2024 IEEE 63rd Conference on Decision and Control (CDC)
December 16-19, 2024. MiCo, Milan, Italy

Risk-Aware Control of Discrete-Time Stochastic Systems: Integrating Kalman Filter and Worst-case CVaR in Control Barrier Functions

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***Abstract*— This paper proposes control approaches for discrete-time linear systems subject to stochastic disturbances. It employs Kalman filter to estimate the mean and covariance of the state propagation, and the worst-case conditional value-at-risk (CVaR) to quantify the tail risk using the estimated mean and covariance. The quantified risk is then integrated into a control barrier function (CBF) to derive constraints for controller synthesis, addressing tail risks near safe set boundaries. Two optimization-based control methods are presented using the obtained constraints for half-space and ellipsoidal safe sets, respectively. The effectiveness of the obtained results is demonstrated using numerical simulations.**

I. Introduction

Safety-critical systems such as autonomous vehicles, aerospace vehicles, and medical devices demand high reliability due to the severe consequences of their failure or malfunction, such as loss of life, significant property damage, or environmental harm. As automation becomes more prevalent in these applications, the need for risk-aware controller design is becoming increasingly essential.

In this context, Control Barrier Functions (CBFs) [1] are now recognized for their pivotal role. CBFs provide conditions for control inputs to ensure system states remain within a given safe set, thereby guaranteeing the satisfaction of safety requirements in various fields [2]–[5].

Recent advancements in CBF approaches, such as those discussed [6]–[8], have incorporated external stochastic disturbances into their considerations. Moreover, some approaches consider risk through chance constraints [9], or bounds on the probability of a collision [10]. In particular, studies [11]–[13] provide ways of considering the safety by using the quantified tail risk to avoid severe consequences. Additionally, although many existing approaches assume that the all the system states can be observed, which may be impractical in some real applications, some [14]–[17] deal with the cases where not all the system states can be directly accessible. Yet, comprehensive solutions addressing both stochastic disturbances and unmeasured states remain limited [14], [18].

The objective of this paper is to introduce a risk-aware control approach tailored for discrete-time linear systems affected by stochastic disturbances, where not all system states are directly observable. The approach integrates Kalman filter [19], [20] with the worst-case conditional value-at-risk (CVaR) [21], [22] and synthesizes it with a CBF framework. This integration is highly synergistic: Kalman filter provides estimates of mean and covariance of the state propagation, essential parameters for the worst-case CVaR to quantify tail risk. This combination effectively enhances controller design using CBFs.

The paper is structured as follows: Section II introduces the notation, definitions and fundamental results. Sections III - V are the main part of this paper: After introducing building blocks in Section III, sets of admissible inputs are characterized using Kalman filter along with the worst-case CVaR and the control barrier function in Section IV, which is followed by the controller design in Section V. After numerical examples in Section VI, the paper is concluded with Section VII.

II. Mathematical Preliminaries

A. Notation

The sets of real numbers, real vectors of length n, real matrices of size $n \times m$, real symmetric matrices of size n, and positive definite matrices of size n are denoted by $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{R}^{n\times m}$, $\mathbb{S}^n$ and $\mathbb{S}^n_+$, respectively. For $M \in \mathbb{R}^{n\times n}$, $M \succ 0$ indicates M is positive definite, $M \succcurlyeq 0$ indicates M is positive semidefinite, and $\mathrm{Tr}(M)$ denotes its the trace. For $M \in \mathbb{R}^{n\times m}$, $M^\top$ denotes its transpose. For a vector $v \in \mathbb{R}^n$, $\|v\|$ denotes its Euclidean norm, $v \geq 0$ its element-wise non-negativity, and $|v|$ the element-wise absolute value.

B. Conditional Value-at-Risk

To present risk quantification in our approach, we begin by showing the (worst-case) CVaR.

Consider a random vector $\xi \in \mathbb{R}^n$ under the true distribution $\mathbb{P}$, characterized by its mean $\mu \in \mathbb{R}^n$ and covariance matrix $\Sigma \in \mathbb{R}^{n\times n}$. The distribution $\mathbb{P}$, representing the probability law of ξ, is assumed to have finite second-order moments. We define $\mathcal{P}$ as the set of all probability distributions on $\mathbb{R}^n$ with identical first- and second-order moments to $\mathbb{P}$. Formally,

$$\mathcal{P} = \left\{ \mathbb{P} : \mathbb{E}_{\mathbb{P}} \left[\begin{bmatrix} \xi_i \\ 1 \end{bmatrix} \begin{bmatrix} \xi_j \\ 1 \end{bmatrix}^\top \right] = \begin{bmatrix} \Sigma\delta_{ij} & \mu \\ \mu^\top & 1 \end{bmatrix}, \forall i,j \right\},$$

where δ_{ij} is the Kronecker delta, and $\mathbb{E}_{\mathbb{P}}[\cdot]$ the expected value under $\mathbb{P}$. Although the exact form of $\mathbb{P}$ is unknown, it is clear that $\mathbb{P} \in \mathcal{P}$.

