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The Model Matching Problem for Periodic Max-Plus Systems

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***Abstract*— Max-plus linear systems are suitable to model discrete event systems with synchronization phenomena, but not competition. In specific situations, competition can be introduced by considering event-varying periodic parameters, which allow us to model shared resources allocated in accordance to a periodic schedule, thus obtaining a periodic max-plus linear system. In this paper, we propose an extension of the geometric approach to systems of such class. The new results can be used to solve the model matching problem, so as to force a given plant to match the output of a given model exactly. A geometric, structural, necessary and sufficient condition for the solvability of such problem is presented.**

I. Introduction

Linear systems over the max-plus algebra, or max-plus linear systems, are a powerful tool to model discrete event systems where synchronization without competition occurs [1]. Essentially, they are equivalent to Petri nets in which each place has exactly one upstream transition and one downstream transition, also called timed event graphs. More information on the theory of max-plus dynamical systems can be found in [2], [3]. Along the lines of [2] and seeking for tools to solve specific control problems, several authors have worked at the development of a structural geometric approach for max-plus dynamical systems [4]–[9].

In this paper, we consider max-plus linear systems whose linear structure is subject to event-varying modifications which follow a periodic schedule and we refer to them as *periodic max-plus linear systems*. Periodic max-plus linear systems can be used to model, e.g., production processes where the order of the actions to be accomplished changes periodically to satisfy a repetitive production schedule or to exploit shared resources. Dynamical systems which have a periodically event-varying structure and are modeled as Petri nets were first considered in [10]. Periodic max-plus dynamical systems were used to model repetitive manufacturing systems in [11], [12], where their evolution for events whose index is a multiple of the period was studied by employing a transformation into event-invariant systems.

Here, we intend to generalize the structural geometric approach, introduced for linear systems over the conventional algebra in [13], [14], to periodic max-plus linear systems and to apply it to the problem of forcing the output of a given plant to match that of a given model. Such problem is an extension of the well-known model matching problem for classical systems, introduced in [15], whose solution provides an efficient and viable control strategy in all the situations in which a desired output behavior can be represented as the output of a suitable model [16]. Since the considered systems are defined over a semiring, results and techniques which were developed in the framework of systems over rings [17] and semirings [5], [9] are relevant to our study. At the same time, since the event-varying structure of max-plus systems is akin to that exhibited by linear periodic systems, the geometric methods developed in that framework [18], [19] and in relation to the model matching problem [20] provide useful hints. On the other hand, periodic max-plus systems can be viewed as switching max-plus systems as considered in [21], [22]. However, the fact that the switching is constrained to follow a periodic schedule leads to introduce geometric notions characterized by periodicity, thus marking a basic difference with the more general switching case.

The contribution of this paper is to introduce novel geometric notions that can be used to analyze, from a structural point of view, the model matching problem for periodic max-plus linear systems. This makes it possible to state conditions for the solvability of the problem and to provide a procedure to synthesize controllers, if any exists, that solves it, namely controllers which forces the output of a given plant to match, for any possible input, that of a given model.

The paper is organized as follows. In Section II, we introduce the basic notions of the max-plus algebra and the structure of periodic max-plus linear systems. In Section III, we introduce the geometric approach for periodic max-plus linear systems. In Section IV, we define the model matching problem for periodic max-plus linear systems. In Section V, we state structural geometric necessary and sufficient solvability conditions for the model matching problems. Section VI contains a practical application of the new results to the matching problem. Section VII contains the conclusions.

II. Periodic Max-plus Systems

The max-plus algebra is constructed by equipping the set $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ with two operations $\oplus$, or sum, and $\otimes$, or product, defined as $a \oplus b = \max\{a, b\}$ for $a, b \in \mathbb{R}_{\max}$ and $a \otimes b = a + b$ if a, b belong to $\mathbb{R}$, or $(-\infty) \otimes a = a \otimes (-\infty) = -\infty$ for any $a \in \mathbb{R}_{\max}$. Neutral elements for $\oplus$ and for $\otimes$ are $\epsilon = -\infty$ and $e = 0 \in \mathbb{R}$, respectively. Since the product $\otimes$ distributes over the sum $\oplus$, the max-plus algebra $\mathbb{R}_{\max}$ is a semiring.

$\mathbb{R}^n_{\max}$ denotes the semimodule over $\mathbb{R}_{\max}$ that consists of the set of n-tuples, or vectors, of elements of $\mathbb{R}_{\max}$, equipped

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with the component-wise sum and the scalar product operations defined in the conventional way in terms of $\oplus$ and $\otimes$. Given two vectors v and w of the same dimension or two matrices A and B of the same dimensions, the partial order relation $v \geq w$, respectively $A \geq B$, holds component-wise. In this context, matrix multiplication, also denoted by $\otimes$, is defined in the conventional way in terms of $\oplus$ and $\otimes$. As in conventional algebra, the symbol $\otimes$ is often omitted.

