

A Robust Dynamic Average Consensus Algorithm that Ensures both Differential Privacy and Accurate Convergence

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Abstract—We propose a new dynamic average consensus algorithm that is robust to information-sharing noise arising from differential-privacy design. Not only is dynamic average consensus widely used in cooperative control and distributed tracking, it is also a fundamental building block in numerous distributed computation algorithms such as multi-agent optimization and distributed Nash equilibrium seeking. We propose a new dynamic average consensus algorithm that is robust to persistent and independent information-sharing noise added for the purpose of differential-privacy protection. In fact, the algorithm can ensure both provable convergence to the exact average reference signal and rigorous ϵ -differential privacy (even when the number of iterations tends to infinity), which, to our knowledge, has not been achieved before in average consensus algorithms. Given that channel noise in communication can be viewed as a special case of differential-privacy noise, the algorithm can also be used to counteract communication imperfections. Numerical simulation results confirm the effectiveness of the proposed approach.

I. INTRODUCTION

Dynamic average consensus addresses the problem where a group of agents cooperate to track the average of multiple time-varying reference signals while each individual agent can only access one reference signal and communicate with its immediate neighbors [2], [3]. Compared with the intensively studied static average consensus problem where the reference signals are confined to be the time-invariant initial values of individual agents, the dynamic average consensus problem allows the tracking targets to evolve with time and fits naturally into applications such as formation control [4], sensor fusion [5], and distributed tracking [6]. More interestingly, in recent years, dynamic average consensus is also proven to be an effective primitive in constructing fully distributed computation algorithms for networked games [7], [8] and large-scale optimization [9], [10], [11].

The vast applicability of dynamic average consensus has spurred intensive interests, and plenty of results have been proposed (see, e.g., the survey paper [3]). However, all of these results require participating agents to exchange and disclose their states explicitly to neighboring agents in every iteration, which is problematic when involved information is sensitive. For example, in sensor network based source

localization, disclosing intermediate states will make individual agents' positions inferrable, which is undesirable when sensors want to keep their positions private in sensitive applications [12], [13]. Another example is smart grids where multiple power generators have to reach agreement on cost while preserving the privacy of their individual generation information, which is sensitive in bidding the right for energy selling [14]. The privacy problem is more pronounced in distributed optimization and learning (many of which are built upon dynamic average consensus) in that the shared states in distributed optimization and learning may contain sensitive information such as salary information or medical records [15]. In fact, it has been shown in [16], [17] that without a privacy mechanism in place, an adversary (e.g., the DLG attacker in [18]) can use shared state information to precisely recover the raw data used for training.

To address the privacy issue in dynamic average consensus, several approaches have been proposed (see, e.g., [19], [20]). In fact, given that the information-sharing mechanism in dynamic average consensus is the same as in static average consensus, many privacy solutions designed for static average consensus (e.g., our prior work [21], [22], [23], [24] as well as others' [25], [26], [27], [28], [29], [30], [31]) can also be applied directly to dynamic average consensus. However, these privacy approaches are restricted in that they either require the communication graph to satisfy certain properties, or can only protect the exact sensitive (initial) value from being *uniquely* inferrable by the adversary. As differential privacy (DP) has become the de facto standard for privacy protection due to its strong resilience against arbitrary post-processing and auxiliary information [32], plenty of results have emerged on differentially-private average consensus (see, e.g. [28], [33], [34], [35], [36]). However, all of these results have to sacrifice provable convergence to the exact consensus value to enable rigorous DP (to ensure a finite privacy budget when the number of iterations tends to infinity). Moreover, these results only address the static average consensus problem, where the information to be protected are time-invariant initial values of individual agents' states.

In this paper, we propose a new dynamic average consensus algorithm that is robust to persistent and independent information-sharing noise injected for the purpose of differential-privacy protection. In fact, the algorithm is proven able to ensure both provable convergence to the desired average reference signal and rigorous ϵ -DP, with the cumulative privacy budget guaranteed to be finite even when the number of iterations tends to infinity. The approach is motivated by the observation that DP-noise enters the algo-

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rithm through inter-agent interaction, and hence, its influence on convergence accuracy can be attenuated by gradually weakening inter-agent interaction. Not only is the proposed algorithm the first to achieve rigorous ϵ -DP in dynamic average consensus, it can also retain provable convergence to the *exact* average reference signal when reference signals' variations decay with time. To our knowledge, ensuring accurate consensus while enabling rigorous ϵ -DP has not been reported before for dynamic average consensus.

Notations: We use \mathbb{R}^d to denote the Euclidean space of dimension d . We write I_d for the identity matrix of dimension d , and $\mathbf{1}_d$ for the d -dimensional column vector with all entries equal to 1; in both cases we suppress the dimension when it is clear from the context. A vector is viewed as a column vector, and for a vector x , $[x]_i$ denotes its i th element. We use $\langle \cdot, \cdot \rangle$ to denote the inner product and $\|x\|$ for the standard Euclidean norm of a vector x . We use $\|x\|_1$ to represent the L_1 norm of a vector x . We write $\|A\|$ for the matrix norm induced by the vector norm $\|\cdot\|$. A^T denotes the transpose of a matrix A . Given vectors x_1, \dots, x_m , we define $\bar{x} = \frac{\sum_{i=1}^m x_i}{m}$. Often, we abbreviate *almost surely* by *a.s.*

II. PROBLEM FORMULATION AND PRELIMINARIES

A. On Dynamic Average Consensus

We consider a dynamic average consensus problem among a set of m agents $[m] = \{1, \dots, m\}$. We index the agents by $1, 2, \dots, m$. Agent i can access fixed-frequency samples of its own reference signal $r_i \in \mathbb{R}^d$, which could be varying with time. Every agent i also maintains a state $x_i \in \mathbb{R}^d$. The aim of dynamic average consensus is for all agents to collaboratively track the average reference signal $\bar{r} \triangleq \frac{\sum_{i=1}^m r_i}{m}$ while every agent can only access discrete-time measurements of its own reference signal and share its state with its immediate neighboring agents.