*This work was supported by JST, PRESTO Grant Number JPMJPR22C3, Japan.

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Definition 2.1 (Conditional Value-at-Risk (CVaR) [22], [23]): Given a measurable loss function $L : \mathbb{R}^n \to \mathbb{R}$, a probability distribution $\mathbb{P}$ on $\mathbb{R}^n$, and a level $\varepsilon \in (0, 1)$, the CVaR at level ε under $\mathbb{P}$ is defined by:

$$\mathbb{P}\text{-CVaR}_\varepsilon[L(\xi)] = \inf_{\beta\in\mathbb{R}} \left\{\beta + \frac{1}{\varepsilon}\mathbb{E}_\mathbb{P}[(L(\xi) - \beta)^+]\right\}.$$

CVaR is the conditional expectation of losses exceeding the $(1-\varepsilon)$-quantile of L, quantifying the tail risk of loss function [22].

Having introduced the definition of CVaR, we now extend this concept to its worst-case scenario, key to the proposed approach. The worst-case CVaR is the supremum of CVaR over a given set of probability distributions as defined below:

Definition 2.2 (Worst-case CVaR [21], [22]): The worst-case CVaR over a set of distributions $\mathcal{P}$ is:

$$\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\text{-CVaR}_\varepsilon[L(\xi)] = \inf_{\beta\in\mathbb{R}} \left\{\beta + \frac{1}{\varepsilon} \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_\mathbb{P}[(L(\xi) - \beta)^+]\right\}.$$

The interchangeability of the supremum and infimum is validated by the stochastic saddle point theorem [24].

The widespread adoption of (worst-case) CVaR in risk quantification is largely attributed to its coherent properties.

Proposition 2.1 (Coherence properties [21], [25]): The worst-case CVaR is a coherent risk measure, i.e., it satisfies the following properties: Let $L_1 = L_1(\xi)$ and $L_2 = L_2(\xi)$ be two measurable loss functions, then the followings hold.

- Sub-additivity: For all L_1 and L_2,
$$\begin{aligned}&\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\text{-CVaR}_\varepsilon[L_1 + L_2]\\ &\le \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\text{-CVaR}_\varepsilon[L_1] + \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\text{-CVaR}_\varepsilon[L_2];\end{aligned}$$
- Positive homogeneity: For a positive constant $c_1 > 0$,
$$\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\text{-CVaR}_\varepsilon[c_1 L_1] = c_1 \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\text{-CVaR}_\varepsilon[L_1];$$
- Monotonicity: If $L_1 \le L_2$ almost surely,
$$\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\text{-CVaR}_\varepsilon[L_1] \le \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\text{-CVaR}_\varepsilon[L_2];$$
- Translation invariance: For a constant c_2,
$$\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\text{-CVaR}_\varepsilon[L_1 + c_2] = \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\text{-CVaR}_\varepsilon[L_1] + c_2.$$

The usefulness of the worst-case CVaR in risk quantification further appears in its computational efficiency for a special, but common, case. When the loss function $L(\xi)$ is quadratic in ξ, the worst-case CVaR can be computed via a semidefinite program. Let the second-order moment matrix of ξ

$$\Omega = \begin{bmatrix} \Sigma + \mu\mu^\top & \mu \\ \mu^\top & 1 \end{bmatrix}.$$

Lemma 2.1 (Quadratic Loss Function [22], [26]): For $L(\xi) = \xi^\top P \xi + 2q^\top \xi + r$, where $P \in \mathbb{S}^n$, $q \in \mathbb{R}^n$, and $r \in \mathbb{R}$, the worst-case CVaR is given by:

$$\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\text{-CVaR}_\varepsilon[L(\xi)] = \inf_{\beta} \left\{\beta + \frac{1}{\varepsilon}\mathrm{Tr}(\Omega N) : N \succcurlyeq 0,\ N - \left[\begin{array}{cc} P & q \\ q^\top & r - \beta \end{array}\right] \succcurlyeq 0\right\}.$$

Similarly to the absolute value of a vector, we denote the element-wise worst-case CVaR by

$$\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\text{-CVaR}_\varepsilon[v] = \begin{bmatrix} \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\text{-CVaR}_\varepsilon[e_1^\top v] \\ \vdots \\ \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\text{-CVaR}_\varepsilon[e_n^\top v] \end{bmatrix},$$

where e_i is the ith column of the identity matrix of size n.