The evolution of a discrete event system in which events of n different types may occur can be modeled using n-dimensional daters. A dater is a function $d(.) : \mathbb{N} \rightarrow \mathbb{R}^n_{\max}$, whose value at $k \in \mathbb{N}$ is a vector $d(k) = (d_1(k), \ldots, d_n(k))^\top$ where the i-th component $d_i(k)$ is the time instant at which an event of the i-th type occurs for the k-th time. Daters must be non-decreasing (i.e. such that $d(k+1) \geq d(k)$ for each $k \in \mathbb{N}$) to have a physical meaning.

A periodic max-plus linear system Σ of period $\omega \in \mathbb{N}$, also said ω-periodic, is a dynamical object whose evolution is defined by equations of the form

$$\Sigma \equiv \begin{cases} x(k) = A(k)x(k-1) \oplus B(k)u(k) \\ y(k) = C(k)x(k) \\ x(0) = \epsilon \end{cases} \tag{1}$$

where $k \in \mathbb{N}$ is the event instance index, $x(.) : \mathbb{N} \rightarrow \mathcal{X} = \mathbb{R}^n_{\max}$ is the dater of internal events, $u(.) : \mathbb{N} \rightarrow \mathcal{U} = \mathbb{R}^m_{\max}$ is the dater of input events, $y(.) : \mathbb{N} \rightarrow \mathcal{Y} = \mathbb{R}^p_{\max}$ is the dater of output events and $A : \mathbb{N} \rightarrow \mathbb{R}^{n\times n}_{\max}$, $B : \mathbb{N} \rightarrow \mathbb{R}^{n\times m}_{\max}$, and $C : \mathbb{N} \rightarrow \mathbb{R}^{p\times n}_{\max}$ are ω-periodic functions. The semi-modules $\mathcal{X}, \mathcal{U}$ and $\mathcal{Y}$ are the state semimodule, the input semimodule and the output semimodule of the system, respectively.

To better understand the behavior of Σ, it is useful to remark that n types of internal events may happen and each one is associated to a component of x. Similarly, the m components of u correspond to the m types of input events, and the p components of y correspond to the p types of output events. More explicitly, if $x(k) = (x_1(k), ..., x_n(k))^\top \in \mathbb{R}^n_{\max}$, then the internal event of type i, for $i = 1, ..., n$, take place for the k-th time in the time instant $x_i(k)$ and a similar interpretation holds for $u(k)$ and for $y(k)$. A sequence $\{u(k)\}_{k\in\mathbb{N}}$ is viewed as the input to Σ, while a sequence $\{y(k)\}_{k\in\mathbb{N}}$ is viewed as the output of Σ.

Recalling that the identity matrix $I_n \in \mathbb{R}^{n\times n}_{\max}$ in the max-plus algebra has all its diagonal elements equal to e and all the other elements equal to ϵ, we say that a system of the form (1) is non-anticipative if $A(k) \geq I_n$, for all $k \in \mathbb{N}$. Every physically realizable max-plus system is non-anticipative, and the free evolution of its state is described by a non-decreasing dater.

III. Geometric Approach

The geometric approach is a formal methodology for the analysis and control of dynamic systems that has provided effective and elegant solutions to many control problems, such as the disturbance decoupling problem and the model matching problem. This approach has been introduced for stationary systems over the conventional algebra in [13], [14] and later extended to systems over rings in [23]–[25] and systems over semirings in [2], [4]–[7].

In this section, we present an extension of the geometric approach to periodic max-plus linear systems. The ideas we develop herein are akin, in the sense specified below, to those employed in the framework of conventional algebra in the case of periodic systems [19], [18] and in the case of switched systems over digraphs [26]. A key concept of the geometric approach is that of invariant semimodule. However, in the case of periodic systems, it is not convenient to consider a single semimodule as an invariant for the system throughout its evolution. Instead, it is less conservative and more effective to introduce a periodic sequence of semimodules which enjoy of a specific invariance property and which are associated to specific event indices. Such idea is formalized in the following definition.

Definition 1: An ω-periodic sequence of semimodules is a function $\mathcal{V}(.)$ that associates to each possible event instance index $k \in \mathbb{N}$ a subsemimodule $\mathcal{V}(k) \subseteq \mathbb{R}^n_{\max}$ and that fulfils the property $\mathcal{V}(k) = \mathcal{V}(k + \omega)$ for all $k \in \mathbb{N}$.

Since ω-periodic sequences of semimodules are widely used in the following, it is convenient to introduce some notation to improve the readability of the provided results. When no confusion arises, we refer to ω-periodic sequences of semimimodules simply as semimodules or ω-periodic semimodules. For instance, we will refer to both $\mathcal{V}(.)$ and $\mathcal{X}$ as semimodules, but the first is a sequence of semimodules, while the latter is a proper semimodule.