We describe the local communication among agents using a weight matrix $L = \{L_{ij}\}$, where $L_{ij} > 0$ if agent j and agent i can directly communicate with each other, and $L_{ij} = 0$ otherwise. For an agent $i \in [m]$, its neighbor set \mathbb{N}_i is defined as the collection of agents j such that $L_{ij} > 0$. We define $L_{ii} \triangleq -\sum_{j \in \mathbb{N}_i} L_{ij}$ for all $i \in [m]$, where \mathbb{N}_i is the neighbor set of agent i . Furthermore, We make the following assumption on L :

Assumption 1. *The matrix $L = \{w_{ij}\} \in \mathbb{R}^{m \times m}$ is symmetric and satisfies $\mathbf{1}^T L = \mathbf{0}^T$, $L\mathbf{1} = \mathbf{0}$, and $\|I + L - \frac{\mathbf{1}\mathbf{1}^T}{m}\| < 1$.*

Assumption 1 ensures that the interaction graph induced by L is connected, i.e., there is a path from each agent to every other agent. It can be verified that $\|I + L - \frac{\mathbf{1}\mathbf{1}^T}{m}\| = \max\{|1 + \rho_2|, |1 + \rho_m|\}$, where $\{\rho_i, i \in [m]\}$ are the eigenvalues of L , with $\rho_m \leq \dots \leq \rho_2 < \rho_1 = 0$.

We also need the following lemma for convergence analysis:

Lemma 1. [16] *Let $\{v^k\}, \{o^k\}$, and $\{p^k\}$ be random nonnegative scalar sequences, and $\{q^k\}$ be a deterministic*

nonnegative scalar sequence satisfying $\sum_{k=0}^{\infty} o^k < \infty$ a.s., $\sum_{k=0}^{\infty} q^k = \infty$, $\sum_{k=0}^{\infty} p^k < \infty$ a.s., and

$$\mathbb{E} [v^{k+1} | \mathcal{F}^k] \leq (1 + o^k - q^k)v^k + p^k, \quad \forall k \geq 0 \quad a.s.$$

where $\mathcal{F}^k = \{v^\ell, o^\ell, p^\ell; 0 \leq \ell \leq k\}$. Then, $\sum_{k=0}^{\infty} q^k v^k < \infty$ and $\lim_{k \rightarrow \infty} v^k = 0$ hold almost surely.

Lemma 2. [16] *Let $\{v^k\}$ be a nonnegative sequence, and $\{\alpha^k\}$ and $\{\beta^k\}$ be positive sequences satisfying $\sum_{k=0}^{\infty} \alpha^k = \infty$, $\lim_{k \rightarrow \infty} \alpha^k = 0$, and $\lim_{k \rightarrow \infty} \frac{\beta^k}{\alpha^k} = 0$. If there exists a $K \geq 0$ such that $v^{k+1} \leq (1 - \alpha^k)v^k + \beta^k$ holds for all $k \geq K$, then we always have $v^k \leq C \frac{\beta^k}{\alpha^k}$ for all k , where C is some constant.*

B. On Differential Privacy

We consider adversaries having full access to all communication channels. Namely, the adversary can peek inside all the messages going back and forth between the agents. Because in dynamic average consensus, the sensitive information of participating agents are time-varying reference signals, we use the notion of ϵ -DP for continuous bit streams [37], which has recently been applied to distributed optimization (see [38] as well as our own work [16]). To enable ϵ -DP, we use the Laplace noise mechanism, i.e., we add Laplace noise to all shared messages. For a constant $\nu > 0$, we use $\text{Lap}(\nu)$ to denote a Laplace distribution of a scalar random variable with the probability density function $x \mapsto \frac{1}{2\nu} e^{-\frac{|x|}{\nu}}$. One can verify that $\text{Lap}(\nu)$'s mean is zero and its variance is $2\nu^2$. Inspired by the formulation of distributed optimization in [38], to facilitate DP analysis, we represent a dynamic average consensus problem \mathcal{P} by $\mathcal{P} \triangleq (\mathcal{J}, L)$, where $\mathcal{J} \triangleq \{r_1, \dots, r_m\}$ are the reference signals of all agents and L is the inter-agent interaction weight matrix. Then, we define adjacency between two dynamic average consensus problems as follows:

Definition 1. *Two dynamic average consensus problems $\mathcal{P} = (\mathcal{J}, L)$ and $\mathcal{P}' = (\mathcal{J}', L')$ are adjacent if the following conditions hold:*

- $L = L'$, i.e., the interaction weight matrices are identical;
- there exists an $i \in [m]$ such that $r_i \neq r'_i$ but $r_j = r'_j$ for all $j \in [m], j \neq i$;
- the different reference signals r_i and r'_i have similar steady-state behaviors. More specifically, there exists some positive C_r and non-negative non-summable but square-summable sequences $\{\chi^k\}$ and $\{\gamma^k\}$ (which will be specified later in Algorithm 1 and Assumption 3, respectively) such that $\|r_i^k - r'_i{}^k\|_1 \leq C_r \chi^k \gamma^k$ holds for all k .

Remark 1. *In Definition 1, since the change from r_i to r'_i in the second condition can be arbitrary, the third condition is added to restrict the change magnitude. It is necessary to enabling rigorous ϵ -DP while maintaining accurate convergence. This is because DP aims to make observations statistically indistinguishable while accurate convergence means*

that the state will stop changing and remain time-invariant after a transient period. Hence, to make the observations of \mathcal{P} and \mathcal{P}' the same after their states converge and remain at their respective converging points, we have to require \mathcal{P} and \mathcal{P}' to have identical converging points.

Given a dynamic average consensus algorithm, we represent an execution of such an algorithm as \mathcal{A} , which is an infinite sequence of the iteration variable ϑ , i.e., $\mathcal{A} = \{\vartheta^0, \vartheta^1, \dots\}$. We consider adversaries that can observe all communicated messages among the agents. Therefore, the observation part of an execution is the infinite sequence of shared messages, which is denoted as \mathcal{O} . Given a dynamic average consensus problem \mathcal{P} and an initial state ϑ^0 , we define the observation mapping as $\mathcal{R}_{\mathcal{P}, \vartheta^0}(\mathcal{A}) \triangleq \mathcal{O}$. Given a dynamic average consensus problem \mathcal{P} , observation sequence \mathcal{O} , and an initial state ϑ^0 , $\mathcal{R}_{\mathcal{P}, \vartheta^0}^{-1}(\mathcal{O})$ is the set of executions \mathcal{A} that can generate the observation \mathcal{O} .