Building on Lemma 2.1, we now present a lemma that provides a bound on linear loss functions.

Lemma 2.2 (A bound on linear $L(\xi)$[12]): Suppose $\mu = 0$. Then, for $q \in \mathbb{R}^n$,

$$\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\text{-CVaR}_\varepsilon[q^\top \xi] \le |q|^\top \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\text{-CVaR}_\varepsilon[\xi]. \tag{1}$$

C. Control Barrier Function

Having discussed the risk quantification, we now turn our attention to CBF, an approach to ensure the system safety. Here, we review the basic CBF with no disturbance.

Define the safe set $\mathcal{C}$ as the superlevel set of a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$,

$$\mathcal{C} = \{x \in \mathbb{R}^n : h(x) \ge 0\}. \tag{2}$$

With $\mathcal{C}$, the CBF is defined below.

Definition 2.3 (Control Barrier Function (CBF) [27]): The function h is a discrete-time CBF for

$$x_{t+1} = f(x_t, u_t) \tag{3}$$

on $\mathcal{C}$ if there exists an $\alpha \in [0, 1)$ such that for all $x_t \in \mathcal{C}$, there exists a $u_t \in \mathbb{R}^m$ such that:

$$h(x_{t+1}) \ge \alpha h(x_t). \tag{4}$$

The existence of a CBF guarantees that the control system is safe assuming that the initial state is in $\mathcal{C}$ [28]. We will define a risk-aware CBF at the end of the next section.

III. Preparation: Integrating Kalman Filter and Worst-case CVaR in CBF

This is the first section of the three main sections of this paper. Here, we set up the foundational elements; system model, state estimation and CBF, for the characterization of the admissible control input and controller synthesis that follow.

A. System Model

This paper deals with the discrete-time linear stochastic system described by:

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t + w_t, \\ z_t &= Hx_t + v_t, \end{aligned} \tag{5}$$

where $x_t \in \mathbb{R}^n$ is the state, $u_t \in \mathbb{R}^m$ is the control input, $w_t \in \mathbb{R}^{n_w}$ is the disturbance, $z_t \in \mathbb{R}^{n_y}$ is the measurement, and $v_t \in \mathbb{R}^{n_v}$ is the noise, respectively, at discrete time instant $t \in \mathbb{Z}_{\ge 0}$. The matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $H \in \mathbb{R}^{n\times n_y}$ are assumed to be constants.

It is assumed that

- the initial state x_0 is a random vector with $\mathbb{E}[x_0] = \bar{x}_{0|0}$ and covariance $\mathbb{E}[(x_0 - \bar{x}_0)(x_0 - \bar{x}_0)^\top] = P_{0|0}$,

- the disturbance w_t are independent and identically distributed random vectors with $\mathbb{E}[w_t] = 0$ and finite covariance $\mathbb{E}[w_t w_t^\top] = Q$,
- the noise v_t are independent and identically distributed random vectors with $\mathbb{E}[v_t] = 0$ and finite covariance $\mathbb{E}[v_t v_t^\top] = R$, and
- the initial state and the noise vectors at each step $\{x_0, w_1, ..., w_t, v_1, ..., v_t\}$ are all mutually independent.

B. Kalman Filter

We now introduce Kalman filter to estimate the mean and covariance of the state propagation.

The Kalman filter, a recursive estimator, optimal for linear systems with Gaussian noise, is adapted here for disturbances and noise, which are not necessarily Gaussians. It operates in two phases: prediction and update. In the prediction phase, the filter forecasts the next state and its uncertainty. Subsequently, during the update phase, it refines these predictions based on new measurements.

We employ standard Kalman filter notation for system (5):

- **Prediction:**
  - Predicted state mean at t: $\bar{x}_{t+1|t} = A\bar{x}_{t|t} + Bu_t$
  - Predicted error covariance: $P_{t+1|t} = AP_{t|t}A^\top + Q$
- **Update (with gain K_{t+1}):**
  - Measurement error covariance: $S_{t+1} = HP_{t+1|t}H^\top + R$ (this is used in Section V-A)
  - Updated state mean: $\bar{x}_{t+1|t+1} = \bar{x}_{t+1|t} + K_{t+1}(z_{t+1} - H\bar{x}_{t+1|t})$
  - Updated error covariance: $P_{t+1|t+1} = (I - K_{t+1}H)P_{t+1|t}$

How to choose the gain K_{t+1} is discussed later in Section V-A.

We also let:

- $\hat{x}_{t|t}$ be the random vector with mean $\bar{x}_{t|t}$ and covariance $P_{t|t}$,

and

$$\hat{x}_{t+1|t} = A\hat{x}_{t|t} + Bu_t + w_t. \tag{6}$$

be the random vector that propagates the state $\hat{x}_{t|t}$ to the next time step according to (5). This is crucial for evaluating the condition for the control input in the next subsection.