We say that an ω-periodic semimodule $\mathcal{E}(.)$ is a subsemimodule of a semimodule $\mathcal{X}$, and denote such relation as $\mathcal{E}(.) \subseteq \mathcal{X}$, if $\mathcal{E}(k) \subseteq \mathcal{X}$ for each $k \in \mathbb{N}$. Moreover, given two ω-periodic subsemimodules $\mathcal{E}(.)$ and $\mathcal{I}(.)$ we say that:

- $\mathcal{E}(.)$ is finitely generated if, for all $k \in \mathbb{N}$, $\mathcal{E}(k)$ is finitely generated;
- $\mathcal{E}(.)$ is equal to $\mathcal{I}(.)$, or $\mathcal{E}(.) = \mathcal{I}(.)$, if $\mathcal{E}(k) = \mathcal{I}(k)$ for all $k \in \mathbb{N}$;
- $\mathcal{E}(.)$ is contained in $\mathcal{I}(.)$, or $\mathcal{E}(.) \subseteq \mathcal{I}(.)$, if $\mathcal{E}(k) \subseteq \mathcal{I}(k)$ for all $k \in \mathbb{N}$;
- an ω-periodic semimodule $\mathcal{S}(.)$ is the sum of $\mathcal{E}(.)$ and $\mathcal{I}(.)$, if $\mathcal{S}(k)$ is the sum of $\mathcal{E}(k)$ and $\mathcal{I}(k)$ for all $k \in \mathbb{N}$;
- an ω-periodic semimodule $\mathcal{R}(.)$ is the intersection of $\mathcal{E}(.)$ and $\mathcal{I}(.)$, or $\mathcal{R}(.) = \mathcal{E}(.) \cap \mathcal{I}(.)$, if $\mathcal{R}(k) = \mathcal{E}(k) \cap \mathcal{I}(k)$ for all $k \in \mathbb{N}$.

Now, we can use the notion of ω-periodic semimodule to develop a geometric approach for periodic max-plus systems. Note that, in doing this, we mark a substantial difference with respect to the geometric approach for classical invariant systems [13], [14] or switching systems [21], [22].

Definition 2 (Invariant semimodule): Given an ω-periodic max-plus linear system Σ of the form (1), an ω-periodic semimodule $\mathcal{V}(.) \subseteq \mathcal{X}$ is said to be an A-invariant semimodule, or, equivalently, an invariant semimodule for Σ, if for all $k \in \mathbb{N}$ and for all $v \in \mathcal{V}(k-1)$, $A(k)v$ belongs to $\mathcal{V}(k)$.

Definition 3 (Controlled invariant semimodule): Given an ω-periodic max-plus linear system Σ of the form (1), an

ω-periodic semimodule $\mathcal{V}(.) \subseteq \mathcal{X}$ is said to be an (A, B)-invariant semimodule, or, equivalently, a controlled invariant semimodule for Σ, if for all $k \in \mathbb{N}$ and for all $v \in \mathcal{V}(k-1)$ there exists $u \in \mathbb{R}^m_{\max}$ such that $A(k)v \oplus B(k)u$ belongs to $\mathcal{V}(k)$.

Given an ω-periodic max-plus linear system Σ of the form (1) and an ω-periodic subsemimodule $\mathcal{K}(.)$ contained in its state semimodule $\mathcal{X}$, the set of all the (A, B)-invariant semimodules contained in $\mathcal{K}(.)$ is a semi-lattice with respect to inclusion and sum of semimodules, so a maximum element of that set, denoted by $\mathcal{V}^*_{\mathcal{K}}(.)$, exists. The following algorithm allows to compute $\mathcal{V}^*_{\mathcal{K}}(.)$ under suitable hypotheses.

Theorem 1: Let $\mathcal{K}(.) \subseteq \mathbb{R}^n_{\max}$ be a ω-periodic semimodule. Letting

$$A^{-1}(k+1)(\mathcal{Y}) = \{v \in \mathbb{R}^n_{\max},\ \text{such that } A(k+1)v \in \mathcal{Y}\}$$

and

$$\mathcal{V}_{r-1}(k+1) \ominus \operatorname{Im} B(k+1) = \{x \in \mathbb{R}^n_{\max} \text{ for which there exists } u \in \mathbb{R}^m_{\max} \text{ such that } x \oplus B(k+1)u \in \mathcal{V}_{r-1}(k+1)\},$$

the sequence of ω-periodic semimodules $\mathcal{V}_r(.)$ defined by

$$\begin{aligned} \mathcal{V}_0(k) &= \mathcal{K}(k) \\ \mathcal{V}_r(k) &= \mathcal{V}_{r-1}(k) \cap A^{-1}(k+1)(\mathcal{V}_{r-1}(k+1) \\ &\quad \ominus \operatorname{Im} B(k+1)) \qquad k \in \mathbb{N} \end{aligned} \tag{2}$$

has the following properties:

1) $\mathcal{V}_r(.) \subseteq \mathcal{V}_{r-1}(.)$ for all $r \in \mathbb{N}$;
2) Denoting $\mathcal{V}_\infty(.) = \lim_{r\to\infty}\mathcal{V}_r(.) = \bigcap_{r\in\mathbb{N}} \mathcal{V}_r(.)$, then every (A, B)-invariant ω-periodic semimodule contained in $\mathcal{K}(.)$ is also contained in $\mathcal{V}_\infty(.)$;
3) If $\mathcal{V}_r(.) = \mathcal{V}_{r-1}(.)$ then $\mathcal{V}_{r-1}(.)$ is an (A, B)-invariant ω-periodic semimodule and, in such case, $\mathcal{V}_\infty(.) = \mathcal{V}_{r-1}(.) = \mathcal{V}^*_{\mathcal{K}}(.)$.

