1\})$. Therefore,

$$c_i A^{i-1} B\bar{u}_i \in conv(\{A^{i-1}B(-\bar{u}_i), A^{i-1}B\bar{u}_i\}), \quad \forall \ i \in \{1, \dots, n\}. \tag{4}$$

Next, we observe: For every $k \in \{1, \dots, n\}$,

$$\sum_{i=1}^k c_i A^{i-1} B\bar{u}_i \in conv(\{\sum_{\tau=1}^k A^{\tau-1} B u_\tau, u_\tau \in \{-\bar{u}_\tau, \bar{u}_\tau\}\}). \tag{5}$$

We use induction to show the above. Recall (4), (5) holds for $k = 1$. Since (4) holds for $i = 1$ and $i = 2$, and note that for any two sets $\mathcal{V}, \mathcal{W}$ in $\mathbb{R}^n$, $conv(\mathcal{V}) + conv(\mathcal{W}) \subseteq conv(\mathcal{V} + \mathcal{W})$, then $c_1 B\bar{u}_1 + c_2 AB\bar{u}_2 \in conv(\{\sum_{\tau=1}^2 A^{\tau-1}Bu_\tau, u_1 \in \{-\bar{u}_1, \bar{u}_1\}, u_2 \in \{-\bar{u}_2, \bar{u}_2\}\})$, and therefore (5) holds for $k = 2$. Assume (5) holds for some $2 \le k \le n-1$, for $\sum_{i=1}^{k+1} c_i A^{i-1}B\bar{u}_i = c_{k+1}A^k B\bar{u}_{k+1} + \sum_{i=1}^k c_i A^{i-1}B\bar{u}_i$, since $c_{k+1}A^kB\bar{u}_{k+1} \in conv(\{A^kB(-\bar{u}_{k+1}), A^kB\bar{u}_{k+1}\})$ and $\sum_{i=1}^k c_i A^{i-1}B\bar{u}_i \in conv(\{\sum_{\tau=1}^k A^{\tau-1}Bu_\tau, u_\tau \in \{-\bar{u}_\tau, \bar{u}_\tau\}\})$, then $\sum_{i=1}^{k+1} c_i A^{i-1}B\bar{u}_i \in conv(\{\sum_{\tau=1}^{k+1} A^{\tau-1}Bu_\tau, u_\tau \in \{-\bar{u}_\tau, \bar{u}_\tau\}\})$. Therefore (5) holds for $k+1$. By induction, (5) holds for all $1 \le k \le n$. Consequently, $x = \sum_{i=1}^n c_i A^{i-1}B\bar{u}_i \in conv(\{\sum_{\tau=1}^n A^{\tau-1}Bu_\tau, u_\tau \in \{-\bar{u}_\tau, \bar{u}_\tau\}\})$. For every $1 \le \tau \le n$, $\bar{u}_\tau \in \mathcal{U}$, then $-\bar{u}_\tau \in \mathcal{U}$ by assumption. Recall $\mathcal{A}$ (2), $\{\sum_{\tau=1}^n A^{\tau-1}Bu_\tau, u_\tau \in \{-\bar{u}_\tau, \bar{u}_\tau\}\} \subseteq \{\sum_{\tau=1}^n A^{\tau-1}Bu_\tau, u_\tau \in \mathcal{U}\} \subseteq \mathcal{A} \subseteq cl(\mathcal{A})$. Consequently, $conv(\{\sum_{\tau=1}^n A^{\tau-1}Bu_\tau, u_\tau \in \{-\bar{u}_\tau, \bar{u}_\tau\}\}) \subseteq conv(cl(\mathcal{A}))$, and $x \in \mathcal{S} = conv(cl(\mathcal{A}))$. Since this holds for every $x \in B_r(0)$, $B_r(0) \subseteq \mathcal{S}$. □

Next we derive the sufficient condition for local asymptotic reachability.

*Proof:* (Theorem 1) Since $\rho(A) < 1$ and $\mathcal{U}$ is finite and therefore bounded, then $\mathcal{A}$ (2) is bounded [6], and therefore $\mathcal{S} = conv(cl(\mathcal{A}))$ is bounded. Let $b > 0$ be such that $\|x\| < b, \forall x \in \mathcal{S}$. For any $\epsilon > 0$, since $\rho(A) < 1$, there is $T \in \mathbb{N}$ such that

$$\|A^T\| b < \epsilon. \tag{6}$$

Next, observe: Given state $x \in \mathcal{S}$, for every $k \in \mathbb{Z}_+$, there is $(u_1, u_2, \dots, u_k) \in \mathcal{U}^k$ such that

$$x - (Bu_1 + ABu_2 + \cdots + A^{k-1}Bu_k) \in A^k \mathcal{S}. \tag{7}$$

We use induction to show the above: For $k = 1$, since $x \in \mathcal{S}$ and $\mathcal{S} \subseteq A\mathcal{S} + B\mathcal{U}$, there is $s \in \mathcal{S}$ and $u_1 \in \mathcal{U}$ such that $x = As + Bu_1$. Then $x - Bu_1 = As \in A\mathcal{S}$, and (7) holds for $k = 1$. Assume (7) holds for some $k \ge 1$, there is $(u_1, u_2, \dots, u_k) \in \mathcal{U}^k$ such that $x - (Bu_1 + ABu_2 + \cdots + A^{k-1}Bu_k) \in A^k\mathcal{S}$. Write $w := x - (Bu_1 + ABu_2 + \cdots + A^{k-1}Bu_k)$, then $w = A^k s$ for some $s \in \mathcal{S}$. Since $\mathcal{S} \subseteq A\mathcal{S} + B\mathcal{U}$, there is $\tilde{s} \in \mathcal{S}$ and $u_{k+1} \in \mathcal{U}$ such that $s = A\tilde{s} + Bu_{k+1}$. Then $w = A^kBu_{k+1} + A^{k+1}\tilde{s}$, and consequently $x - (Bu_1 + ABu_2 + \cdots + A^{k-1}Bu_k + A^kBu_{k+1}) = A^{k+1}\tilde{s}$. Therefore, there is $(u_1, u_2, \dots, u_k, u_{k+1}) \in \mathcal{U}^{k+1}$ such that $x - (Bu_1 + ABu_2 + \cdots + A^{k-1}Bu_k + A^kBu_{k+1}) \in A^{k+1}\mathcal{S}$, and (7) holds for $k+1$. By induction, (7) holds for all $k \in \mathbb{Z}_+$. Consequently, for the integer $T$ satisfying (6), with a slight change of notation, there is $(\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_T) \in \mathcal{U}^T$ such that $x - \sum_{k=1}^T A^{k-1}B\tilde{u}_k \in A^T\mathcal{S}$. Given system (1) with $x_0 = 0$, and with the control sequence $\{u_t\}_{t=0}^{T-1}$ where $u_t = \tilde{u}_{T-t}, 0 \le t \le T-1$, then $x_T = \sum_{k=1}^T A^{k-1}B\tilde{u}_k$. Therefore, $x - x_T \in A^T\mathcal{S}$, and there is some $s \in \mathcal{S}$ such that $x - x_T = A^T s$. Then $\|x - x_T\| = \|A^T s\| \le \|A^T\|\|s\| < \|A^T\| b < \epsilon$. Recall Definition 1, every state $x \in \mathcal{S}$ is asymptotically reachable. Based on Lemma 1, there is $r > 0$ such that $B_r(0) \subseteq \mathcal{S}$. Recall Definition 2, system (1) is locally asymptotically reachable. □