Definition 2. (ϵ -differential privacy, adapted from [38]). For a given $\epsilon > 0$, an iterative distributed algorithm is ϵ -differentially private if for any two adjacent \mathcal{P} and \mathcal{P}' , any set of observation sequences $\mathcal{O}_s \subseteq \mathbb{O}$ (with \mathbb{O} denoting the set of all possible observation sequences), and any initial state ϑ^0 , the following relationship always holds

$$\mathbb{P}[\mathcal{R}_{\mathcal{P}, \vartheta^0}^{-1}(\mathcal{O}_s)] \leq e^\epsilon \mathbb{P}[\mathcal{R}_{\mathcal{P}', \vartheta^0}^{-1}(\mathcal{O}_s)], \quad (1)$$

with the probability \mathbb{P} taken over the randomness over iteration processes.

ϵ -DP ensures that an adversary having access to all shared messages cannot gain information with a significant probability of any participating agent's reference signal. It can be seen that a smaller ϵ means a higher level of privacy protection. It is worth noting that the considered notion of ϵ -DP is more stringent than other relaxed (approximate) DP notions such as (ϵ, δ) -DP [39], zero-concentrated DP [40], or Rényi DP [41].

III. DIFFERENTIALLY-PRIVATE DYNAMIC AVERAGE CONSENSUS

In this section, we present our new dynamic average consensus algorithm that is robust to DP-noise, which is summarized in Algorithm 1. Our basic idea is to use a decaying factor χ^k to gradually attenuate inter-agent interaction, and hence, attenuate the influence of DP-noise. Of course, to enable necessary information fusion among the agents, we have to judiciously design the decaying factor. Moreover, in our proposed algorithm, another fundamental difference from conventional dynamic average consensus algorithms is the introduction of a stepsize α^k . The two features enable Algorithm 1 to avoid the accumulation and explosion of noise variance in conventional dynamic average consensus algorithms in the presence of noise. In fact, the proposed new algorithm can guarantee convergence to the exact average reference signal even when the DP-noise is allowed to increase with time to ensure rigorous ϵ -DP, which will be elaborated later in detail in Sec. V. To our knowledge,

this is the first time that both provable convergence and rigorous ϵ -DP are achieved in dynamic average consensus algorithms.

Algorithm 1: Robust dynamic average consensus

Parameters: Weakening factor $\chi^k > 0$ and stepsize $\alpha^k > 0$. Every agent i 's reference signal is r_i^k . Every agent i maintains one state variable x_i^k , which is initialized as $x_i^0 = r_i^0$.
for $k = 1, 2, \dots$ **do**

- a) Every agent j adds persistent DP-noise ζ_j^k to its state x_j^k , and then sends the obscured state $x_j^k + \zeta_j^k$ to agent $i \in \mathbb{N}_j$.
- b) After receiving $x_j^k + \zeta_j^k$ from all $j \in \mathbb{N}_i$, agent i updates its state as follows:

$$x_i^{k+1} = (1 - \alpha^k)x_i^k + \chi^k \sum_{j \in \mathbb{N}_i} L_{ij}(x_j^k + \zeta_j^k - x_i^k) + r_i^{k+1} - (1 - \alpha^k)r_i^k. \quad (2)$$

We make the following assumption on the DP-noise:

Assumption 2. For every $i \in [m]$ and every k , conditional on x_i^k , the DP-noise ζ_i^k satisfies $\mathbb{E}[\zeta_i^k | x_i^k] = 0$ and $\mathbb{E}[\|\zeta_i^k\|^2 | x_i^k] = (\sigma_i^k)^2$ for all $k \geq 0$, and

$$\sum_{k=0}^{\infty} (\chi^k)^2 \max_{i \in [m]} (\sigma_i^k)^2 < \infty, \quad (3)$$

where $\{\chi^k\}$ is the sequence from Algorithm 1. The initial random vectors satisfy $\mathbb{E}[\|x_i^0\|^2] < \infty$, $\forall i \in [m]$.

IV. CONVERGENCE ANALYSIS

In this section, we prove that when the following assumption holds, the proposed algorithm can ensure every x_i^k to converge almost surely to the exact average reference signal $\bar{r}^k \triangleq \frac{1}{m} \sum_{i=1}^m r_i^k$:

Assumption 3. For every $i \in [m]$, there exist some nonnegative sequence $\{\gamma^k\}$ and a constant C such that

$$\|r_i^{k+1} - (1 - \alpha^k)r_i^k\| \leq \gamma^k C \quad (4)$$

holds, where α^k is from Algorithm 1 and $\{\gamma^k\}$ satisfies $\lim_{k \rightarrow \infty} \frac{\alpha^k}{\gamma^k} < \infty$.

It can be seen that the condition $\lim_{k \rightarrow \infty} \frac{\alpha^k}{\gamma^k} < \infty$ is necessary since otherwise (4) will not hold for constant r_i^k .

To prove that the state x_i^k converges to the precise average \bar{r}^k , we first prove that $\bar{x}^k = \frac{\sum_{i=1}^m x_i^k}{m}$ converges to \bar{r}^k . Such a property holds for the conventional dynamic average consensus in the absence of DP-noise. However, it does not hold any more in the presence of information-sharing noise since the noise will accumulate on \bar{x}^k in the conventional dynamic average consensus algorithm, leading to an exploding variance [42], [43] (see also the blue curves in the numerical simulation results in Fig. 3). Here, we prove that our proposed algorithm can ensure the converge of \bar{x}^k to \bar{r}^k even when all agents add independent DP-noise to their shared messages.

Lemma 3. Under Assumptions 1, 2, \bar{x}^k in the proposed algorithm converges a.s. to \bar{r}^k if $\sum_{k=0}^{\infty} \alpha^k = \infty$ and $\sum_{k=0}^{\infty} (\alpha^k)^2 < \infty$ hold.

Proof. According to the definitions of \bar{x}^k and \bar{r}^k , we have the following relationship based on (2):

$$\bar{x}^{k+1} = (1 - \alpha^k)\bar{x}^k + \chi^k \bar{\zeta}^k + \bar{r}^{k+1} - (1 - \alpha^k)\bar{r}^k, \quad (5)$$

where $\bar{\zeta}^k \triangleq \frac{\sum_{i=1}^m |L_{ii}| \zeta_i^k}{m}$ and we have used the symmetric property of L and the definition $L_{ii} \triangleq -\sum_{j \in \mathbb{N}_i} L_{ij}$.