Proof: (1) Follows from the definition of the sequence of ω-periodic semimodules.

(2) Let $\mathcal{P}(.) \subseteq \mathcal{K}(.) = \mathcal{V}_0(.)$ be an (A, B)-invariant ω-periodic semimodule and assume that, for some $r \in \mathbb{N}$, we have $\mathcal{P}(.) \subseteq \mathcal{V}_{r-1}(.)$. Then, since $\mathcal{P}(k) \subseteq A_i^{-1}(k+1)(\mathcal{P}(k+1) \ominus \operatorname{Im}B(k+1)) \subseteq A_i^{-1}(k+1)(\mathcal{V}_{r-1}(k+1) \ominus \operatorname{Im}B(k+1))$, we also have $\mathcal{P}(k) \subseteq \mathcal{V}_r(k)$ and the conclusion follows by induction.

(3) If $\mathcal{V}_r(.) = \mathcal{V}_{r-1}(.)$, the invariance of $\mathcal{V}_r(.)$ is a direct consequence of equation (2). In this case the equality $\mathcal{V}_\infty(.) = \mathcal{V}_{r-1}(.)$ is obvious and $\mathcal{V}_\infty(.) = \mathcal{V}^*_{\mathcal{K}}(.)$ follows from (2). ■

Remark 1: The sequence (2) does not provide a general algorithm for the computation of $\mathcal{V}^*_{\mathcal{K}}(.)$, as it does not necessarily converge in a finite number of steps. This fact, that applies also to systems over rings, marks an important difference with respect to systems over a field, for which a construction algorithm is available [13], [14].

Remark 2: If $\mathcal{K}(.)$ is finitely generated, it follows from [27, Corollary 86] that all the semimodules $\mathcal{V}_k(.)$ in the sequence (2) are finitely generated. Their generators can be computed by using a general elimination algorithm [28] as the set of solutions of appropriate equations of the form $Dx = Cx$ [5].

Definition 4: Given an ω-periodic max-plus linear system Σ of the form (1), an ω-periodic semimodule $\mathcal{V}(.) \subseteq \mathcal{X}$ is said to be an (A, B)-invariant semimodule of feedback type for Σ if there exists an ω-periodic matrix $F(.) : \mathbb{N} \to \mathbb{R}^{m\times n}_{\max}$ such that $(A(k) \oplus B(k)F(k))v$ belongs to $\mathcal{V}(k)$ for all $v \in \mathcal{V}(k-1)$, for all $k \in \mathbb{N}$.

In the framework of systems with coefficients in a field, controlled invariance of feedback type and controlled invariance are equivalent properties [13], [14]. However, for systems with coefficients in a semiring, or in a ring, invariance of feedback type is a stronger condition [24], [5].

IV. Problem Statement

In this section, we introduce the problem we tackle in this paper. The objective is to control the input of a max-plus periodic plant to obtain an output equal to that of a given model of the same kind for each possible input of the model. We give two formulations of the problem. The first one does not impose any constraints on the structure of the control sequence that possibly solves the problem, while the second formulation requires a solution which consists of a feedback-forward scheme.

Problem 1 (Model Matching Problem): Given a non-anticipative ω-periodic max-plus linear system

$$\Sigma_P \equiv \left\{ \begin{aligned} x_P(k) &= A_P(k)x_P(k-1) \oplus B_P(k)u_P(k) \\ y_P(k) &= C_P(k)x_P(k) \\ x_P(0) &= \epsilon \end{aligned} \right. \tag{3}$$

of the form (1), called the plant, and a non-anticipative ω-periodic max-plus linear system

$$\Sigma_M \equiv \left\{ \begin{aligned} x_M(k) &= A_M(k)x_M(k-1) \oplus B_M(k)u_M(k) \\ y_M(k) &= C_M(k)x_M(k) \\ x_M(0) &= \epsilon \end{aligned} \right. \tag{4}$$

of the form (1), called the model, with $x_P : \mathbb{N} \to \mathbb{R}^{n_P}_{\max}$, $x_M : \mathbb{N} \to \mathbb{R}^{n_M}_{\max}$, $u_P : \mathbb{N} \to \mathbb{R}^{m_P}_{\max}$, $u_M : \mathbb{N} \to \mathbb{R}^{m_M}_{\max}$ and $y_P, y_M : \mathbb{N} \to \mathbb{R}^p_{\max}$, the Model Matching Problem (MMP) consists in finding, for all possible non-decreasing input sequences $\{u_M(k)\}_{k\in\mathbb{N}}$ of the model, an appropriate non-decreasing control input sequence $\{u_P(k)\}_{k\in\mathbb{N}}$ for the plant, such that the output $\{y_P(k)\}_{k\in\mathbb{N}}$ of the plant equals the output $\{y_M(k)\}_{k\in\mathbb{N}}$ of the model, i.e. $y_P(k) = y_M(k)$ for all $k \in \mathbb{N}$.