We now shift our attention to the necessary condition for local asymptotic reachability. First we make several observations that will be useful in deriving the necessary condition.

*Lemma* 2. Given a finite input set $\mathcal{U} \subset \mathbb{R}^m$, $|\mathcal{U}| < \infty$, and a sequence of control segments $\{\mathbf{u_k}\}_{k=1}^\infty$, where $\mathbf{u_k} = \{u_t^{(k)}\}_{t=0}^{T_{(k)}-1}$, $T_{(k)} \in \mathbb{Z}_+$, $u_t^{(k)} \in \mathcal{U}, \forall \ 0 \le t \le T_{(k)} - 1$. For any $l \in \mathbb{Z}_+$, if $T_{(k)} \ge l$, $\forall \ k \in \mathbb{Z}_+$, then there is $(u_1^*, u_2^*, \dots, u_l^*) \in \mathcal{U}^l$ such that

$$|\{k \in \mathbb{Z}_+ | \ u_{T_{(k)}-1}^{(k)} = u_1^*, u_{T_{(k)}-2}^{(k)} = u_2^*, \dots, u_{T_{(k)}-l}^{(k)} = u_l^*\}| = \infty. \tag{8}$$

*Proof:* Since $T_{(k)} \ge l$, $\forall \ k \in \mathbb{Z}_+$, we first note that $\mathbb{Z}_+ = \{k \in \mathbb{Z}_+ | (u_{T_{(k)}-1}^{(k)}, u_{T_{(k)}-2}^{(k)}, \dots, u_{T_{(k)}-l}^{(k)}) \in \mathcal{U}^l\} = \bigcup_{\mathbf{u} \in \mathcal{U}^l} \{k \in \mathbb{Z}_+ | (u_{T_{(k)}-1}^{(k)}, u_{T_{(k)}-2}^{(k)}, \dots, u_{T_{(k)}-l}^{(k)}) = \mathbf{u}\}$. Next, assume

$$|\{k \in \mathbb{Z}_+ | (u^{(k)}_{T_{(k)}-1}, u^{(k)}_{T_{(k)}-2}, \ldots, u^{(k)}_{T_{(k)}-l}) = \mathbf{u}\}| < \infty, \quad \forall\, \mathbf{u} \in \mathcal{U}^l. \tag{9}$$

Since $|\mathcal{U}^l| < \infty$, we have $\sum_{\mathbf{u}\in\mathcal{U}^l} |\{k \in \mathbb{Z}_+ | (u^{(k)}_{T_{(k)}-1}, u^{(k)}_{T_{(k)}-2}, \ldots, u^{(k)}_{T_{(k)}-l}) = \mathbf{u}\}| < \infty$, which contradicts with $|\mathbb{Z}_+| = \infty$. Therefore, assumption (9) is false, and there is $\mathbf{u}^* = (u^*_1, u^*_2, \ldots, u^*_l) \in \mathcal{U}^l$ such that $|\{k \in \mathbb{Z}_+ |\ u^{(k)}_{T_{(k)}-1} = u^*_1, u^{(k)}_{T_{(k)}-2} = u^*_2, \ldots, u^{(k)}_{T_{(k)}-l} = u^*_l\}| = \infty$. □

*Lemma* 3. Given system (1), assume $A^{-1}$ exists, and $0 \in \mathcal{U}$. Recall $\mathcal{S}$ defined in (3). If a state $x \in \mathbb{R}^n$ is asymptotically reachable, then $x \in A\mathcal{S} + B\mathcal{U}$.

*Proof:* Since $x$ is asymptotically reachable, recall Definition 1, for every $k \in \mathbb{Z}_+$, there is $T_{(k)} \in \mathbb{N}$ and an input sequence $\mathbf{u}_k = \{u^{(k)}_t\}_{t=0}^{T_{(k)}-1}$ such that $\|\sum_{t=0}^{T_{(k)}-1} A^{T_{(k)}-1-t} B u^{(k)}_t - x\| < 1/k$. Consequently, $x = \lim_{k\to\infty} \sum_{t=0}^{T_{(k)}-1} A^{T_{(k)}-1-t} B u^{(k)}_t$. Since $0 \in \mathcal{U}$, without loss of generality, let $T_{(k)} \geq 1, \forall k \in \mathbb{Z}_+$. Recall Lemma 2, there is $u^* \in \mathcal{U}$ such that

$$|\{k \in \mathbb{Z}_+ |\ u^{(k)}_{T_{(k)}-1} = u^*\}| = \infty. \tag{10}$$

Consider the infinite subsequence of $\{\mathbf{u}_k\}_{k=1}^\infty$ corresponding to (10) and renaming it $\{\tilde{\mathbf{u}}_k\}_{k=1}^\infty$, then for $\tilde{\mathbf{u}}_k = \{\tilde{u}^{(k)}_t\}_{t=0}^{\tilde{T}_{(k)}-1}$, have $x = \lim_{k\to\infty} \sum_{t=0}^{\tilde{T}_{(k)}-1} A^{\tilde{T}_{(k)}-1-t} B \tilde{u}^{(k)}_t, \tilde{u}^{(k)}_{\tilde{T}_{(k)}-1} = u^*, \quad \forall\, k \in \mathbb{Z}_+$. Recall $A^{-1}$ exists by assumption, then

$$x = Bu^* + A \lim_{k\to\infty} \sum_{t=0}^{\tilde{T}_{(k)}-2} A^{\tilde{T}_{(k)}-2-t} B \tilde{u}^{(k)}_t. \tag{11}$$