C. Risk-Aware CBF

This subsection integrates Kalman filter and worst-case CVaR to introduce Kalman filter-based risk-aware discrete-time CBF. This function is key for defining admissible control inputs in the subsequent section.

Consider a random vector $\xi_{t|t}$ and its shifted version $\xi_{t|t}^c$, defined as:

$$\xi_{t|t} = \begin{bmatrix} \hat{x}_{t|t}^\top & w_t^\top \end{bmatrix}^\top, \ \xi_{t|t}^c = \xi_{t|t} - \begin{bmatrix} \bar{x}_{t|t}^\top & 0 \end{bmatrix}^\top. \tag{7}$$

Define

$$\mathcal{P}_{t|t}^c = \left\{ \mathbb{P} : \mathbb{E}_{\mathbb{P}} \left[\begin{bmatrix} \xi_i \\ 1 \end{bmatrix} \begin{bmatrix} \xi_j \\ 1 \end{bmatrix}^\top \right] = \begin{bmatrix} \Sigma_{\xi_{t|t}^c} \delta_{ij} & 0 \\ 0^\top & 1 \end{bmatrix}, \forall i, j \right\}, \tag{8}$$

where $\Sigma_{\xi_{t|t}^c} = \begin{bmatrix} P_{t|t} & 0 \\ 0 & Q \end{bmatrix}$. Then, the measure $\mathbb{P}$ of $\xi_{t|t}^c$ is $\mathbb{P} \in \mathcal{P}_{t|t}^c$.

With $\mathcal{C}$ in (2), we define Kalman filter-based risk-aware discrete-time CBF as below.

Definition 3.1 (Risk-Aware Control Barrier Function): A function h is a Kalman filter-based risk-aware discrete-time CBF for system (5) on $\mathcal{C}$ if there exists an $\alpha \in [0, 1)$ such that for all $\hat{x}_{t|t}$ that satisfies $\sup_{\mathbb{P} \in \mathcal{P}_{t|t}^c} \mathbb{P}\text{-CVaR}_\varepsilon[-h(\hat{x}_{t|t})] \leq 0$, there exists a $u_t \in \mathbb{R}^m$ such that:

$$\sup_{\mathbb{P} \in \mathcal{P}_{t|t}^c} \mathbb{P}\text{-CVaR}_\varepsilon[-h(\hat{x}_{t+1|t}) + \alpha h(\hat{x}_{t|t})] \leq 0. \tag{9}$$

Here, $\hat{x}_{t|t}$ and $\hat{x}_{t+1|t}$ are computed by Kalman filter.

As we have integrated Kalman filter into CBF, Section IV will proceed to characterize sets of admissible control inputs using this, paving the way for controller synthesis.

IV. Charactering the Set of Admissible Control Input Using Kalman Filter

This is the second section of the three main sections. Here, we consider characterizing the set of admissible control input by quantifying the left-hand-side of (9) to determine the control input u_t at time t in the next section. Here, we still assume that the gain K_t is given.

A. Half-space safe set

We start with a half-space safe set scenario, where the set is defined by an affine function $h(x)$:

$$\mathcal{C}_{\text{hs}} = \{x \in \mathbb{R}^n : h(x) = q^\top x + r \geq 0\}. \tag{10}$$

This scenario allows us to reformulate the constraint (9) as a linear function of the control input u:

Theorem 4.1: If $h(x) = q^\top x + r$, $q \in \mathbb{R}^n$ and $r \in \mathbb{R}$, then the constraint (9) holds if and only if

$$-q^\top Bu_t \leq \phi(\hat{x}_{t|t}), \tag{11}$$

where

$$\begin{aligned} \phi(\hat{x}_{t|t}) = &- \sup_{\mathbb{P} \in \mathcal{P}_{t|t}^c} \mathbb{P}\text{-CVaR}_\varepsilon \left[-q^\top \begin{bmatrix} A - \alpha I & I \end{bmatrix} \xi_{t|t} \right] \\ &+ q^\top (A - \alpha I)\bar{x}_{t|t} + (1 - \alpha)r. \end{aligned} \tag{12}$$

Proof: To express the constraint in terms of the control input u_t, substitute (6) into the constraint (9):

$$\begin{aligned} &\sup_{\mathbb{P} \in \mathcal{P}_{t|t}^c} \mathbb{P}\text{-CVaR}_\varepsilon[-q^\top(\alpha I - A)(\hat{x}_{t|t} - \bar{x}_{t|t} + \bar{x}_{t|t} + w_t)] \\ &\quad - q^\top Bu_t - r + \alpha r \\ &= \sup_{\mathbb{P} \in \mathcal{P}_{t|t}^c} \mathbb{P}\text{-CVaR}_\varepsilon[-q^\top(\alpha I - A)(\hat{x}_{t|t} - \bar{x}_{t|t} + w_t)] \\ &\quad - q^\top Bu_t - q^\top(\alpha I - A)\bar{x}_{t|t} - (1 - \alpha)r \leq 0. \end{aligned} \tag{13}$$