If we require that the control input $u_P(k)$ be a linear function, with ω-periodic coefficients, of the state of the plant $x_P(k-1)$, of the state of the model $x_M(k-1)$, and of the input of the model $u_M(k)$, we get a more restrictive formulation of the MMP. In this case, the control law consists of a state feedback term and a feedforward term as in the classical case of linear systems over a field.

Problem 2 (Feedback Model Matching Problem): Given a plant of the form (3) and a model of the form (4), the Feedback Model Matching Problem (FMMP) consists in finding, for all possible non-decreasing input sequences $\{u_M(k)\}_{k\in\mathbb{N}}$ of the model, two ω-periodic matrices

$F : \mathbb{N} \to \mathbb{R}_{\max}^{m_P \times (n_P+n_M)}$ and $G : \mathbb{N} \to \mathbb{R}_{\max}^{m_P \times m_M}$, such that the control input sequence defined by

$$u_P(k) = \begin{cases} F(1) \begin{pmatrix} x_P(0) \\ x_M(0) \end{pmatrix} \oplus G(1) u_M(1) \text{ for } k{=}1 \\ F(k) \begin{pmatrix} x_P(k-1) \\ x_M(k-1) \end{pmatrix} \oplus G(k) u_M(k) \oplus \\ \qquad \oplus u_P(k-1) \text{ for } k{>}1 \end{cases} \tag{5}$$

is a solution for the corresponding MMP.

Remark 3: The dynamic component $u_P(k-1)$ at the second member of equation (5) for $k > 1$ is needed to assure that the resulting control sequence is non-decreasing.

Remark 4: In the given formulation of the feedback model matching problem, no constraints are imposed on the matrices $F(.)$ and $G(.)$. The solution can correspond to an anticipative feedback if some entries of the matrices are negative real numbers. In this case, in order to implement the control strategy, it is necessary to know the input of the model with some advance. However, not necessarily the entire sequence of future inputs has to be known.

V. Problem Solution

Given a plant Σ_P of the form (3) and a model Σ_M of the form (4), we can consider the extended ω-periodic system Σ_E described by

$$\Sigma_E \equiv \begin{cases} x_E(k) = A_E(k) x_E(k-1) \oplus B_1(k) u_P(k) \\ \qquad\qquad \oplus B_2(k) u_M(k) \\ x_E(0) = \epsilon \end{cases} \tag{6}$$

where $x_E(.) = \begin{pmatrix} x_P(.) \\ x_M(.) \end{pmatrix} : \mathbb{N} \to \mathcal{X}_E = \mathbb{R}_{\max}^{(n_P+n_M)}$ is the internal event dater, $A_E(k) = \begin{pmatrix} A_P(k) & \epsilon \\ \epsilon & A_M(k) \end{pmatrix}$, $B_1(k) = \begin{pmatrix} B_P(k) \\ \epsilon \end{pmatrix}$, and $B_2(k) = \begin{pmatrix} \epsilon \\ B_M(k) \end{pmatrix}$.

Then, Problem 1 can be reformulated as that of finding, for any input $\{u_M(k)\}_{k\in\mathbb{N}}$, a control sequence $\{u_P(k)\}_{k\in\mathbb{N}}$ that forces $x_E(k)$ to evolve inside the *output equalizer* ω-periodic semimodule $\mathcal{K}(.) \subseteq \mathcal{X}_E$ defined by

$$\mathcal{K}(k) = \left\{ \begin{pmatrix} x_P \\ x_M \end{pmatrix} \in \mathcal{X}_E \text{ s.t. } C_P(k) x_P = C_M(k) x_M \right\} \tag{7}$$

Definition 5: A periodic max-plus linear system Σ of the form (1) is said to be strongly non-anticipative if it is non-anticipative (i.e. $A(k) \geq I_n$ for all $k \in \mathbb{N}$) and

$$A(k+1)B(k) \geq B(k+1) \text{ for all } k \in \mathbb{N} \tag{8}$$

Lemma 1: If a periodic max-plus linear system Σ of the form (1) is strongly non-anticipative and $u(k+1) = u(k)$ for some $k \in \mathbb{N}$, then the term $B(k+1)u(k+1)$ does not affect the evolution of the system.