Recall (2), $\sum_{t=0}^{\tilde{T}_{(k)}-2} A^{\tilde{T}_{(k)}-2-t} B \tilde{u}^{(k)}_t \in \mathcal{A}$, and consequently $\lim_{k\to\infty} \sum_{t=0}^{\tilde{T}_{(k)}-2} A^{\tilde{T}_{(k)}-2-t} B \tilde{u}^{(k)}_t \in cl(\mathcal{A}) \in conv(cl(\mathcal{A})) = \mathcal{S}$. Therefore, $x \in A\mathcal{S} + B\mathcal{U}$. □

*Lemma* 4. Given system (1), assume $A^{-1}$ exists, and $0 \in \mathcal{U}$. Recall $\mathcal{S}$ defined in (3). If a state $x \in \mathbb{R}^n$ is not asymptotically reachable, and $Ax \notin (B(\mathcal{U} \setminus \{0\}) + A\mathcal{S})$, then $Ax$ is not asymptotically reachable.

*Proof:* We prove by contradiction. Assume $Ax$ is asymptotically reachable, recall the derivation of Lemma 3, there is a sequence of control segments $\{\mathbf{u}_k\}_{k=1}^\infty$, where $\mathbf{u}_k = \{u^{(k)}_t\}_{t=0}^{T_{(k)}-1}$, and $u^* \in \mathcal{U}$ such that $Ax = \lim_{k\to\infty} \sum_{t=0}^{T_{(k)}-1} A^{T_{(k)}-1-t} B u^{(k)}_t$, and $u^{(k)}_{T_{(k)}-1} = u^*, \quad \forall\, k \in \mathbb{Z}_+$. Since $A^{-1}$ exists by assumption, similar to (11), have

$$Ax = Bu^* + A \lim_{k\to\infty} \sum_{t=0}^{T_{(k)}-2} A^{T_{(k)}-2-t} B u^{(k)}_t. \tag{12}$$

If $u^* \in (\mathcal{U} \setminus \{0\})$, recall $\mathcal{S}$ (3), then $\lim_{k\to\infty} \sum_{t=0}^{T_{(k)}-2} A^{T_{(k)}-2-t} B u^{(k)}_t \in cl(\mathcal{A}) \subseteq \mathcal{S}$, and therefore $Ax \in (B(\mathcal{U} \setminus \{0\}) + A\mathcal{S})$, which contradicts with the hypothesis that $Ax \notin (B(\mathcal{U} \setminus \{0\}) + A\mathcal{S})$. Consequently, $u^* \notin (\mathcal{U} \setminus \{0\})$ and therefore $u^* = 0$. Recall (12), and since $A^{-1}$ exists, have $x = \lim_{k\to\infty} \sum_{t=0}^{T_{(k)}-2} A^{T_{(k)}-2-t} B u^{(k)}_t$, which contradicts with $x$ being not asymptotically reachable. Consequently, the assumption of $Ax$ being asymptotically reachable is false. □

Now we are ready to prove Theorem 2.

*Proof:* (Theorem 2) Given system (1), we first define an augmented input set $\tilde{\mathcal{U}}$ as

$$\tilde{\mathcal{U}} = \mathcal{U} \cup \{u_1 - u_2 : u_1 \in \mathcal{U}, u_2 \in \mathcal{U}\}, \tag{13}$$

and consider the following augmented system

$$x_{t+1} = Ax_t + Bu_t, \quad u_t \in \tilde{\mathcal{U}}, \tag{14}$$

where matrices $A, B$ are identical to that of the original system (1), and $\tilde{\mathcal{U}}$ is defined in (13). Given system (14), we observe: There is $\bar{x} \in \mathbb{R}^n$ such that

$$A^k \bar{x} \text{ is not asymptotically reachable}, \quad \forall\, k \in \mathbb{Z}_+, \tag{15}$$

where asymptotic reachability in the above is with respect to system (14). We show the above observation by considering the eigenvalue-eigenvector pairs $(\lambda, v)$ of $A$.

*Case 1):* $\lambda \in \mathbb{R}$. Without loss of generality, let $v \in \mathbb{R}^n$, $\|v\| = 1$, and $span(v) = \{av \mid a \in \mathbb{R}\}$. Recall sets $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{S}}$ defined in the statement of Theorem 2,

$$\tilde{\mathcal{A}} = \{\alpha \in \mathbb{R}^n : \alpha = \sum_{\tau=0}^{t} A^{t-\tau} B u_\tau, u_\tau \in \tilde{\mathcal{U}}, t \in \mathbb{N}\}, \tag{16}$$

$$\tilde{\mathcal{S}} = conv(cl(\tilde{\mathcal{A}})). \tag{17}$$

Since $\rho(A) < 1$ and $\tilde{\mathcal{U}}$ (13) is finite, $A\tilde{\mathcal{S}}$ is bounded. Define $\bar{c} \in \mathbb{R}$ as:

$$\bar{c} = \sup\{c \in \mathbb{R} | c = \langle x, v\rangle, x \in span(v) \cap A\tilde{\mathcal{S}}\}. \tag{18}$$

where $\langle\cdot,\cdot\rangle$ denotes the dot product in $\mathbb{R}^n$. Since $span(v) \cap A\tilde{\mathcal{S}}$ is bounded, $\bar{c}$ is well-defined. Recall Lemma 1, and note that for every $u \in \tilde{\mathcal{U}}$ (13), $-u \in \tilde{\mathcal{U}}$, and that $\mathcal{U} \subseteq \tilde{\mathcal{U}}$, then there is $r > 0$ such that $B_r(0) \subseteq \tilde{\mathcal{S}}$. Consequently $AB_r(0) \subseteq A\tilde{\mathcal{S}}$. Since $A^{-1}$ exists by assumption, there is $\tilde{r} > 0$ such that $B_{\tilde{r}}(0) \subseteq AB_r(0)$. Rewrite $\tilde{r}$ as $r$, then there is $r > 0$ such that

$$B_r(0) \subseteq A\tilde{\mathcal{S}}. \tag{19}$$

Since $(r/2)v \in span(v) \cap A\tilde{\mathcal{S}}$, note $\langle (r/2)v, v\rangle = r/2 > 0$, and recall (18), we have