This is equivalent to

$$\begin{aligned} -q^\top Bu_t \leq &- \sup_{\mathbb{P} \in \mathcal{P}_{t|t}^c} \mathbb{P}\text{-CVaR}_\varepsilon \left[-q^\top \begin{bmatrix} A - \alpha I & I \end{bmatrix} \xi_{t|t} \right] \\ &+ q^\top (A - \alpha I)\bar{x}_{t|t} + (1 - \alpha)r = \phi(\hat{x}_{t|t}). \end{aligned} \tag{14}$$

■

B. Ellipsoidal safe set

Next, we consider a scenario with an ellipsoidal safe set, defined by a positive definite matrix $E \in \mathbb{S}_+^n$ and a vector $x_c \in \mathbb{R}^n$:

$$\mathcal{C}_{\text{ell}} = \{x \in \mathbb{R}^n : h(x) = -(x - x_c)^\top E(x - x_c) + r \geq 0\}. \tag{15}$$

In this case, the constraint (9) can be expressed as follows:

Proposition 4.1: If $h(x) = -(x - x_c)^\top E(x - x_c) + r$, $E \in \mathbb{S}_+^n$, $x_c \in \mathbb{R}^n$ and $r \in \mathbb{R}$, then the constraint (9) is satisfied if and only if

$$\sup_{\mathbb{P}\in\mathcal{P}^c_{t|t}} \mathbb{P}\text{-CVaR}_\varepsilon \left[(\xi^c_{t|t})^\top \bar{P}\xi^c_{t|t} + 2\bar{q}^\top \xi^c_{t|t} + \bar{r}\right] \leq 0, \tag{16}$$

where $\bar{P}$, $\bar{q}$, and $\bar{r}$ are defined as follows:

$$\begin{aligned}
\bar{P} &= \begin{bmatrix} A^\top EA - \alpha E & A^\top E \\ EA & E \end{bmatrix}, \\
\bar{q} &= \bar{q}_1 + \bar{q}_2, \\
\bar{q}_1 &= \begin{bmatrix} A^\top E(A\bar{x}_{t|t} - x_c) - \alpha E(x - x_c) \\ E(A\bar{x}_{t|t} - x_c) \end{bmatrix}, \\
\bar{q}_2 &= \begin{bmatrix} A^\top EBu_t \\ EBu_t \end{bmatrix}, \\
\bar{r} &= (A\bar{x}_{t|t} + Bu_t - x_c)^\top E(A\bar{x}_{t|t} + Bu_t - x_c) \\
&\quad - \alpha(\bar{x}_{t|t} - x_c)^\top E(\bar{x}_{t|t} - x_c) - (1-\alpha)r.
\end{aligned} \tag{17}$$

Although checking the satisfaction of (16) for a given u_t is straightforward, characterizing the exact set of u_t that satisfies (16) is not except for some simple cases.

However, a sufficient condition for the constraint (9) to be satisfied can be expressed using a quadratic function of the control input u as follows:

Theorem 4.2: Let $h(x) = -(x - x_c)^\top E(x - x_c) + r$ with $E \in \mathbb{S}_+^n$, $x_c \in \mathbb{R}^n$, and $r \in \mathbb{R}$. Define $\bar{u}_t = [u_t^\top, v_t^\top]^\top$ as a combination of the control input $u_t \in \mathbb{R}^m$ and an auxiliary variable $v_t \in \mathbb{R}^m$. Then, the constraint (9) holds if:

$$\bar{u}_t^\top \tilde{P}\bar{u}_t + 2\tilde{q}^\top(x)\bar{u}_t + \tilde{r}(x) \leq 0 \text{ and } \tilde{A}\bar{u}_t \leq 0, \tag{18}$$