Proof: Given a periodic max-plus linear system Σ of the form (1), let $u(k+1) = u(k)$. Then, since $x(k) = A(k)x(k-1) \oplus B(k)u(k)$, we can write

$$\begin{aligned} x(k+1) &= A(k+1)x(k) \oplus B(k+1)u(k+1) \\ &= A(k+1)x(k) \oplus B(k+1)u(k) \\ &= A(k+1)A(k)x(k-1) \\ &\quad \oplus A(k+1)B(k)u(k) \oplus B(k+1)u(k). \end{aligned}$$

By strong non-anticipativeness, $A(k+1)B(k) \geq B(k+1)$. Hence $x(k+1) = A(k+1)A(k)x(k-1) \oplus A(k+1)B(k)u(k)$ ■

Strong non-anticipativeness can be viewed as the property of the system dynamics to be slow enough to filter the effects of the switching in the input matrix if the input is constant.

Lemma 2: A non-anticipative system with constant input matrix (i.e. such that $B(k) = \bar{B} \in \mathbb{R}_{\max}^{n \times m}$ for all $k \in \mathbb{N}$) is strongly non-anticipative.

We can now state a necessary and sufficient condition for the solvability of the MMP.

Theorem 2: Given a strongly non-anticipative ω-periodic plant Σ_P of the form (3) and a strongly non-anticipative ω-periodic model Σ_M of the form (4), consider the extended system Σ_E given by (6). Then, the related MMP is solvable if and only if for each $k \in \mathbb{N}$ and for each $x_{uM} \in \operatorname{Im} B_2(k) = \operatorname{Im} \begin{pmatrix} \epsilon \\ B_M(k) \end{pmatrix} \subseteq \mathcal{X}_E$ there exists $x_{uP} \in \operatorname{Im} B_1(k) = \operatorname{Im} \begin{pmatrix} B_P(k) \\ \epsilon \end{pmatrix} \subseteq \mathcal{X}_E$ such that $x_{uM} \oplus x_{uP}$ belongs to $\mathcal{V}^*(k) \subseteq \mathcal{X}_E$, where $\mathcal{V}^*(.)$ is the maximum (A_E, B_1)-invariant semimodule for Σ_E contained in the output equalizer semimodule $\mathcal{K}(.) \subseteq \mathcal{X}_E$ defined by (7).

Proof: *If.* By the hypotheses it is possible to find, for each $k \in \mathbb{N}$ and for each $x_E \in \mathcal{V}^*(k-1)$, a vector $u_1(k) \in \mathbb{R}_{\max}^{m_P}$ such that $A_E(k)x_E \oplus B_1(k)u_1(k)$ belongs to $\mathcal{V}^*(k)$ and, for each $k \in \mathbb{N}$ and each $u_M \in \mathbb{R}_{\max}^{m_M}$, a vector $u_2(k) \in \mathbb{R}_{\max}^{m_P}$ such that $B_2(k)u_M \oplus B_1(k)u_2(k) \in \mathcal{V}^*(k)$. Then, for each input $\{u_M(k)\}_{k\in\mathbb{N}}$, one can construct recursively a control input for Σ_E as

$$u_P(k) = \begin{cases} u_2(1) \text{ for } k = 1 \\ u_1(k) \oplus u_2(k) \oplus u_P(k-1) \text{ for } k > 1 \end{cases}$$

The corresponding state evolution is given by

$$x_E(k) = \begin{cases} B_1(k)u_2(k) \oplus B_2(k)u_M(k) \quad \text{for } k = 1 \\ (A_E(k)x_E(k-1) \oplus B_1(k)u_1(k)) \oplus \\ \quad (B_1(k)u_2(k) \oplus B_2(k)u_M(k)) \oplus \\ \quad B_1(k)u_P(k-1) \quad \text{for } k > 1 \end{cases}$$

and, since the plant is strongly non-anticipative, we can show by induction that $x_E(k)$ belongs to $\mathcal{V}^*(k)$ for all $k \in \mathbb{N}$.

Only if. Assume that the condition of the theorem does not hold and let $\bar{k} \in \mathbb{N}$ and $u_M(\bar{k}) = \bar{u}_M$ be such that $B_2(\bar{k})u_M(\bar{k}) \oplus B_1(\bar{k})u_P \notin \mathcal{V}^*(\bar{k})$ for any $u_P \in \mathbb{R}_{\max}^{m_P}$. The same property holds for $u_M(\bar{k}) = \alpha\bar{u}_M$ with arbitrary $\alpha \in \mathbb{R}$. Hence, taking the constant input sequence $u_M(k) = \alpha\bar{u}_M$, one can choose α sufficiently big to have that $x_E(\bar{k}) = A_E(\bar{k})x_E(\bar{k}-1) \oplus B_1(\bar{k})u_P(\bar{k}) \oplus B_2(\bar{k})\alpha\bar{u}_M$ does not belong to $\mathcal{V}^*(\bar{k})$ for any possible $u_P(\bar{k}) \in \mathbb{R}_{\max}^{m_P}$. Writing, recursively, for $k > \bar{k}$, $x_E(k) = A_E(k)x_E(k-1) \oplus B_1(k)u_P(k) \oplus B_2(k)u_M(k) = A_E(k)x_E(k-1) \oplus B_1(k)u_P(k) \oplus B_2(k)\alpha\bar{u}_M$, since the model is strongly non-anticipative, we have $x_E(k) = A_E(k)x_E(k-1) \oplus B_1(k)u_P(k)$. Now, as $x_E(\bar{k})$ does not belong to $\mathcal{V}^*(\bar{k})$, for any input sequence $\{u_P(k)\}_{k\in\mathbb{N}}$ there exists $q \in \mathbb{Z}$ such that $x_E(q) \notin \mathcal{K}(q)$. This says that $x_E(.)$ cannot be forced to evolve inside $\mathcal{K}(.)$ and the MMP cannot be solved. ■