$$\bar{c} > 0. \tag{20}$$

Note $A^{-1}$ exists and $\rho(A) < 1$ by assumption, have $0 < |\lambda| < 1$. Recall $d(A\tilde{\mathcal{S}}, B(\tilde{\mathcal{U}} \setminus \{0\}) + A\tilde{\mathcal{S}}) > 0$ as stated in the hypotheses of Theorem 2, let $\bar{\epsilon}$ be

$$\bar{\epsilon} = \min\{1/2, \frac{1-|\lambda|}{2|\lambda|}, \frac{d(A\tilde{\mathcal{S}}, B(\tilde{\mathcal{U}} \setminus \{0\}) + A\tilde{\mathcal{S}})}{4\bar{c}}\}, \tag{21}$$

then $0 < \bar{\epsilon} < 1$. Recall (15), choose $\bar{x}$ as

$$\bar{x} = (1 + \bar{\epsilon})\bar{c}v. \tag{22}$$

Next we show that $\bar{x}$ satisfies (15) by induction. For $k = 1$ in

(15), we first show that $\bar{x} = (1+\bar{\epsilon})\bar{c}v \notin A\tilde{\mathcal{S}} + B\tilde{\mathcal{U}}$. Assume $(1+\bar{\epsilon})\bar{c}v \in A\tilde{\mathcal{S}}$, then $(1+\bar{\epsilon})\bar{c}v \in span(v) \cap A\tilde{\mathcal{S}}$. Recall (18), $\bar{c} \geq (1+\bar{\epsilon})\bar{c}$, and therefore $\bar{\epsilon}\bar{c} \leq 0$, which contradicts with $\bar{\epsilon} > 0$ (21) and $\bar{c} > 0$ (20). Therefore the assumption $(1+\bar{\epsilon})\bar{c}v \in A\tilde{\mathcal{S}}$ is false, and consequently

$$\bar{x} = (1+\bar{\epsilon})\bar{c}v \notin A\tilde{\mathcal{S}}. \tag{23}$$

Next, consider the distance $d((1+\bar{\epsilon})\bar{c}v, A\tilde{\mathcal{S}})$. Recall (18), for $(1-\bar{\epsilon})\bar{c} < \bar{c}$, there is $c_1$ with $(1-\bar{\epsilon})\bar{c} < c_1 < \bar{c}$ such that $c_1 \in \{c \in \mathbb{R} | c = \langle x, v\rangle, x \in span(v) \cap A\tilde{\mathcal{S}}\}$. Consequently there is $x_1 = c_1 v \in span(v) \cap A\tilde{\mathcal{S}}$. Since $x_1 \in A\tilde{\mathcal{S}}$, we have $d((1+\bar{\epsilon})\bar{c}v, A\tilde{\mathcal{S}}) \leq d((1+\bar{\epsilon})\bar{c}v, x_1) = (1+\bar{\epsilon})\bar{c} - c_1$. Since $c_1 > (1-\bar{\epsilon})\bar{c}$, we have $(1+\bar{\epsilon})\bar{c} - c_1 < 2\bar{\epsilon}\bar{c}$. Recall (21), (20), we have $2\bar{\epsilon}\bar{c} \leq d(A\tilde{\mathcal{S}}, B(\tilde{\mathcal{U}} \setminus \{0\}) + A\tilde{\mathcal{S}})/2$. Therefore, $d((1+\bar{\epsilon})\bar{c}v, A\tilde{\mathcal{S}}) < d(A\tilde{\mathcal{S}}, B(\tilde{\mathcal{U}} \setminus \{0\}) + A\tilde{\mathcal{S}})/2$. Consequently, $d((1+\bar{\epsilon})\bar{c}v, B(\tilde{\mathcal{U}} \setminus \{0\}) + A\tilde{\mathcal{S}}) \geq d(A\tilde{\mathcal{S}}, B(\tilde{\mathcal{U}} \setminus \{0\}) + A\tilde{\mathcal{S}}) - d((1+\bar{\epsilon})\bar{c}v, A\tilde{\mathcal{S}}) > d(A\tilde{\mathcal{S}}, B(\tilde{\mathcal{U}} \setminus \{0\}) + A\tilde{\mathcal{S}})/2 > 0$. Consequently,

$$(1+\bar{\epsilon})\bar{c}v \notin (B(\tilde{\mathcal{U}} \setminus \{0\}) + A\tilde{\mathcal{S}}). \tag{24}$$

Note that $A\tilde{\mathcal{S}} + B\tilde{\mathcal{U}} = A\tilde{\mathcal{S}} \cup (B(\tilde{\mathcal{U}} \setminus \{0\}) + A\tilde{\mathcal{S}})$, by (23), (24), we have $\bar{x} = (1+\bar{\epsilon})\bar{c}v \notin A\tilde{\mathcal{S}} + B\tilde{\mathcal{U}}$. Recall Lemma 3, $\bar{x} = (1+\bar{\epsilon})\bar{c}v$ is not asymptotically reachable w.r.t. (14). Next, we show that $A\bar{x} \notin (B(\tilde{\mathcal{U}} \setminus \{0\}) + A\tilde{\mathcal{S}})$. Note $A\bar{x} = \lambda(1+\bar{\epsilon})\bar{c}v$, recall (21), $\bar{\epsilon} \leq \frac{1-|\lambda|}{2|\lambda|}$, and consequently $|\lambda|(1+\bar{\epsilon})\bar{c} < \bar{c}$. Recall (18), there is $c_2$ with

$$|\lambda|(1+\bar{\epsilon})\bar{c} < c_2 < \bar{c} \tag{25}$$

such that $c_2 \in \{c \in \mathbb{R} | c = \langle x, v\rangle, x \in span(v) \cap A\tilde{\mathcal{S}}\}$. Consequently there is $x_2 = c_2 v \in span(v) \cap A\tilde{\mathcal{S}}$. Since $\tilde{\mathcal{S}}$ is convex, $A\tilde{\mathcal{S}}$ is also convex. Since $0 \in \tilde{\mathcal{U}}$, and $\forall\ u \in \tilde{\mathcal{U}}$, $-u \in \tilde{\mathcal{U}}$, we have