where

$$\begin{aligned}
\tilde{P} &= \begin{bmatrix} B^\top EB & 0 \\ 0 & 0 \end{bmatrix}, \\
\tilde{q}(x) &= \begin{bmatrix} B^\top E(A\bar{x}_{t|t} - x_c) \\ \sup_{\mathbb{P}\in\mathcal{P}_{t|t}} \mathbb{P}\text{-CVaR}_\varepsilon \left[\begin{bmatrix} A^\top EB \\ EB \end{bmatrix}^\top \xi^c_{t|t}\right] \end{bmatrix}, \\
\tilde{r}(x) &= \sup_{\mathbb{P}\in\mathcal{P}_{t|t}} \mathbb{P}\text{-CVaR}_\varepsilon \left[(\xi^c_{t|t})^\top \bar{P}\xi^c_{t|t} + 2\bar{q}_1^\top \xi^c_{t|t}\right] \\
&\quad + (A\bar{x}_{t|t} - x_c)^\top E(A\bar{x}_{t|t} - x_c) \\
&\quad - \alpha(\bar{x}_{t|t} - x_c)^\top E(\bar{x}_{t|t} - x_c) - (1-\alpha), \\
\tilde{A} &= \begin{bmatrix} I & -I \\ -I & -I \end{bmatrix}.
\end{aligned} \tag{19}$$

Proof: Applying Proposition 2.1:

$$\begin{aligned}
&\sup_{\mathbb{P}\in\mathcal{P}_{t|t}} \mathbb{P}\text{-CVaR}_\varepsilon \left[(\xi^c_{t|t})^\top \bar{P}\xi^c_{t|t} + 2\bar{q}^\top \xi^c_{t|t} + \bar{r}\right] \\
&\leq \sup_{\mathbb{P}\in\mathcal{P}_{t|t}} \mathbb{P}\text{-CVaR}_\varepsilon \left[(\xi^c_{t|t})^\top \bar{P}\xi^c_{t|t} + 2\bar{q}_1^\top \xi^c_{t|t}\right] \\
&\quad + 2\sup_{\mathbb{P}\in\mathcal{P}_{t|t}} \mathbb{P}\text{-CVaR}_\varepsilon \left[\bar{q}_2^\top \xi^c_{t|t}\right] + \bar{r}.
\end{aligned} \tag{20}$$

Further, Lemma 2.2 implies:

$$\begin{aligned}
&\sup_{\mathbb{P}\in\mathcal{P}_{t|t}} \mathbb{P}\text{-CVaR}_\varepsilon \left[\bar{q}_2^\top \xi^c_{t|t}\right] \\
&\leq |u_t|^\top \sup_{\mathbb{P}\in\mathcal{P}_{t|t}} \mathbb{P}\text{-CVaR}_\varepsilon \left[\begin{bmatrix} A^\top EB \\ EB \end{bmatrix}^\top \xi^c_{t|t}\right].
\end{aligned} \tag{21}$$

Thus, a sufficient condition for constraint satisfaction is:

$$\begin{aligned}
&\sup_{\mathbb{P}\in\mathcal{P}_{t|t}} \mathbb{P}\text{-CVaR}_\varepsilon \left[(\xi^c_{t|t})^\top \bar{P}\xi^c_{t|t} + 2\bar{q}_1^\top \xi^c_{t|t}\right] \\
&+ 2\,|u_t|^\top \sup_{\mathbb{P}\in\mathcal{P}_{t|t}} \mathbb{P}\text{-CVaR}_\varepsilon \left[\begin{bmatrix} A^\top EB \\ EB \end{bmatrix}^\top \xi^c_{t|t}\right] + \bar{r} \leq 0.
\end{aligned} \tag{22}$$

Introducing $v_t \geq 0$ such that $-v_t \leq u_t \leq v_t$ and completing squares leads to the conditions in (18)-(19). ∎

V. Controller Synthesis

This section, the final of the three main sections, presents the synthesis of controllers utilizing results from previous sections, and discusses the gain K_{t+1} in our approach.

A. The gain K_{t+1}

This subsection consider the role of the gain K_{t+1} in relation to the control input u_t. Basically, the control input should satisfy the constraint (9). By looking back Lemma 2.1, this left-hand-side is minimized when Ω, or Σ is small. Consequently, we employ the gain K_{t+1}, known as the Kalman gain, which minimizes the error covariance $P_{t+1|t+1}$, which corresponds to Σ. The Kalman gain is given by:

$$K_{t+1} = P_{t+1|t}H^\top S^{-1}_{t+1}. \tag{23}$$

With this, the error covariance of updated state estimate is

$$P_{t+1|t+1} = (I - K_{t+1}H)P_{t+1|t}. \tag{24}$$

Together with Section III-B, the Kalman filter formula is completed.