The preceding relationship implies

$$\begin{aligned} \|\bar{x}^{k+1} - \bar{r}^{k+1}\|^2 &= \|(1 - \alpha^k)(\bar{x}^k - \bar{r}^k) + \chi^k \bar{\zeta}^k\|^2 \\ &= (1 - \alpha^k)^2 \|\bar{x}^k - \bar{r}^k\|^2 + (\chi^k)^2 \|\bar{\zeta}^k\|^2 \\ &\quad + 2 \langle (1 - \alpha^k)(\bar{x}^k - \bar{r}^k), \chi^k \bar{\zeta}^k \rangle. \end{aligned}$$

Using the assumption that the DP-noise ζ_i^k has zero mean and variance $(\sigma_i^k)^2$ conditional on x_i^k (see Assumption 2), taking the conditional expectation, given $\mathcal{F}^k = \{x^0, \dots, x^k\}$, we obtain the following inequality for all $k \geq 0$:

$$\begin{aligned} \mathbb{E} [\|\bar{x}^{k+1} - \bar{r}^{k+1}\|^2 | \mathcal{F}^k] &\leq (1 + (\alpha^k)^2 - 2\alpha^k) \|\bar{x}^k - \bar{r}^k\|^2 \\ &\quad + \frac{\sum_{i=1}^m L_{ii}^2 (\sigma_i^k)^2 (\chi^k)^2}{m}, \end{aligned}$$

where we have used the relationship $\|\bar{\zeta}^k\|^2 \leq \frac{\sum_{i=1}^m L_{ii}^2 \|\zeta_i^k\|^2}{m}$.

Under the conditions in the lemma statement, it can be seen that $\|\bar{x}^{k+1} - \bar{r}^{k+1}\|^2$ satisfies the conditions for v^k in Lemma 1, with $o^k = (\alpha^k)^2$, $q^k = 2\alpha^k$, and $p^k = \frac{\sum_{i=1}^m L_{ii}^2 (\sigma_i^k)^2 (\chi^k)^2}{m}$. Therefore, we have $\|\bar{x}^{k+1} - \bar{r}^{k+1}\|^2$ converging *a.s.* to zero, and hence, \bar{x}^k converging *a.s.* to \bar{r}^k . ■

Lemma 4. *Under Assumption 1 and two positive sequences $\{\chi^k\}$ and $\{\alpha^k\}$ satisfying $\sum_{k=0}^{\infty} (\alpha^k)^2 < \infty$ and $\sum_{k=0}^{\infty} (\chi^k)^2 < \infty$, there always exists a $T > 0$ such that $\|(1 - \alpha^k)(I - \frac{\mathbf{1}\mathbf{1}^T}{m}) + \chi^k L\| < 1 - \chi^k |\rho_2|$ holds for all $k \geq T$, where ρ_2 is the second largest eigenvalue of L .*

Proof. Because L is symmetric according to Assumption 1, all of its eigenvalues are real numbers. Since L ' off-diagonal entries are non-negative and its diagonal entries L_{ii} are given by $L_{ii} = -\sum_{j \in \mathbb{N}_i} L_{ij}$, the Gershgorin circle theorem implies that all eigenvalues of L are non-positive, with one of them being equal to 0. In fact, the 0 eigenvalue is simple (i.e., all the other eigenvalues are strictly negative) from Assumption 1. Arrange the eigenvalues of L as $\rho_m \leq \rho_{m-1} \leq \dots \leq \rho_2 < \rho_1 = 0$. It can be verified that the eigenvalues of $(1 - \alpha^k)I + \chi^k L$ are equal to $1 - \alpha^k + \chi^k \rho_m \leq 1 - \alpha^k + \chi^k \rho_{m-1} \leq \dots \leq 1 - \alpha^k + \chi^k \rho_2 < 1 - \alpha^k + \chi^k \rho_1 = 1 - \alpha^k$, and the eigenvalues of $(1 - \alpha^k)(I - \frac{\mathbf{1}\mathbf{1}^T}{m}) + \chi^k L = (1 - \alpha^k)I + \chi^k L + (1 - \alpha^k) \frac{\mathbf{1}\mathbf{1}^T}{m}$ are given by $\{1 - \alpha^k + \chi^k \rho_m, 1 - \alpha^k + \chi^k \rho_{m-1}, \dots, 1 - \alpha^k + \chi^k \rho_2, 0\}$, with $|\rho_m| \geq |\rho_{m-1}| \geq \dots \geq |\rho_2| > 0$. Hence, the norm of $\|(1 - \alpha^k)(I - \frac{\mathbf{1}\mathbf{1}^T}{m}) + \chi^k L\|$ is no larger than $\max\{|1 - \alpha^k + \chi^k \rho_m|, |1 - \alpha^k + \chi^k \rho_2|\}$. Further taking into account the fact that α^k and χ^k decay to zero because they are square summable, we conclude that there always exists a $T > 0$ such that $|1 - \alpha^k + \chi^k \rho_m| = 1 - \alpha^k - \chi^k |\rho_m| < 1 - \chi^k |\rho_m|$ and $|1 - \alpha^k + \chi^k \rho_2| = 1 - \alpha^k - \chi^k |\rho_2| < 1 - \chi^k |\rho_2|$ hold for all $k \geq T$. Given $|\rho_m| \geq |\rho_2|$, we have the stated result. ■

Theorem 1. *Under Assumptions 1, 2, 3, if the nonnegative sequences $\{\alpha^k\}$ and $\{\chi^k\}$ in Algorithm 1 and the nonnegative sequence $\{\gamma^k\}$ in Assumption 3 satisfy*

$$\sum_{k=0}^{\infty} \alpha^k = \infty, \quad \sum_{k=0}^{\infty} \chi^k = \infty, \quad \sum_{k=0}^{\infty} (\chi^k)^2 < \infty, \quad \sum_{k=0}^{\infty} \frac{(\gamma^k)^2}{\chi^k} < \infty, \quad (6)$$

and $\lim_{k \rightarrow \infty} \frac{\alpha^k}{\gamma^k} < \infty$, then, the following results hold *a.s.* :

- 1) every x_i^k in Algorithm 1 converges to $\bar{r}^k = \frac{\sum_{i=1}^m r_i^k}{m}$;
- 2) $\sum_{k=0}^{\infty} \chi^k \sum_{i=1}^m \|x_i^k - \bar{x}^k\|^2 < \infty$;
- 3) $\sum_{k=0}^{\infty} \gamma^k \sum_{i=1}^m \|x_i^k - \bar{x}^k\| < \infty$.