$$\{cv | |c| \leq c_2\} \subseteq A\tilde{\mathcal{S}}. \tag{26}$$

Consequently $A\bar{x} \in A\tilde{\mathcal{S}}$. Since $d(A\tilde{\mathcal{S}}, B(\tilde{\mathcal{U}} \setminus \{0\}) + A\tilde{\mathcal{S}}) > 0$, have $A\bar{x} \notin (B(\tilde{\mathcal{U}} \setminus \{0\}) + A\tilde{\mathcal{S}})$. Since $\bar{x}$ is not asymptotically reachable w.r.t. (14), by Lemma 4 $A\bar{x}$ is not asymptotically reachable, and $\bar{x}$ (22) satisfies (15) for $k = 1$. Next, assume $A^k\bar{x}$ is not asymptotically reachable for some $k \geq 1$. Since $|\lambda| < 1$, we have $|\lambda^{k+1}(1+\bar{\epsilon})\bar{c}| < |\lambda|(1+\bar{\epsilon})\bar{c}$. Recall (25) (26), we have $A^{k+1}\bar{x} \in A\tilde{\mathcal{S}}$, and therefore $A^{k+1}\bar{x} \notin (B(\tilde{\mathcal{U}} \setminus \{0\}) + A\tilde{\mathcal{S}})$. Recall Lemma 4, $A^{k+1}\bar{x}$ is not asymptotically reachable. By induction, $\bar{x}$ (22) satisfies (15) for all $k \in \mathbb{Z}_+$ when $\lambda \in \mathbb{R}$.

*Case 2): $\lambda \in \mathbb{C} \setminus \mathbb{R}$.* Let $v^*$ be the complex conjugate of the eigenvector $v$. Write $\lambda = re^{i\theta}, v = w + iu$, then

$$A^k(v + v^*) = 2r^k(w\cos(k\theta) - u\sin(k\theta)),\ k \in \mathbb{Z}_+ \tag{27}$$

where $0 < r < 1$, $-\pi \leq \theta < \pi$, $w \in \mathbb{R}^n$, $u \in \mathbb{R}^n$. Since $A\tilde{\mathcal{S}}$ is bounded, without loss of generality, let $A^k(v+v^*) \notin A\tilde{\mathcal{S}}$. Define $\bar{k} \in \mathbb{Z}_+$ as

$$\bar{k} = \max\{k \in \mathbb{Z}_+ | A^k(v+v^*) \notin A\tilde{\mathcal{S}}\}. \tag{28}$$

Recall (19) (27), $\{k \in \mathbb{Z}_+ | A^k(v+v^*) \notin A\tilde{\mathcal{S}}\}$ is finite and $\bar{k}$ is well-defined. If $d(A^{\bar{k}}(v+v^*), A\tilde{\mathcal{S}}) > d(A\tilde{\mathcal{S}}, B(\tilde{\mathcal{U}} \setminus \{0\}) + A\tilde{\mathcal{S}})/2$, define a function $f : [0,1] \to \mathbb{R}$ as: $f(s) = d(s \cdot A^{\bar{k}}(v+v^*), A\tilde{\mathcal{S}}), s \in [0,1]$. Then it can be shown that $f : [0,1] \to \mathbb{R}$ is continuous. Thus the range of $f$ is an interval $[c,d]$ for some $c, d \in \mathbb{R}$. Since $f(0) = 0$, $f(1) = d(A^{\bar{k}}(v+v^*), A\tilde{\mathcal{S}})$, there is $\bar{s} \in (0,1)$ such that $f(\bar{s}) = d(\bar{s} \cdot A^{\bar{k}}(v+v^*), A\tilde{\mathcal{S}}) = \frac{1}{2}d(A\tilde{\mathcal{S}}, B(\tilde{\mathcal{U}} \setminus \{0\}) + A\tilde{\mathcal{S}})$. Let $\bar{\alpha}$ be

$$\bar{\alpha} = \begin{cases} \bar{s}, & \text{if } d(A^{\bar{k}}(v+v^*), A\tilde{\mathcal{S}}) > \\ & \qquad d(A\tilde{\mathcal{S}}, B(\tilde{\mathcal{U}} \setminus \{0\}) + A\tilde{\mathcal{S}})/2, \\ 1, & \text{otherwise.} \end{cases} \tag{29}$$

Recall (15), choose $\bar{x}$ as

$$\bar{x} = \bar{\alpha}A^{\bar{k}}(v+v^*), \tag{30}$$

where $\bar{k}$ is given in (28) and $\bar{\alpha}$ is given in (29). We will show $\bar{x}$ satisfies (15) by induction. For $k = 1$, recall (29), have $d(\bar{x}, A\tilde{\mathcal{S}}) \leq \frac{1}{2}d(A\tilde{\mathcal{S}}, B(\tilde{\mathcal{U}} \setminus \{0\}) + A\tilde{\mathcal{S}})$, and $\bar{x} \notin A\tilde{\mathcal{S}}$. Consequently $d(\bar{x}, B(\tilde{\mathcal{U}} \setminus \{0\}) + A\tilde{\mathcal{S}}) \geq d(A\tilde{\mathcal{S}}, B(\tilde{\mathcal{U}} \setminus \{0\}) + A\tilde{\mathcal{S}}) - d(\bar{x}, A\tilde{\mathcal{S}}) > 0$, and therefore $\bar{x} \notin B(\tilde{\mathcal{U}} \setminus \{0\}) + A\tilde{\mathcal{S}}$. Consequently $\bar{x} \notin B\tilde{\mathcal{U}} + A\tilde{\mathcal{S}}$. Recall Lemma 3, $\bar{x}$ is not asymptotically reachable w.r.t. (14). Recall (28), for every $m \in \mathbb{Z}_+$, $A^{\bar{k}+m}(v+v^*) \in A\tilde{\mathcal{S}}$. Since $A\tilde{\mathcal{S}}$ is convex, $0 \in A\tilde{\mathcal{S}}$, and $\bar{\alpha}$ (29) satisfies $0 < \bar{\alpha} \leq 1$, we have

$$A^m\bar{x} = \bar{\alpha}A^{\bar{k}+m}(v+v^*) \in A\tilde{\mathcal{S}}, \quad \forall\ m \in \mathbb{Z}_+. \tag{31}$$