B. Method 1: Modifying a nominal controller

One approach to designing a controller using (9) is to adapt a nominal controller $u_{\text{nom}}(x)$, which does not account for safety constraints, to comply with derived safety conditions by minimally modifying it so as to ensure the safety [8], [29]. The control input at t can be computed by

$$\begin{aligned}
&u^*(x) = \text{argmin}_u \|u - u_{\text{nom}}(x)\|^2 \\
&\quad \text{s.t. CBF constraint (9) e.g., (11), and (18).}
\end{aligned} \tag{25}$$

This method prioritizes safety while minimizing deviation from the nominal controller. If infeasible, a modified optimization problem incorporating a penalty parameter $\rho > 0$ to balance safety and controller deviation can be used. In case of the ellipsoidal safe set, for example, the constraint (18) is revised:

$$\begin{aligned} u^*(x) &= \mathrm{argmin}_u \|u - u_{\mathrm{nom}}(x)\|^2 + \rho\delta \\ \text{s.t. } &\bar{u}_t^\top \tilde{P}\bar{u}_t + 2\tilde{q}^\top(x)\bar{u}_t + \tilde{r}(x) \le \delta \text{ and } \tilde{A}\bar{u}_t \le 0. \end{aligned} \tag{26}$$

C. Method 2: CLF-CBF-based optimization

Another approach is to combine Control Lyapunov Functions (CLF) and CBF to obtain the control inputs [28], [30].

Let us first define the CLF:

Definition 5.1 (Control Lyapunov Function [2]): A map $V : \mathbb{R}^n \to \mathbb{R}$ is an exponential control Lyapunov function for the system $x_{t+1} = f(x_t, u_t)$ if there exists:

- positive constants c_1 and c_2 such that $c_1\|x_t\|^2 \le V(x_t) \le c_2\|x_t\|^2$, and
- a control input $u_t : \mathbb{R}^m \to \mathbb{R}$, $\forall x_t \in \mathbb{R}^n$ and $c_3 > 0$ such that $V(x_{t+1}) - V(x_t) + c_3\|x_t\|^2 \le 0$.

We consider aiming at stabilizing the estimated mean states $\bar{x}_{t|t}$, $\bar{x}_{t+1|t}$ by choosing an appropriate V and forcing the condition $V(\bar{x}_{t+1|t}) - V(\bar{x}_{t|t}) + c_3\|\bar{x}_{t|t}\|^2 \le 0$.

Choose

$$V(x_t) = x_t^\top \Phi x_t \tag{27}$$

for some $\Phi \in \mathbb{S}_+^n$. Choosing $c_1 = \sigma_{\min}$ and $c_2 = \sigma_{\max}$, where $\sigma_{\min}$ and $\sigma_{\max}$ are the smallest and largest singular values of Φ, respectively, satisfies the first condition of CLF.

For the second condition to be satisfied, u_t must exist such that satisfies the quadratic constraint.

$$\begin{aligned} &V(\bar{x}_{t+1|t}) - V(\bar{x}_{t|t}) + c_3\|\bar{x}_{t|t}\|^2 \\ =&(A\bar{x}_{t|t} + Bu_t)^\top \Phi (A\bar{x}_{t|t} + Bu_t) - \bar{x}_{t|t}^\top \Phi \bar{x}_{t|t} + c_3\|\bar{x}_{t|t}\|^2 \\ =&u_t^\top B^\top \Phi B u_t + 2(A\bar{x}_{t|t})^\top \Phi B u_t \\ &\quad + \bar{x}_{t|t}^\top (A^\top \Phi A - \Phi + c_3 I)\bar{x}_{t|t} \le 0. \end{aligned} \tag{28}$$

Combining this with the CBF constraint, at each time, the control input can be computed by

$$\begin{aligned} \mathbf{v}^*(x) = &\underset{\mathbf{v}=[u_t,\delta]^\top \in \mathbb{R}^{m+1}}{\arg\min} \frac{1}{2}\mathbf{v}^\top \Theta \mathbf{v} + \eta^\top \mathbf{v} \\ \text{s.t. } &u_t^\top B^\top \Phi B u_t + 2(A\bar{x}_{t|t})^\top \Phi B u_t \\ &\quad + \bar{x}_{t|t}^\top (A^\top \Phi A - \Phi + c_3 I)\bar{x}_{t|t} \le \delta, \\ &\text{CBF constraint (9) e.g., (11), and (18)} \end{aligned} \tag{29}$$

where a positive definite matrix $\Theta \in \mathbb{S}_+^{m+1}$ and a positive vector $\eta \in \mathbb{R}^{m+1}$ are weights and δ is a relaxation variable that ensures the solvability of the optimization problem by relaxing the constraint (28).

VI. Numerical Examples

In this section, the effectiveness of the control strategies developed in Section V is demonstrated through numerical simulations, using an example of vehicle navigation.