Proof. For the convenience of analysis, we write the iterates of x_i^k on per-coordinate expressions. More specifically, for all $\ell = 1, \dots, d$, and $k \geq 0$, we define $x^k(\ell) = [[x_1^k]_{\ell}, \dots, [x_m^k]_{\ell}]^T$ where $[x_i^k]_{\ell}$ represents the ℓ th element of the vector x_i^k . Similarly, we define $r^k(\ell) = [[r_1^k]_{\ell}, \dots, [r_m^k]_{\ell}]^T$ and $\zeta^k(\ell) = [[\zeta_1^k]_{\ell}, \dots, [\zeta_m^k]_{\ell}]^T$. In this per-coordinate view, (2) has the following form for all $\ell = 1, \dots, d$, and $k \geq 0$:

$$\begin{aligned} x^{k+1}(\ell) &= (1 - \alpha^k)x^k(\ell) + \chi^k Lx^k(\ell) + \chi^k L^0 \zeta^k(\ell) \\ &\quad + r^{k+1}(\ell) - (1 - \alpha^k)r^k(\ell). \end{aligned} \quad (7)$$

where L^0 is the same as L except that all of its diagonal entries are zero.

The dynamics of \bar{x}^k is governed by

$$\begin{aligned} \bar{x}^{k+1} t_{\ell} &= \frac{\mathbf{1}^T}{m} x^{k+1}(\ell) = \frac{\mathbf{1}^T}{m} ((1 - \alpha^k)x^k(\ell) + \chi^k Lx^k(\ell)) \\ &\quad + \frac{\mathbf{1}^T}{m} (\chi^k L^0 \zeta^k(\ell) + r^{k+1}(\ell) - (1 - \alpha^k)r^k(\ell)), \end{aligned} \quad (8)$$

where $[\bar{x}^{k+1}]_{\ell}$ represents the ℓ -th element of \bar{x}^{k+1} .

Assumption 1 implies $\mathbf{1}^T L = 0$, which can further simplify (8) as:

$$\begin{aligned} \bar{x}^{k+1} t_{\ell} &= \frac{\mathbf{1}^T}{m} ((1 - \alpha^k)x^k(\ell) + \chi^k L^0 \zeta^k(\ell) \\ &\quad + r^{k+1}(\ell) - (1 - \alpha^k)r^k(\ell)). \end{aligned} \quad (9)$$

Combining the relations in (7) and (9) yields

$$\begin{aligned} x^{k+1}(\ell) - \mathbf{1}[\bar{x}^{k+1}]_{\ell} &= ((1 - \alpha^k)\Pi + \chi^k L) x^k(\ell) + \chi^k \Pi L^0 \zeta^k(\ell) \\ &\quad + \Pi (r^{k+1}(\ell) - (1 - \alpha^k)r^k(\ell)). \end{aligned} \quad (10)$$

where $\Pi \triangleq I - \frac{\mathbf{1}\mathbf{1}^T}{m}$.

To simplify the notation, we define

$$W^k \triangleq (1 - \alpha^k)\Pi + \chi^k L = (1 - \alpha^k)(I - \frac{\mathbf{1}\mathbf{1}^T}{m}) + \chi^k L. \quad (11)$$

Assumption 1 ensures $W^k \mathbf{1} = 0$, and hence $W^k \mathbf{1}[\bar{x}^k]_{\ell} = 0$ for any $1 \leq \ell \leq d$. Subtracting $W^k \mathbf{1}[\bar{x}^k]_{\ell} = 0$ from the right hand side of (10) yields

$$\begin{aligned} &\|x^{k+1}(\ell) - \mathbf{1}[\bar{x}^{k+1}]_{\ell}\|^2 \\ &\leq \|W^k(x^k(\ell) - \mathbf{1}[\bar{x}^k]_{\ell}) + \Pi(r^{k+1}(\ell) - (1 - \alpha^k)r^k(\ell))\|^2 \\ &\quad + 2 \langle W^k(x^k(\ell) - \mathbf{1}[\bar{x}^k]_{\ell}) + \Pi(r^{k+1}(\ell) - (1 - \alpha^k)r^k(\ell)), \\ &\quad \chi^k \Pi L^0 \zeta^k(\ell) \rangle + (\chi^k)^2 \|L^0\|^2 \|\zeta^k(\ell)\|^2. \end{aligned} \quad (12)$$

where we used $\|\Pi\| = 1$. Using the assumption that the DP-noise ζ_i^k has zero mean and variance $(\sigma_i^k)^2$ conditional on x_i^k (see Assumption 2), taking the conditional expectation, given $\mathcal{F}^k = \{x^0, \dots, x^k\}$, we obtain the following inequality for all $k \geq 0$:

$$\begin{aligned} & \mathbb{E} [\|x^{k+1}(\ell) - \mathbf{1}[\bar{x}^{k+1}]_\ell\|^2 | \mathcal{F}^k] \\ & \leq (\|W^k\| \|x^k(\ell) - \mathbf{1}[\bar{x}^k]_\ell\| + \|r^{k+1}(\ell) - (1 - \alpha^k)r^k(\ell)\|)^2 \\ & \quad + (\chi^k)^2 \|L^0\|^2 \mathbb{E} [\|\zeta^k(\ell)\|^2]. \end{aligned} \quad (13)$$

For the first term on the right hand side of the preceding inequality, we bound it using the fact that there exists a $T \geq 0$ such that $0 < \|W^k\| \leq 1 - \chi^k |\rho_2|$ holds for all $k \geq T$ (see Lemma 4). Hence, equation (13) implies the following relationship for all $k \geq T$:

$$\begin{aligned} & \mathbb{E} [\|x^{k+1}(\ell) - \mathbf{1}[\bar{x}^{k+1}]_\ell\|^2 | \mathcal{F}^k] \leq \\ & ((1 - \chi^k |\rho_2|) \|x^k(\ell) - \mathbf{1}[\bar{x}^k]_\ell\| + \|r^{k+1}(\ell) - (1 - \alpha^k)r^k(\ell)\|)^2 \\ & \quad + (\chi^k)^2 \|L^0\|^2 \mathbb{E} [\|\zeta^k(\ell)\|^2]. \end{aligned} \quad (14)$$