Therefore $A\bar{x} \in A\tilde{\mathcal{S}}$, and therefore $A\bar{x} \notin B(\tilde{\mathcal{U}} \setminus \{0\}) + A\tilde{\mathcal{S}}$. Recall Lemma 4, $A\bar{x}$ is not asymptotically reachable, i.e. $\bar{x}$ (30) satisfies (15) for $k = 1$. Next, assume $A^k\bar{x}$ is not asymptotically reachable for some $k \geq 1$, recall (31), $A^{k+1}\bar{x} \in A\tilde{\mathcal{S}}$, and therefore $A^{k+1}\bar{x} \notin B(\tilde{\mathcal{U}} \setminus \{0\}) + A\tilde{\mathcal{S}}$. Recall Lemma 4, $A^{k+1}\bar{x}$ is not asymptotically reachable. By induction, $\bar{x}$ (30) satisfies (15) for all $k \in \mathbb{Z}_+$ when $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Up to this point, we have shown (15) by choosing $\bar{x}$ according to either (22) or (30), depending on whether the eigenvalue $\lambda$ is real or not. Next, we show that $\mathcal{A}$ (2) is nowhere dense. For every $\alpha \in cl(\mathcal{A})$, given $\epsilon > 0$, since $\rho(A) < 1$, there is $K \in \mathbb{Z}_+$ such that

$$\|A^K\bar{x}\| < \epsilon/3. \tag{32}$$

Since $\mathcal{A}$ (2) is bounded, there is $b > 0$ such that $\|\alpha\| < b, \forall\ \alpha \in \mathcal{A}$. Since $\rho(A) < 1$, there is $L \in \mathbb{Z}_+$ such that

$$\|A^L\|b < \epsilon/3. \tag{33}$$

Since $\alpha \in cl(\mathcal{A})$, there is $\{\mathbf{u}_k\}_{k=1}^{\infty}$, where $\mathbf{u}_k = \{u_t^{(k)}\}_{t=0}^{T_{(k)}-1}$, $u_t^{(k)} \in \mathcal{U}$, $0 \leq t \leq T_{(k)} - 1$, such that $\alpha = \lim_{k\to\infty} \sum_{t=0}^{T_{(k)}-1} A^{T_{(k)}-1-t}Bu_t^{(k)}$. Since $0 \in \mathcal{U}$, w.l.o.g. let $T_{(k)} \geq L, \forall\ k \in \mathbb{Z}_+$. Recall Lemma 2, and similar to the derivation of Lemma 3, there is $\{\mathbf{u}_k\}_{k=1}^{\infty}$, where $\mathbf{u}_k = \{u_t^{(k)}\}_{t=0}^{T_{(k)}-1}$, $u_t^{(k)} \in \mathcal{U}$, $0 \leq t \leq T_{(k)} - 1$, and $(u_1^*, u_2^*, \ldots, u_L^*) \in \mathcal{U}^L$ such that

$$\alpha = \lim_{k\to\infty} \sum_{t=0}^{T_{(k)}-1} A^{T_{(k)}-1-t}Bu_t^{(k)},$$
$$u_{T_{(k)}-1}^{(k)} = u_1^*, u_{T_{(k)}-2}^{(k)} = u_2^*, \ldots, u_{T_{(k)}-L}^{(k)} = u_L^*, \forall\ k \in \mathbb{Z}_+. \tag{34}$$

Let $\delta \in \mathbb{R}^n$ be

$$\delta = \sum_{l=1}^{L} A^{l-1}Bu_l^* + A^K\bar{x}. \tag{35}$$

Recall (34), there is $N \in \mathbb{Z}_+$ such that $\|\sum_{t=0}^{T_{(k)}-1} A^{T_{(k)}-1-t}Bu_t^{(k)} - \alpha\| < \epsilon/3, \forall\ k \geq N$. For any $k \geq N$, $\|\delta - \alpha\| \leq \|\sum_{t=0}^{T_{(k)}-1} A^{T_{(k)}-1-t}Bu_t^{(k)} - \alpha\| + \|\delta - \sum_{t=0}^{T_{(k)}-1} A^{T_{(k)}-1-t}Bu_t^{(k)}\| < \|\delta - \sum_{t=0}^{T_{(k)}-1} A^{T_{(k)}-1-t}Bu_t^{(k)}\| + \epsilon/3$. Recall (34), (35), we have $\delta - \sum_{t=0}^{T_{(k)}-1} A^{T_{(k)}-1-t}Bu_t^{(k)} = A^K\bar{x} - A^L\sum_{t=0}^{T_{(k)}-L-1} A^{T_{(k)}-L-1-t}Bu_t^{(k)}$. Note that $\sum_{t=0}^{T_{(k)}-L-1} A^{T_{(k)}-L-1-t}Bu_t^{(k)} \in \mathcal{A}$, and recall (32), (33), we have $\|\delta - \sum_{t=0}^{T_{(k)}-1} A^{T_{(k)}-1-t}Bu_t^{(k)}\| < 2\epsilon/3$. Thus $\delta \in B_\epsilon(\alpha)$, where $\delta$ is given in (35). Next, we show $\delta \notin cl(\mathcal{A})$ by contradiction. Assume $\delta \in cl(\mathcal{A})$, then as in previous arguments, there is $\{\mathbf{w}_k\}_{k=1}^\infty$, where $\mathbf{w}_k = \{w_t^{(k)}\}_{t=0}^{T_{(k)}-1}$, $w_t^{(k)} \in \mathcal{U}$, $0 \leq t \leq T_{(k)} - 1$, and $(w_1^*, w_2^*, \ldots, w_L^*) \in \mathcal{U}^L$ such that

$$\delta = \lim_{k\to\infty} \sum_{t=0}^{T_{(k)}-1} A^{T_{(k)}-1-t}Bw_t^{(k)},$$
$$w_{T_{(k)}-1}^{(k)} = w_1^*, w_{T_{(k)}-2}^{(k)} = w_2^*, \ldots, w_{T_{(k)}-L}^{(k)} = w_L^*,\ k \in \mathbb{Z}_+. \tag{36}$$