Consider the state vector $x_t = \begin{bmatrix} x_{t,1} & x_{t,2} \end{bmatrix}^\top$, representing the vehicle's position $x_{t,1}$ and velocity $x_{t,2}$ at time t. The control input u_t denotes the commanded acceleration, while the output z_t denotes the measured position at time t. With the sampling time 0.05, the system's dynamics are modeled as follows:

$$\begin{aligned} x_{t+1} &= \begin{bmatrix} 1 & 0.05 \\ 0 & 1 \end{bmatrix} x_t + \begin{bmatrix} 0.0125 \\ 0.05 \end{bmatrix} u_t + w_t, \\ z_t &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_t + v_t. \end{aligned} \tag{30}$$

The disturbance and noise covariances are:

$$\begin{aligned} Q &= \begin{bmatrix} 7.66 \times 10^{-5} & 3.06 \times 10^{-3} \\ 3.06 \times 10^{-3} & 1.23 \times 10^{-1} \end{bmatrix}, \\ R &= 0.09. \end{aligned} \tag{31}$$

Methods 1 and 2 are illustrated for a half-space safe set:

$$\mathcal{C} = \{x \in \mathbb{R}^2 : h(x) = \begin{bmatrix} 0.4 & 0.4 \end{bmatrix} x + 1 \ge 0\}. \tag{32}$$

The design parameters $\varepsilon = 0.3$ and $\alpha = 0.7$, and the initial conditions $\bar{x}_0 = \begin{bmatrix} 7 & 0 \end{bmatrix}^\top$ and $P_0 = Q$ are used. The nominal controller used in Method 1 is

$$u(x) = -15x_1 - 5x_2, \tag{33}$$

and the parameters used in Method 2 are

$$\Psi = \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix}, \Theta = \begin{bmatrix} 10 & 0 \\ 0 & 0.1 \end{bmatrix}, \eta = \begin{bmatrix} 0 & 100 \end{bmatrix}^\top, c_3 = 10. \tag{34}$$

For the duration of time 4, the performances of the proposed controllers (Proposed) are compared with the performances of the controllers that

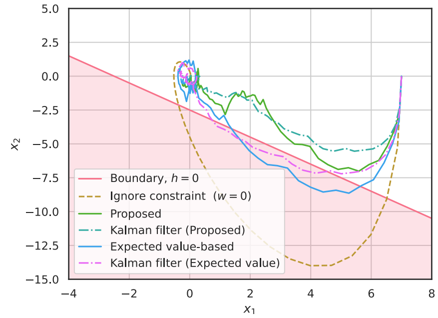
- disregard safety constraints assuming no disturbances (Ignore constraint ($w = 0$)), and
- use the expected value instead of the worst-case CVaR (Expected value-based).

These comparisons are illustrated in Figs. 1a and 1b for Methods 1 and 2, respectively. The trajectories of the Kalman filter's estimated state mean discussed in III-B are also included.

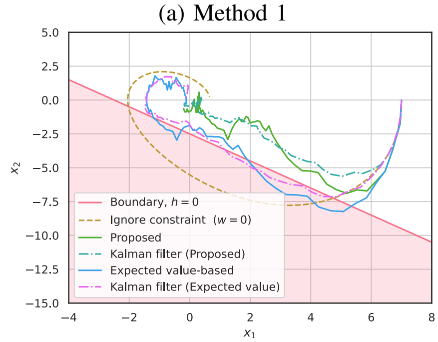
We observe in both figures that ignoring safety constraints leads to trajectories entering unsafe regions even without disturbances. This underscores the importance of employing CBFs to maintain safety. In Fig. 1a, this is the trajectory resulting from using the controller (33).

The Kalman filter's estimation, which are used to obtain the control inputs, align closely with true dynamics for both the proposed and expected value-based controllers.

On the other hand, the most part of the trajectories of expected value-based controller enters the unsafe region while resulting that of the proposed controller remain in the safe region regardless of the stochastic disturbance. This shows the use of expected value is not sufficient to remain in the safe region. Moreover, in these cases, we see the trajectories also successfully approach to the origin as desired.



(a) Method 1



(b) Method 2

Fig. 1: Phase portrait for vehicle navigation: the red regions indicate the unsafe region $h(x) < 0$

VII. Conclusions

This paper developed control approaches for discrete-time linear stochastic systems with partially observable states. Key contributions include:

- Employment of Kalman filter with the worst-case CVaR for effective incorporation of the tail risk into CBFs
- Development of risk-aware constraints for the controller synthesis that effectively manage tail risks at the boundaries of safe sets
- Integration of risk-aware constraints into two control methods: modification of nominal controllers and CLF-CBF-based optimization
- Demonstration of improved safety and reliability through numerical examples of vehicle navigation

The proposed approaches contribute to the field of risk-aware control, offering a foundation for future research in enhancing safety measures. Future research directions include investigating the applicability of other state observers within this framework to potentially broaden the approach's versatility, as well as expanding these techniques to nonlinear systems, thereby enhancing their scope and impact.

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