Next, we apply the inequality $(a + b)^2 \leq (1 + \epsilon)a^2 + (1 + \epsilon^{-1})b^2$ (which is valid for any scalars a, b , and $\epsilon > 0$) to (14) and set $\epsilon = \frac{\chi^k |\rho_2|}{1 - \chi^k |\rho_2|}$ (which implies $(1 + \epsilon) = \frac{1}{1 - \chi^k |\rho_2|}$ and $1 + \epsilon^{-1} = \frac{1}{\chi^k |\rho_2|}$):

$$\begin{aligned} & \mathbb{E} [\|x^{k+1}(\ell) - \mathbf{1}[\bar{x}^{k+1}]_\ell\|^2 | \mathcal{F}^k] \leq (\chi^k)^2 \|L^0\|^2 \mathbb{E} [\|\zeta^k(\ell)\|^2] \\ & \quad + (1 - \chi^k |\rho_2|) \|x^k(\ell) - \mathbf{1}[\bar{x}^k]_\ell\|^2 \\ & \quad + \frac{1}{\chi^k |\rho_2|} \|r^{k+1}(\ell) - (1 - \alpha^k)r^k(\ell)\|^2. \end{aligned} \quad (15)$$

Note that the following relations always hold: $\sum_{\ell=1}^d \|x^k(\ell) - [\bar{x}^k]_\ell \mathbf{1}\|^2 = \sum_{i=1}^m \|x_i^k - \bar{x}^k\|^2$, $\sum_{\ell=1}^d \|r^{k+1}(\ell) - (1 - \alpha^k)r^k(\ell)\|^2 = \sum_{i=1}^m \|r_i^{k+1} - (1 - \alpha^k)r_i^k\|^2$, $\sum_{\ell=1}^d \|\zeta^k(\ell)\|^2 = \sum_{i=1}^m \|\zeta_i^k\|^2$.

Hence, summing (15) over $\ell = 1, \dots, d$ leads to

$$\begin{aligned} & \mathbb{E} \left[\sum_{i=1}^m \|x_i^{k+1} - \bar{x}^{k+1}\|^2 | \mathcal{F}^k \right] \\ & \leq (\chi^k)^2 \|L^0\|^2 \sum_{i=1}^m (\sigma_i^k)^2 + (1 - \chi^k |\rho_2|) \sum_{i=1}^m \|x_i^k - \bar{x}^k\|^2 \\ & \quad + \frac{1}{\chi^k |\rho_2|} \sum_{i=1}^m \|r_i^{k+1} - (1 - \alpha^k)r_i^k\|^2. \end{aligned} \quad (16)$$

Using the assumption $\|r_i^{k+1} - (1 - \alpha^k)r_i^k\| \leq \gamma^k C$ for all $i \in [m]$ from Assumption 3, we obtain

$$\frac{1}{\chi^k |\rho_2|} \sum_{i=1}^m \|r_i^{k+1} - (1 - \alpha^k)r_i^k\|^2 \leq \frac{(\gamma^k)^2}{\chi^k |\rho_2|} m C^2.$$

Submitting the preceding relationship into (16) yields

$$\begin{aligned} & \mathbb{E} \left[\sum_{i=1}^m \|x_i^{k+1} - \bar{x}^{k+1}\|^2 | \mathcal{F}^k \right] \leq (\chi^k)^2 \|L^0\|^2 \sum_{i=1}^m (\sigma_i^k)^2 \\ & \quad + (1 - \chi^k |\rho_2|) \sum_{i=1}^m \|x_i^k - \bar{x}^k\|^2 + \frac{(\gamma^k)^2}{\chi^k |\rho_2|} m C^2. \end{aligned} \quad (17)$$

Therefore, under Assumption 2 and the conditions for χ^k and γ^k in (6), we have that the sequence $\{\sum_{i=1}^m \|x_i^k - \bar{x}^k\|^2\}$ satisfies the conditions for $\{v^k\}$ in Lemma 1, and hence, converges to zero almost surely. So x_i^k converges to \bar{x}^k almost surely. Further recalling \bar{x}^k converging a.s. to \bar{r}^k in Lemma 3 yields that x_i^k converges a.s. to \bar{r}^k .

Moreover, Lemma 1 also implies the following relation a.s.:

$$\sum_{k=0}^{\infty} \chi^k \sum_{i=1}^m \|x_i^k - \bar{x}^k\|^2 < \infty. \quad (18)$$

To prove the last statement, we invoke the Cauchy–Schwarz inequality, which ensures

$$\begin{aligned} & \sum_{k=0}^{\infty} \sqrt{\chi^k \sum_{i=1}^m \|x_i^k - \bar{x}^k\|^2} \sqrt{\frac{(\gamma^k)^2}{\chi^k}} \\ & \leq \sqrt{\sum_{k=0}^{\infty} \chi^k \sum_{i=1}^m \|x_i^k - \bar{x}^k\|^2} \sqrt{\sum_{k=0}^{\infty} \frac{(\gamma^k)^2}{\chi^k}}. \end{aligned}$$

Noting that the summand in the left hand side of the preceding inequality is actually $\gamma^k \sqrt{\sum_{i=1}^m \|x_i^k - \bar{x}^k\|^2}$, and the right hand side of the preceding inequality is less than infinity almost surely under the proven result in (18) and the assumption in (6), we have that $\sum_{k=0}^{\infty} \gamma^k \sqrt{\sum_{i=1}^m \|x_i^k - \bar{x}^k\|^2} < \infty$ holds almost surely. Further utilizing the relationship $\sum_{i=1}^m \|x_i^k - \bar{x}^k\| \leq \sqrt{m \sum_{i=1}^m \|x_i^k - \bar{x}^k\|^2}$ yields the stated result. ■

Remark 2. In Theorem 1, $\{\gamma^k\}$ decays to zero. In combination with Assumption 3, this requires r_i^k 's variations to gradually decay with time, which is necessary to ensure every x_i^k to track the exact average reference signal \bar{r}^k . This is because when individual reference signals are unknown and varying persistently, in general it is impossible for agents to precisely track their average using discrete fixed-frequency samples of these unknown signals [3]. Note that this assumption is satisfied in many applications of dynamic average consensus. For example, in dynamic-average-consensus based distributed optimization, the reference signals of individual agents are gradients of individual objective functions, which gradually converge to constant values as the iterates converge to the optimal solution [44]. This is also the case in distributed Nash equilibrium seeking, where the reference signals of individual agents (usually called players in games) are individual pseudogradients, which will converge to constant values when the players converge to the Nash equilibrium [7].

Remark 3. In distributed systems, usually communication imperfections can be modeled as channel noises [45], which can be regarded as a special case of DP-noise. Therefore, Algorithm 1 can also be used to counteract communication imperfections in distributed computation.

Next, we prove that Algorithm 1 can ensure ϵ -DP for individual agents' reference signals r_i^k with the cumulative

privacy budget guaranteed to be finite, even when the number of iterations tends to infinity.