Recall (35), we have $A^K\bar{x} = \delta - \sum_{l=1}^{L} A^{l-1}Bu_l^* = \lim_{k\to\infty}(\sum_{l=1}^{L} A^{l-1}B(w_l^* - u_l^*) + \sum_{t=0}^{T_{(k)}-L-1} A^{T_{(k)}-1-t}Bw_t^{(k)})$. Recall $\tilde{\mathcal{U}}$ (13), define $\{\tilde{\mathbf{w}}_k\}_{k=1}^\infty$, where $\tilde{\mathbf{w}}_k = \{\tilde{w}_t^{(k)}\}_{t=0}^{\tilde{T}_{(k)}-1}$, $\tilde{w}_t^{(k)} \in \tilde{\mathcal{U}}$, $0 \leq t \leq \tilde{T}_{(k)} - 1$, as

$$\begin{cases} \tilde{T}_{(k)} = T_{(k)}, \\ \tilde{w}_{\tilde{T}_{(k)}-l}^{(k)} = (w_l^* - u_l^*) \in \tilde{\mathcal{U}}, \quad l \in \{1, 2, \ldots, L\}, \\ \tilde{w}_{\tilde{T}_{(k)}-l}^{(k)} = w_{T_{(k)}-l}^{(k)}, \quad l \in \{L+1, \ldots, \tilde{T}_{(k)}\}. \end{cases}$$

Then we have $A^K\bar{x} = \lim_{k\to\infty}\sum_{t=0}^{\tilde{T}_{(k)}-1} A^{\tilde{T}_{(k)}-1-t}B\tilde{w}_t^{(k)}$, consequently $A^K\bar{x}$ is asymptotically reachable with respect the augmented system (14), which contradicts with (15), and therefore the assumption $\delta \in cl(\mathcal{A})$ is false. Consequently, $\delta \notin cl(\mathcal{A})$. To briefly summarize, for every $\alpha \in cl(\mathcal{A})$, and given any $\epsilon > 0$, there is $\delta \in B_\epsilon(\alpha)$ such that $\delta \notin cl(\mathcal{A})$. Therefore $int(cl(\mathcal{A})) = \varnothing$, and $\mathcal{A}$ (2) is nowhere dense. Similarly, since $0 \in \mathcal{A}$, for any open ball $B_\epsilon(0)$, there is $\delta \in B_\epsilon(0)$ that is not asymptotically reachable. Recall Definition 2, system (1) is not asymptotically reachable.

## IV. Algorithmic Verification of the Conditions for Local Asymptotic Reachability

In this section, we present a computational procedure to verify the conditions for local asymptotic reachability proposed in Section III. Particularly, we present an algorithm to approximate the convex set $\mathcal{S} = conv(cl(\mathcal{A}))$ arbitrarily closely. In this section, we assume that the matrix $A$ of system (1) is Schur stable. The computational procedure is shown in the following.

**Algorithm 1** Compute approximations of $\mathcal{S} = conv(cl(\mathcal{A}))$

**Input**: Matrices $A$, $B$, set $\mathcal{U}$, approximation tolerance $\epsilon > 0$
**Compute**: $b > 0$ such that $\|x\|_\infty < b, \forall x \in cl(\mathcal{A})$
**Compute**: $N \in \mathbb{N}$ such that $\|A^N\|_\infty b < \epsilon/4$
**Compute**: $\mathcal{A}_N = \{x \in \mathbb{R}^n : x = \sum_{\tau=0}^{t} A^{t-\tau}Bu_\tau, u_\tau \in \mathcal{U}, t \in \{0, 1, \ldots, N\}\}$
**Compute**: $\mathcal{B} = \{x \in \mathbb{R}^n : \|x\|_\infty \leq \epsilon/2\}$
**Compute**: $\underline{\mathcal{S}} = conv(\mathcal{A}_N)$
**Compute**: $\bar{\mathcal{S}} = conv(\mathcal{A}_N + \mathcal{B})$
**Return**: $\underline{\mathcal{S}}$, $\bar{\mathcal{S}}$

We make the following observation of the computed polytopes $\underline{\mathcal{S}}$ and $\bar{\mathcal{S}}$.

*Observation:* Given system (1) with Schur stable matrix $A$ and $\mathcal{S}$ defined in (3), the $\underline{\mathcal{S}}$ and $\bar{\mathcal{S}}$ returned by Algorithm 1 satisfy the following:

- $\underline{\mathcal{S}} \subset \mathcal{S} \subset \bar{\mathcal{S}}$.
- For any $x \in \mathcal{S}$, there is $x' \in \underline{\mathcal{S}}$ such that $\|x - x'\|_\infty < \epsilon$.
- For any $x \in \bar{\mathcal{S}}$, there is $x'' \in \mathcal{S}$ such that $\|x - x''\|_\infty < \epsilon$.

Essentially, $\underline{\mathcal{S}}$ is an inner approximation of $\mathcal{S}$ and $\bar{\mathcal{S}}$ is an outer approximation of $\mathcal{S}$. We can also specify the difference between $\underline{\mathcal{S}}$ and $\mathcal{S}$ as well as the difference between $\bar{\mathcal{S}}$ and $\mathcal{S}$ to be arbitrarily small by choosing the approximation tolerance $\epsilon$. With the computed approximations $\underline{\mathcal{S}}$ and $\bar{\mathcal{S}}$, we could use them to verify the conditions for local asymptotic reachability. Specifically, for Theorem 1: If $\bar{\mathcal{S}} \subseteq A\underline{\mathcal{S}} + B\mathcal{U}$, then $\mathcal{S} \subseteq A\mathcal{S} + B\mathcal{U}$; for Theorem 2: Let $\underline{\tilde{\mathcal{S}}}$, $\bar{\tilde{\mathcal{S}}}$ be the returned values from Algorithm 1 with the input $(A, B, \tilde{\mathcal{U}})$, where $\tilde{\mathcal{U}}$ is given in (13), if $d(A\bar{\tilde{\mathcal{S}}}, B(\tilde{\mathcal{U}} \setminus \{0\}) + A\bar{\tilde{\mathcal{S}}}) > 0$, then $d(A\tilde{\mathcal{S}}, B(\tilde{\mathcal{U}} \setminus \{0\}) + A\tilde{\mathcal{S}}) > 0$.

Here we comment on the complexity of Algorithm 1: The upper bound $b$ could be computed in constant time [5]. Given an approximation tolerance $\epsilon$, the input sequence length $N$ is in the order of $\log_{\rho(A)} \epsilon$, where $\rho(A) < 1$ is the spectral radius of matrix $A$. The time complexity to compute the set $\mathcal{A}_N$ is in the order of $|\mathcal{U}|^N$, and therefore the complexity to compute the convex hulls $\underline{\mathcal{S}}$ and $\bar{\mathcal{S}}$ is in the order of $N|\mathcal{U}|^N$.