Remark 5: The condition provided by Theorem 2 can be equivalently written, using the $\ominus$ operator defined in Theorem 1, as $\mathrm{Im}B_2(k) \subseteq \mathcal{V}^*(k) \ominus \mathrm{Im}B_1(k)\ \forall k \in \mathbb{N}$ and it can be checked using the numeric techniques mentioned in Remark 2 and [5, Remark 1].

Remark 6: If the plant is not strongly non-anticipative, the condition expressed in Theorem 2 is necessary but not sufficient. Dually, if the model is not strongly non-anticipative the condition is sufficient, but not necessary. In fact, these assumptions are used in the proof of the Theorem only in the corresponding sections.

The analogous result about the FMMP is stated in the following theorem.

Theorem 3: Given a strongly non-anticipative ω-periodic plant Σ_P of the form (3) and a strongly non-anticipative ω-periodic model Σ_M of the form (4), consider the extended system Σ_E given by (6). Then, the related FMMP is solvable if and only if there exists an (A_E, B_1)-invariant ω-periodic semimodule $\mathcal{V}(.)$ of feedback type contained in the output equalizer semimodule $\mathcal{K}(.)$ defined by (7) such that, for each $k \in \mathbb{N}$ and for each $x_{uM} \in \mathrm{Im}\, B_2(k) = \mathrm{Im}\begin{pmatrix}\epsilon \\ B_M(k)\end{pmatrix} \subseteq \mathcal{X}_E$ there exists $x_{uP} \in \mathrm{Im}\, B_1(k) = \mathrm{Im}\begin{pmatrix}B_P(k) \\ \epsilon\end{pmatrix} \subseteq \mathcal{X}_E$ with $x_{uM} \oplus x_{uP} \in \mathcal{V}(k)$.

Proof: If. Let $\mathcal{V}(.) \subseteq \mathcal{K}(.)$ be an (A_E, B_1)-invariant semimodule of feedback type for which the condition of the theorem holds. Then, by controlled invariance of feedback type, there exists an ω-periodic matrix $F(.)$ such that for each $k \in \mathbb{N}$ and $x_E(k-1) \in \mathcal{V}^*(k-1)$, $(A_E(k) \oplus B_1(k)F(k))x_E(k-1)$ belongs to $\mathcal{V}^*(k)$. The condition of the theorem implies the existence of an ω-periodic matrix $G(.)$ such that the columns of the matrix $\begin{pmatrix}\epsilon \\ B_M(k)\end{pmatrix} I_{m_M} \oplus \begin{pmatrix}B_P(k) \\ \epsilon\end{pmatrix} G(k) = \begin{pmatrix}B_P(k)G(k) \\ B_M(k)\end{pmatrix}$ belong to $\mathcal{V}(k)$ for each $k \in \mathbb{N}$. Then, applying a control law recursively defined as in equation (5), with $F(.)$ and $G(.)$ defined as above, the compensated dynamics becomes

$$\begin{aligned} x_E(k) \;=\; & (A_E(k) \oplus B_1(k)F(k))x_E(k-1)\oplus \\ & \oplus \begin{pmatrix}B_P(k)G(k) \\ B_M(k)\end{pmatrix} u_M(k) \oplus \\ & \oplus B_1(k)u_P(k-1) \end{aligned} \tag{9}$$

where $u_P(0) = \epsilon$. By strong non-anticipativeness of the plant, we have $A_E(k)x_E(k-1) \geq A_E(k)B_1(k-1)u_P(k-1) \geq B_1(k)u_P(k-1)$ and the last summand of the right-hand term of equation (9) does not affect the state of the system, which therefore evolves in $\mathcal{V}(.) \subseteq \mathcal{K}(.)$.

Only if. Assume that the FMMP is solved by a control law of the form (5). Then, for each $k \in \mathbb{N}$, the set of reachable states for the dynamics (9) at event instance k, is an (A_E, B_1)-invariant periodic semimodule of feedback type contained in $\mathcal{K}(k)$ that contains all the columns of the matrix $\begin{pmatrix}B_P(k)G(k) \\ B_M(k)\end{pmatrix} = \begin{pmatrix}\epsilon \\ B_M(k)\end{pmatrix} I_{m_M} \oplus \begin{pmatrix}B_P(k) \\ \epsilon\end{pmatrix} G(k)$. This clearly implies the condition of the theorem. ■

Remark 7: The conditions of Theorems 2 and 3 can be practically checked by solving the linear equations considered in their proofs by means of general elimination methods (see [5] and [28]). A toolbox that implements such methods was originally developed for Scilab® [29], and is now integrated in the Scicoslab software [30].