V. DIFFERENTIAL-PRIVACY ANALYSIS

We have to characterize the sensitivity of Algorithm 1 in order to quantify the level of enabled privacy strength. Similar to the sensitivity definition of iterative optimization algorithms in [38], we define the sensitivity of a dynamic average consensus algorithm as follows:

Definition 3. *At each iteration k , for any initial state ϑ^0 and any adjacent dynamic average consensus problems \mathcal{P} and \mathcal{P}' , the sensitivity of Algorithm 1 is*

$$\Delta^k \triangleq \sup_{\mathcal{O} \in \mathcal{O}} \left\{ \sup_{\vartheta \in \mathcal{R}_{\mathcal{P}, \vartheta^0}^{-1}(\mathcal{O}), \vartheta' \in \mathcal{R}_{\mathcal{P}', \vartheta^0}^{-1}(\mathcal{O})} \|\vartheta^k - \vartheta'^k\|_1 \right\}. \quad (19)$$

Lemma 5. *In Algorithm 1, if each agent's DP-noise vector $\zeta_i^k \in \mathbb{R}^d$ consists of d independent Laplace noises with parameter ν^k such that $\sum_{k=1}^{T_0} \frac{\Delta^k}{\nu^k} \leq \bar{\epsilon}$, then Algorithm 1 is ϵ -differentially private with the cumulative privacy budget from iteration $k = 0$ to $k = T_0$ less than $\bar{\epsilon}$.*

Proof. The lemma follows the same line of reasoning of Lemma 2 in [38] (also see Theorem 3 in [46]). ■

Theorem 2. *Under the conditions of Theorem 1, if all elements of ζ_i^k are drawn independently from Laplace distribution $\text{Lap}(\nu^k)$ with $(\sigma_i^k)^2 = 2(\nu^k)^2$ satisfying Assumption 2, then all agents will converge a.s. to the average reference signal \bar{r}^k . Moreover,*

- 1) *For any finite number of iterations T , Algorithm 1 is ϵ -differentially private with the cumulative privacy budget bounded by $\epsilon \leq \sum_{k=1}^T \frac{2C_r \zeta^k}{\nu^k}$ where $\zeta^k \triangleq \sum_{p=1}^{k-1} \prod_{q=p}^{k-1} (1 - \alpha^q - \bar{L}\gamma^q) \Lambda^{p-1} + \Lambda^{k-1}$, $\bar{L} \triangleq \min_i \{|L_{ii}|\}$, $\Lambda^k \triangleq \gamma^{k+1} \chi^{k+1} + (1 - \alpha^k) \gamma^k \chi^k$, and C_r is given in Definition 1;*
- 2) *The cumulative privacy budget is finite for $T \rightarrow \infty$ when the sequence $\{\frac{\gamma^k}{\nu^k}\}$ is summable.*

Proof. Since the Laplace noise satisfies Assumption 2, the convergence result follows directly from Theorem 1.

To prove the two statements on the strength of ϵ -DP, we first analyze the sensitivity of the algorithm. Given two adjacent dynamic average consensus problems \mathcal{P} and \mathcal{P}' , for any given fixed observation \mathcal{O} and initial state $\vartheta^0 = x^0$, the sensitivity depends on $\|x^k - x'^k\|_1$ according to Definition 3. Since in \mathcal{P} and \mathcal{P}' , only one reference signal is different, we represent this different reference signal as the i th one, i.e., r_i in \mathcal{P} and r'_i in \mathcal{P}' , without loss of generality.

Because the initial conditions, reference signals, and observations of \mathcal{P} and \mathcal{P}' are identical for $j \neq i$, we have $x_j^k = x'_j{}^k$ for all $j \neq i$ and k . Therefore, $\|x^k - x'^k\|_1$ is always equal to $\|x_i^k - x'_i{}^k\|_1$.

Algorithm 1 implies

$$\begin{aligned} x_i^{k+1} - x'_i{}^{k+1} &= (1 - \alpha^k - |L_{ii}|\chi^k)(x_i^k - x'_i{}^k) \\ &\quad + (r_i^{k+1} - r'_i{}^{k+1}) - (1 - \alpha^k)(r_i^k - r'_i{}^k). \end{aligned}$$

Note that we have used the fact that the observations $x_j^k + \zeta_j^k$ and $x'_j{}^k + \zeta_j{}^k$ are the same.

Hence, using the third condition in Definition 1, the sensitivity Δ^k satisfies

$$\begin{aligned} \Delta^{k+1} &\leq (1 - \alpha^k - |L_{ii}|\chi^k)\Delta^k + C_r \gamma^{k+1} \chi^{k+1} + C_r (1 - \alpha^k) \gamma^k \chi^k \\ &\leq (1 - \alpha^k - \bar{L}\chi^k)\Delta^k + C_r \Lambda^k, \end{aligned} \quad (20)$$

where \bar{L} and Λ^k are defined in the theorem statement. Then, by iteration, we can arrive at the first privacy statement using Lemma 5 (note $\Delta^0 = 0$ as \mathcal{P} and \mathcal{P}' have identical initial conditions).

For the infinity-time-horizon result in the second statement, we exploit Lemma 2 and the third condition in Definition 1. More specifically, from (20), according to Lemma 2, we can always find some \bar{C} such that $\Delta^k \leq \bar{C}\gamma^k$ holds (note that α^k decays faster than χ^k). Using Lemma 5, we can easily obtain $\epsilon \leq \sum_{k=1}^T \frac{\bar{C}\gamma^k}{\nu^k}$. Hence, ϵ will be finite even when T tends to infinity if the sequence $\{\frac{\gamma^k}{\nu^k}\}$ is summable. ■

Note that in dynamic average consensus applications such as Nash equilibrium seeking, [46] achieves ϵ -DP by enforcing the tracking input signal to be summable (by multiplying it with a geometrically-decreasing factor), which, however, also makes it impossible to ensure accurate convergence to the desired equilibrium. In our approach, by allowing the tracking input $r_i^{k+1} - (1 - \alpha^k)r_i^k$ to be non-summable (since we allow $\sum_{k=0}^{\infty} \gamma^k = \infty$), we achieve both accurate convergence and finite cumulative privacy budget, even when the number of iterations goes to infinity. To our knowledge, this is the first dynamic average consensus algorithm that can achieve both almost sure convergence to the *exact* average reference signal and rigorous ϵ -DP, even with the number of iterations going to infinity.