*Remark*: From another point of view, we are presenting a method to approximately identify the reachable set of system (1). Particularly, if the hypotheses in Theorem 1 are satisfied, then every state in $\mathcal{S} = conv(cl(\mathcal{A}))$ is asymptotically reachable. On the other hand, any state outside of $\mathcal{S}$ is not asymptotically reachable. Essentially, $\mathcal{S}$ contains exactly the asymptotically reachable states. Therefore, Algorithm 1 approximately identifies all the asymptotically reachable states of system (1).

## V. Illustrative Examples

In this section, we present a locally asymptotically reachable system, Example 3, and another one, Example 2, that is not asymptotically reachable.

*Example* 2. Consider a system (1) with parameters with parameters: $A = \begin{bmatrix} 0.4 & 0.3 \\ 0 & 0.5 \end{bmatrix}, B = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}, \mathcal{U} = \{0, 1\}$. We use Algorithm 1 to verify the hypotheses in Theorem 2.

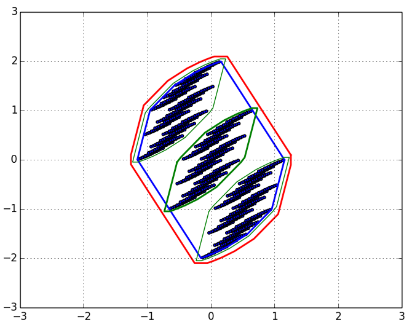


Fig. 1: A plant that is not locally asymptotically reachable.

As shown in the above figure, with approximation tolerance $\epsilon = 0.2$, the blue polytope represents $\underline{\tilde{\mathcal{S}}}$ and the red polytope represents $\bar{\tilde{\mathcal{S}}}$ as stated in Section IV. Note that $\underline{\tilde{\mathcal{S}}}$ and $\bar{\tilde{\mathcal{S}}}$ are inner and outer approximations of $\tilde{\mathcal{S}}$ (17) respectively. The scattered points represent elements of $\tilde{\mathcal{A}}$ (16). The green polytopes represent the set $B\tilde{\mathcal{U}} + A\bar{\tilde{\mathcal{S}}}$, and the green polytope with thicker lines represent the set $A\bar{\tilde{\mathcal{S}}}$. As shown in Figure 1, $d(A\bar{\tilde{\mathcal{S}}}, B(\tilde{\mathcal{U}} \setminus \{0\}) + A\bar{\tilde{\mathcal{S}}}) > 0$. Consequently, the hypotheses of Theorem 2 are satisfied and the reachable set $\mathcal{A}$ (2) is nowhere dense in $\mathbb{R}^2$.

*Example* 3. Consider a system (1) with parameters: $A = \begin{bmatrix} 0.91 & 0.1 \\ 0 & 0.92 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathcal{U} = \{[0\ 0]^T, [0\ 1]^T, [0\ -1]^T, [-0.2\ 0.2]^T, [0.2\ -0.2]^T\}$. We run Algorithm 1 and present the following result:

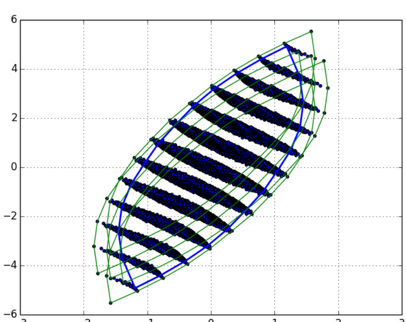


Fig. 2: A locally asymptotically reachable plant.

In the above figure, the blue polytope $\underline{\mathcal{S}}$ represents an inner approximation of $\mathcal{S}$ (3) with $N = 5$ in Algorithm 1. The green polytopes represent the set $B\mathcal{U} + A\underline{\mathcal{S}}$, and the scattered points represent elements of $\mathcal{A}$ (2). As shown in the figure $\underline{\mathcal{S}} \subseteq B\mathcal{U} + A\underline{\mathcal{S}}$, and consequently, by the derivation of Theorem 1, the system is locally asymptotically reachable and the reachable set $\mathcal{A}$ (2) is dense in some neighborhood of the origin.

## VI. Conclusion and Future Work

In this paper, we motivate and propose a notion of local asymptotic reachability for linear systems with finite input alphabet. We characterize the proposed local asymptotic reachability by deriving both a sufficient condition and a necessary condition for reachability. The proposed conditions involve considering the convex hull of the closure of the reachable states and its transition under the linear dynamics. We also present an algorithm to help verify the proposed conditions. Future work will aim to further develop and generalize the proposed conditions such that they could be applied to identify local asymptotic reachability for more systems, and will try to potentially arrive at a necessary and sufficient condition for reachability. We would also like to investigate how likely it is that no reachability guarantees can be given for an arbitrary system using the conditions proposed in this manuscript. We also plan to study reachability with finite length input sequences whereas the current manuscript mainly considers the asymptotic behaviors. Towards a more practical end, we wish to identify real life scenarios where the proposed results in this manuscript could be applied. Finally, we would like to study the impact of disturbances in the system and if the proposed conditions may be extended for non-Schur systems.

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## References


[1] Y. Anzai. A note on reachability of discrete-time quantized conrol systems. *IEEE Transactions on Automatic Control*, 19(5):575–577, 1974.

[2] A. Bicchi, A. Marigo, and B. Piccoli. On the reachability of quantized control systems. *IEEE Transactions on Automatic Control*, 47(4):546–563, 2002.

[3] Y. Chitour and B. Piccoli. Controllability for discrete systems with a finite control set. *Mathematics of Control, Signals and Systems*, 14:173–193, 2001.

[4] D. F. Delchamps. Stabilizing a linear system with quantized state feedback. *IEEE transactions on automatic control*, 35(8):916–924, 1990.

[5] D. Fan and D. C. Tarraf. On finite memory observability of a class of systems over finite alphabets with linear dynamics. In *Proceedings of the 53rd IEEE Conference on Decision and Control*, pages 3884–3891, Los Angeles, CA, 2014.

[6] D. Fan and D. C. Tarraf. Finite uniform bisimulations for linear systems with finite input alphabets. *IEEE Transactions on Automatic Control*, 62(8):4144–4150, 2017.

[7] D. Fan and D. C. Tarraf. Output observability of systems over finite alphabets with linear internal dynamics. *IEEE Transactions on Automatic Control*, 63(10):3404–3417, 2018.

[8] J. Lunze and B. Nixdorf. Discrete reachability of hybrid systems. *International Journal of Control*, 76(14):1453–1468, 2003.