VI. An Illustrative Example

This example shows how the previous results can be used to tackle the model matching problem of a manufacturing plant with a periodic configuration. Let us consider an ω-periodic plant of the form (3) with $\omega = 3$ and

$$A_P(1) = A_P(2) = \begin{pmatrix} 2 & \epsilon & \epsilon & \epsilon & \epsilon \\ \epsilon & 1 & \epsilon & \epsilon & \epsilon \\ 4 & \epsilon & 2 & \epsilon & \epsilon \\ \epsilon & 5 & \epsilon & 4 & \epsilon \\ 5 & 6 & 3 & 5 & 1 \end{pmatrix}$$

$$B_P(1) = B_P(2) = \begin{pmatrix} 2 & 1 & 4 & 5 & 6 \end{pmatrix}^\top$$

$$A_P(3) = \begin{pmatrix} 2 & \epsilon & \epsilon & \epsilon & \epsilon \\ \epsilon & 3 & \epsilon & \epsilon & \epsilon \\ \epsilon & 5 & 2 & \epsilon & \epsilon \\ 6 & \epsilon & \epsilon & 4 & \epsilon \\ 7 & 6 & 3 & 5 & 1 \end{pmatrix} \quad B_P(3) = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 6 \\ 7 \end{pmatrix}$$

$$C_P(k) = \begin{pmatrix} \epsilon & \epsilon & \epsilon & \epsilon & e \end{pmatrix}\ \forall k \in \mathbb{N}$$

As a model, we take the following system of the form (4):

$$\Sigma_M \equiv \begin{cases} x_M(k) \;=\; 5x_M(k-1) \oplus 5u_M(k) \\ y_M(k) \;=\; x_M(k) \\ x_M(0) \;=\; \epsilon \end{cases} \tag{10}$$

Both the plant and the model are strongly non-anticipative. The output equalizer semimodule is

$$\mathcal{K}(k) = \mathrm{Im}\begin{pmatrix} e & \epsilon & \epsilon & \epsilon & \epsilon \\ \epsilon & e & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & e & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & e & \epsilon \\ \epsilon & \epsilon & \epsilon & \epsilon & e \\ \epsilon & \epsilon & \epsilon & \epsilon & e \end{pmatrix} \text{ for all } k \in \mathbb{N}. \tag{11}$$

In this case, the output equalizer semimodule is constant for each possible value of k, because both the plant and the model have stationary output matrices. The maximal (A_E, B_1)-invariant subspace $\mathcal{V}^*(.)$ for the 3-periodic joint dynamics can be computed by a suitable Scicoslab procedure. The sequence of semimodules considered in Theorem 1 converges after two iterations (i.e. $\mathcal{V}_1(.) = \mathcal{V}_2(.) = \mathcal{V}^*(.)$) and we get

$$\mathcal{V}^*(1) = \mathcal{V}^*(3) = \mathrm{Im}\begin{pmatrix} e & \epsilon & \epsilon & \epsilon & \epsilon \\ \epsilon & e & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & 2 & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & e & \epsilon \\ e & 1 & e & e & e \\ e & 1 & e & e & e \end{pmatrix}$$

$$\mathcal{V}^*(2) = \mathrm{Im}\begin{pmatrix} e & \epsilon & \epsilon & \epsilon & \epsilon \\ \epsilon & e & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & 2 & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & e & \epsilon \\ 2 & 1 & e & e & e \\ 2 & 1 & e & e & e \end{pmatrix}$$

The condition of Theorem 2 is satisfied, so the MMP is solvable. Moreover, $\mathcal{V}^*(.)$ is of feedback type, so the FMMP is also solvable by Theorem 3 and the control sequence $\{u_P(k)\}_{k\in\mathbb{N}}$ of the form (5), with

$$\begin{aligned} &F(1) = F(2) = \begin{pmatrix}\epsilon & \epsilon & \epsilon & \epsilon & -1 & \epsilon\end{pmatrix}, \\ &G(1) = G(2) = -1, \\ &F(3) = \begin{pmatrix}\epsilon & \epsilon & \epsilon & \epsilon & \epsilon & -2\end{pmatrix}, \\ &G(3) = -2 \end{aligned}$$

solves the problem.

VII. Conclusions

An extension of the geometric approach and a formulation of the model matching problem for periodic max-plus linear systems were presented. The geometric approach was shown to be effective in providing a solution, if any exists, to the model matching problem, under suitable hypotheses. Among the required hypotheses, the strong non-anticipativeness of the plant and the model deserves particular mention. Moreover, to assure the feasibility of the obtained state feedback regulator, it is required, in general, that the input sequence of the model be known with appropriate advance. This may also occur in the nonperiodic case [9, Remark 2], the length of the preview depending on the matrices F and G. These assumptions are often not restrictive in practice. However, they may limit the applicability of the provided results. Therefore, the construction of more general procedures will be the object of future investigation.

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