Remark 4. *To ensure that the cumulative privacy budget $\sum_{k=1}^{\infty} \frac{C_r \gamma^k}{\nu^k}$ is bounded, we employ Laplace noise with parameter ν^k increasing with time (since we require the sequence $\{\frac{\gamma^k}{\nu^k}\}$ to be summable while the sequence $\{\gamma^k\}$ is non-summable). Because the strength of the shared signal is time-invariant (i.e., x_i^k), an increasing ν^k makes the relative level between noise ζ_i^k and signal x_i^k increase with time. However, since it is $\chi^k \text{Lap}(\nu^k)$ that is actually fed into the algorithm, and the increase in the noise level ν^k is outweighed by the decrease of χ^k (see Assumption 2), the actual noise fed into the algorithm still decays with time, which explains why Algorithm 1 can ensure every agent's accurate convergence. Moreover, according to Theorem 1, the convergence will not be affected if we scale ν^k by any constant $\frac{1}{\epsilon} > 0$ to achieve any desired level of ϵ -DP, as long as ν^k (with associated variance $(\sigma_i^k)^2 = 2(\nu^k)^2$) satisfies Assumption 2.*

Remark 5. *It is worth noting that our simultaneous achievement of both provable accurate convergence and ϵ -DP does not contradict the fundamental limitations of the DP theory [32]. In fact, the DP theory indicates that a query mechanism on a dataset can achieve ϵ -DP only by sacrificing the*

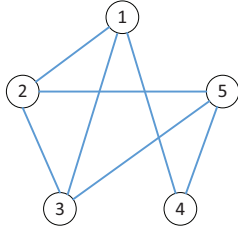


Fig. 1. The interaction network used in numerical simulation.

accuracy of query. However, in dynamic average consensus, what are queried in every iteration are individual reference signals but not the average reference signal, and revealing the value of the average reference signal at steady state to an observer is not equivalent to revealing entire individual reference signals (the actual query target).

VI. NUMERICAL SIMULATIONS

We evaluate the performance of the proposed algorithm using a network of $m = 5$ agents. In the simulation, we set the reference signal of agent i as $a_i + \frac{b_i}{10k} \sin(0.05k)$ where a_i and b_i are randomly selected from a uniform distribution on $(0, 10)$ for all $i \in [m]$. The used reference signals for all agents as well as the average reference signal are depicted in Fig. 2. To enable DP, we inject Laplace noise with parameter $\nu^k = 1 + 0.1k^{0.2}$. We set the diminishing sequence as $\chi^k = \frac{2}{1+k^{0.5}}$. The stepsize is set as $\alpha^k = \frac{0.01}{1+k}$. It can be verified that the conditions in Theorem 1 are satisfied under the setting. In the evaluation, we run our algorithm for 100 times, and calculate the average and the variance of the tracking error $\sum_{i=1}^m \|x_i^k - \bar{x}^k\|$ against the iteration index k . The result is given by the red curve and error bars in Fig. 3. For comparison, we also run the existing dynamic average consensus algorithm proposed by Zhu et al. in [2] under the same noise, and the existing DP approach proposed by Huang et al. in [38] under the same cumulative privacy budget ϵ . Note that the DP approach in [38] addresses distributed optimization problems, which can be viewed as a dynamic average consensus problem if we regard the gradient of an agent's objective function as the input to the agent. We adapt its DP mechanism (geometrically decreasing stepsizes for input gradients and geometrically decreasing DP-noise) to the dynamic average consensus problem. The evolution of the average error/variance of the approaches in [2] and [38] are given by the blue and black curves/error bars in Fig. 3, respectively. It can be seen that the proposed algorithm has a much better accuracy.

VII. CONCLUSIONS

This paper proposes a robust dynamic average consensus algorithm that can ensure provable convergence to the exact average reference signal even in the presence of persistent differential-privacy noise. Different from existing differential privacy solutions for average consensus that have to trade convergence accuracy for differential privacy, the proposed

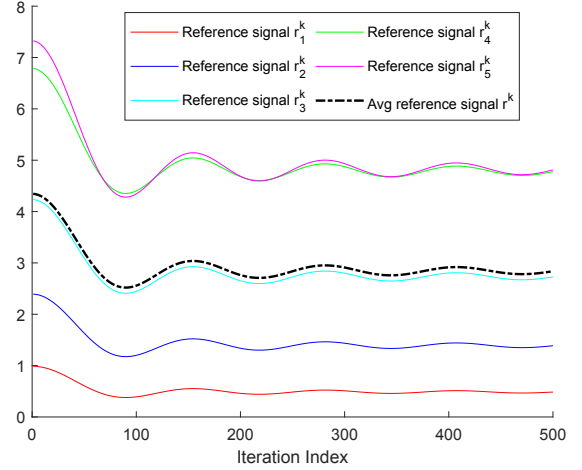


Fig. 2. The individual reference signals of five agents and the average reference signal (dashed black curve).

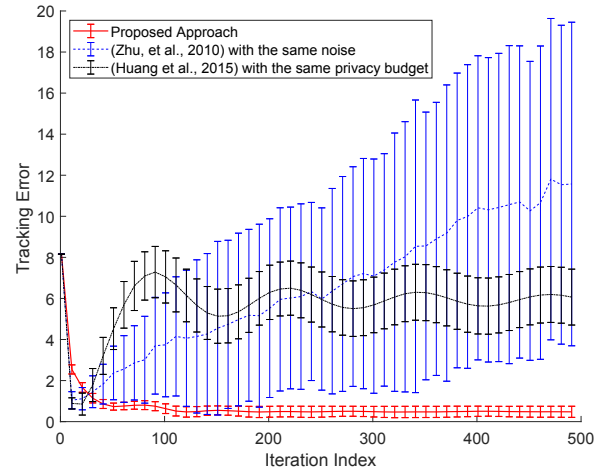


Fig. 3. Comparison of the proposed algorithm with existing DP approach for distributed optimization from Huang et al. 2015 [38] which can be adapted to dynamic average consensus (under the same privacy budget ϵ) and the conventional dynamic average consensus approach in Zhu et al., 2010 [2] (under the same noise). The tracking error is defined as $(\sum_{i=1}^m \|x_i^k - \bar{x}^k\|)$.

approach can ensure both provable convergence to the exact desired average signal and rigorous ϵ -differential privacy. To our knowledge, the simultaneous achievement of both ϵ -differential privacy and provable convergence accuracy has not been achieved in average consensus before. Since channel noise in communication can be viewed as a special case of differential-privacy noise, our algorithm can also counteract communication imperfections in dynamic average consensus. Numerical simulation results confirm the effectiveness of the proposed algorithm.

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